



# Polynomial (including Quadratic), Rational, and Radical Equations: A Guide to High School Algebra II Standards

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# Table of Contents

<b>Part 1: What do the standards say?</b>	<b>4</b>
REASONING ABOUT QUADRATIC EQUATIONS AND THE STRUCTURE OF THE NUMBER SYSTEM	6
REASONING ABOUT POLYNOMIAL EQUATIONS	9
REASONING ABOUT RATIONAL AND RADICAL EQUATIONS	14
THE ROLE OF MATHEMATICAL PRACTICES	21
<b>Part 2: Reasoning about graphical solutions to equations</b>	<b>22</b>
SOLVING A SIMPLE CUBIC EQUATION	22
COLLEGE FUND	23
IDEAL GAS LAW	24
GRAPHING FROM FACTORS III	25
<b>Part 3: Where does reasoning with equations come from?</b>	<b>26</b>
ALGEBRA I: SOLVING QUADRATIC EQUATIONS	26
SUGGESTIONS FOR STUDENTS WHO ARE BEHIND	31
<b>Endnotes</b>	<b>32</b>

**A-APR.B | Understand the relationship between zeros and factors of polynomials**

**A-APR.D | Rewrite rational expressions**

**A-REI.A | Understand solving equations as a process of reasoning and explain the reasoning**

**A-REI.B | Solve equations and inequalities in one variable: Solve quadratic equations in one variable**

**A-REI.D | Represent and solve equations and inequalities graphically**

**A-SSE.A | Interpret the structure of expressions**

**N-RN.A | Extend the properties of exponents to rational exponents**

**N-CN.A | Perform arithmetic operations with complex numbers**

**N-CN.C | Use complex numbers in polynomial identities and equations**

Welcome to the UnboundEd Mathematics Guide series! These guides are designed to explain what new, high standards for mathematics say about what students should learn in each grade, and what they mean for curriculum and instruction. This guide, the first for Algebra II, includes two parts. The first part gives a “tour” of the standards focused on reasoning about polynomial (and quadratic), radical and rational equations using freely available online resources that you can use or adapt for your class. It then explains how reasoning about equations relates to other mathematical content in Algebra II, especially graphing, and how to use understandings from prior grades to support students who enter Algebra II with gaps in their learning.

# Part 1: What do the standards say?

Algebra II is full of rich content, much of which is important for success in higher mathematics and in a range of careers. So why should this series, and probably your year, begin with reasoning about equations? Like Algebra I and Grade 6 even earlier, Algebra II represents a crucial inflection point in students' learning. In Grade 6, we saw the culmination of fractions and the launching of ratios and proportional relationships. In Algebra I, we saw the culmination of reasoning about linear equations and the maturation of reasoning about quadratic equations. In Algebra II, and in this guide in particular, we see the culmination of reasoning about quadratic equations and the launching of reasoning about higher-order polynomial equations, radical equations and rational equations. Reasoning about equations, therefore, is a good way for students to "ease into" Algebra II with material that relates strongly to what they already know. At the same time, the core idea of reasoning (starting with a claim and making a precise argument to support that claim using a series of logical steps supported by detail) is used again and again in other course work—like literary analysis, social studies, science and later mathematics—and in many careers, so it's good that students continue to build expertise with it.

Another reason to focus on reasoning with equations is that five of the nine clusters covered in this guide (A-SSE.A, A-APR.B, A-REI.A, A-REI.D, and N-RN.A) are also recognized as "major" by PARCC's Model Content Frameworks, meaning they deserve a significant amount of attention over the course of the school year.<sup>1</sup> (It's generally a good idea to prioritize major standards within the year to make sure they get the attention they deserve.)

The high school standards are organized into five "categories," and within each category are a number of "domains." The standards involving reasoning about polynomial and quadratic, radical, rational equations are spread across three domains in the Algebra category—"Seeing Structure in Expressions" (A-SSE), "Arithmetic with Polynomials and Rational Expressions" (A-APR) and "Reasoning with Equations and Inequalities" (A-REI). We also have two domains from the Number & Quantity category ("The Real Number System" and "The Complex Number System," or N-RN and N-CN, respectively) that will impact work with radical equations and quadratic equations in particular. Before we get started with the content in these standards, let's pause and take a look at the standards themselves. As you read, think about:

- Where do these standards emphasize conceptual understanding of important ideas?
- Where do these standards include opportunities to develop key procedural skills?

## A-APR.B | Understand the relationship between zeros and factors of polynomials

### A-APR.B.2

Know and apply the Remainder Theorem: For a polynomial  $p(x)$  and a number  $a$ , the remainder on division by  $x - a$  is  $p(a)$ , so  $p(a) = 0$  if and only if  $x - a$  is a factor of  $p(x)$ .

### A-APR.B.3

Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

## A-APR.D | Rewrite rational expressions

### A-APR.D.6

Rewrite simple rational expressions in different forms: write  $a(x)/b(x)$  in the form  $q(x) + r(x)/b(x)$ , where  $a(x)$ ,  $b(x)$ ,  $q(x)$ , and  $r(x)$  are polynomials with the degree of  $r(x)$  less than the degree of  $b(x)$ , using inspection, long division, or, for the more complicated examples, a computer algebra system.

## **A-REI.A | Understand solving equations as a process of reasoning and explain the reasoning**

### **A-REI.A.1**

Explain each step in solving a simple [polynomial, quadratic, rational, and radical] equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

## **A-REI.B | Solve equations and inequalities in one variable: Solve quadratic equations in one variable**

### **A-REI.B.4.B**

Solve quadratic equations by inspection, taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as  $a \pm bi$  for real numbers  $a$  and  $b$ .

## **A-REI.D | Represent and solve equations and inequalities graphically**

### **A-REI.D.11**

Explain why the  $x$ -coordinates of the points where the graphs of the equations  $y = f(x)$  and  $y = g(x)$  intersect are the solutions of the equations  $f(x) = g(x)$ ; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where  $f(x)$  and/or  $g(x)$  are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.

## **A-SSE.A | Interpret the structure of expressions**

### **A-SSE.A.2**

Use the structure of an expression to identify ways to rewrite it.

## **N-RN.A | Extend the properties of exponents to rational exponents**

### **N-RN.A.1**

Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents.

### **N-RN.A.2**

Rewrite expressions involving radicals and rational exponents using the properties of exponents.

## **N-CN.A | Perform arithmetic operations with complex numbers**

### **N-CN.A.1**

Know there is a complex number  $i$  such that  $i^2 = -1$ , and every complex number has the form  $a + bi$  with  $a$  and  $b$  real.

**N-CN.C.7**

Solve quadratic equations with real coefficients that have complex solutions.

The order of the standards doesn't indicate the order in which they have to be taught. Standards are only a set of expectations for what students should know and be able to do by the end of each year; they don't prescribe an exact sequence or curriculum. The high school standards can be sequenced in a variety of ways that result in a coherent experience for students.<sup>2</sup>

**The importance of coherence**

Historically, Algebra II has been organized more like a checklist than a coherent inflection point in students' mathematical journey. In this guide, we suggest that working with a variety of different equations is not an end unto itself, but rather a way for students to understand and use mathematical reasoning. The standards above (and their associated clusters, domains and categories) all relate to the idea of reasoning about equations. We begin with quadratic equations, which completes a progression from Algebra I. Then we turn to higher order polynomial equations, a natural extension from quadratic equations. Next, we consider rational equations which follow from the long division necessary to solve polynomial equations. We finish with radical equations, where we study in more detail the reversibility and irreversibility of certain reasoning, which can lead to extraneous solutions.

*Reasoning about quadratic equations and the structure of the number system*

We begin our reasoning work by considering our well-worn friend, the quadratic equation. As with all equations, students should treat solving equations a process of reasoning, transforming one equation into another equation with the same solutions and justifying their thinking at each step. A-REI.A.1 The idea is to make equation-solving a conceptual undertaking, focusing on why the process works while learning how to complete the necessary calculations. If students only learn algorithmic steps, they run the risk of forgetting why their methods work or making up invalid moves.

While the methods used in Algebra I to reason through to solutions—factoring, completing the square, and the quadratic formula—continue to be helpful, the kinds of quadratic equations that have been solvable to this point have been constrained to those that exist within the set of real numbers ( $x^2 - 3x - 12 = 0$ , for example). However, there are quadratic equations (like  $x^2 + 6x + 10 = 0$ ) that can be solved, but only by working within a superset of the real numbers: the complex numbers. A-REI.B.4, N-CN.C.7 In other words, by considering the structure of number systems, and by working within a larger system, we are able to solve more problems. This is particularly helpful with engineering applications and modeling certain kinds of physics problems.

In Algebra II, students are introduced to complex numbers in service of solving quadratic equations that otherwise would be unsolvable. The introduction of the new idea of complex numbers to a problem type with which they are already familiar is a good way to link students' prior knowledge of quadratics to new learning. The example below is taken from a lesson that does just this. The lesson begins by positing an "unsolvable" equation ( $x^2 + 1 = 0$ ), and this is the conclusion of the resulting discussion. In the process, they come to see that the equation is only unsolvable *over the real numbers*, but has a solution *within the complex numbers*. (It's definitely worth viewing the whole lesson plan to see how imaginary numbers are derived from real ones via rotations of the number line, but for now we'll focus on the moment in which complex numbers emerge.)

## Algebra II, Module 1, Lesson 37: Discussion

When we perform two  $90^\circ$  rotations, it is the same as performing a  $180^\circ$  rotation, so multiplying by  $i$  twice results in the same rotation as multiplying by  $-1$ . Since two rotations by  $90^\circ$  is the same as a single rotation by  $180^\circ$ , two rotations by  $90^\circ$  is equivalent to multiplication by  $i$  twice, and one rotation by  $180^\circ$  is equivalent to multiplication by  $-1$ , we have

$$i^2 * x = -1 * x$$

for any real number  $x$ ; thus,

$$i^2 * x = -x$$

Why might this new number  $i$  be useful?

- Recall from the Opening Exercise that there are no real solutions to the equation

$$x^2 + 1 = 0$$

- However, this new number is a solution.

$$(i)^2 + 1 = -1 + 1 = 0$$

In fact, “solving” the equation  $x^2 + 1 = 0$ , we get

$$x^2 = -1$$

$$\sqrt{x^2} = \sqrt{-1}$$

$$x = \sqrt{-1} \text{ or } x = -\sqrt{-1}$$

However, because we know from above that  $i^2 = -1$ , and  $(-i)^2 = (-1)^2(i)^2 = -1$ , we have two solutions to the quadratic equation  $x^2 = -1$ , which are  $i$  and  $-i$ .

These results suggests that  $i = \sqrt{-1}$ . That seems a little weird, but this new imagined number  $i$  already appears to solve problems we could not solve before.

Algebra II, Module 1, Lesson 37 Available from [engageny.org/resource/algebra-ii-module-1-topic-d-lesson-37](https://www.engageny.org/resource/algebra-ii-module-1-topic-d-lesson-37); accessed 2015-05-29. Copyright © 2015 Great Minds. UnboundEd is not affiliated with the copyright holder of this work.

This lesson (as with many in Algebra II) relies on nuanced reasoning to derive new ideas from old ones. The process starts with a few assumptions: that the square of a number is always positive, based on the properties of integers and exponents, and that every number has two square roots (positive and negative). If you believe your students might struggle to articulate these key ideas, it might help to begin the lesson with a series of simple review problems, similar to those they first encountered in Grades 6-8. You might, for example, have them evaluate several pairs of integer expressions, such as  $3^2$  and  $(-3)^2$  and ask them what they notice. And you might have them solve a few equations, such as  $x^2 = 25$ , and have students explain why there are two solutions. Having these prerequisite ideas at the forefront will help students make sense of the reasoning involved with the rest of the lesson, and can help them answer more advanced questions. For example, can there be a square root of  $-1$ ? Why or why not? What if there were a number that allowed us to solve  $x^2 = -1$ ? What would the properties of that number need to be?

## Working with complex numbers

Once students see how complex numbers arise, they should have opportunities to understand the structure of complex numbers, discovering and articulating patterns along the way. N-CN.A.1, N-CN.A.2 However, consider spending time judiciously on this topic, as precious instructional time in Algebra II is best spent on the major work of the course. With complex numbers in their toolboxes, students have the tools necessary to reason about the solutions to *any* quadratic equation. And because reasoning about equations is the main goal here, the following task helps give students an opportunity to apply previous approaches in new ways:

### N-CN, A- REI Completing the Square

Renee reasons as follows to solve the equation  $x^2 + x + 1 = 0$

- First I will rewrite this as a square plus some number.

$$x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$$

- Now I can subtract  $\frac{3}{4}$  from both sides of the equation

$$\left(x + \frac{1}{2}\right)^2 = -\frac{3}{4}$$

- But I can't take the square root of a negative number so I can't solve this equation.
  - a. Show how Renee might have continued to find the complex solutions of  $x^2 + x + 1 = 0$
  - b. Apply Renee's reasoning to find the solutions to  $x^2 + 4x + 6 = 0$

N-CN, A-REI by Illustrative Mathematics is licensed under CC BY 4.0.

What's nice about this task is both its connection back to a well-known approach from Algebra I—completing the square—and, its connection to the new idea, complex numbers. Notice also the transparency of the reasoning in Renee's example, and how it shows her thinking and her work, including the moment she could no longer solve the problem. Another approach to building reasoning with quadratic equations is to introduce the discriminant, which allows students to reason first about the nature of the solutions that they are looking for, before they start looking.



## Reasoning about polynomial equations

Having examined quadratic equations, let's look at equations of a higher degree, namely polynomial equations. Students learned about polynomial arithmetic and solving quadratic equations in Algebra I, and extending that learning to higher order polynomials. Note, though, that we are not just "doing more work with polynomials." Rather, we are deepening, extending and becoming more expert in reasoning with equations. The goal, as indicated by standard A-REI.A.1, is to "explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method." In other words, rather than focusing solely on the procedures involved, the standards describe that students should understand *why* the process of solving an equation works, and what each step in that process means. A-REI.A.1 Throughout this guide, we are going to return to these ideas repeatedly. Reasoning, as the standards tell us, is about solving with intent and purpose, justifying solution steps, and taking steps that are logically connected.

In the past, reasoning through solutions for polynomial equations has often been taught as rote procedure, with little or no connection to learning from prior grades: Think FOIL or Synthetic Division. As a result, students were often able to perform the procedures to add, subtract, multiply and divide polynomials, but lacked any conceptual understanding of why those procedures worked, or even when they should be used in a solution process. These methods also had limited utility. FOIL, for example, is a method that works to find the product of  $(2x + 5)$  and  $(3x - 8)$ , but what about multiplying  $(x + 5)$  by  $(x^2 - 3x + 10)$ ? When framed within the context of structures that students already know, however, reasoning about polynomial equations—and division of polynomials in particular—makes sense conceptually and is easier for students to retain. Moreover, students are also able to solve a greater variety of problems.

### The Remainder Theorem and polynomial division

In Algebra II, reasoning about polynomial equations such as  $ax^3 - 3x^2 - x = -3$  depends greatly on a student's ability to use both the Remainder Theorem and polynomial division. These can seem like complicated ideas, but both have their structural roots in the division of integers. In Grade 6, students are expected to master the standard algorithm for division after years of coming to understand the concept of division using strategies based on place value, the properties of operations, and the relationship between multiplication and division. (6.NS.B.2) Later, in Algebra I, students learn that polynomials have a structure similar to the integers. As with integers, polynomial division uses the same structure, and they can also be divided using "long division." In the elementary grades, students learn that if one number divided by another leads to no remainder ( $8 \div 4 = 2$ ), then the divisor (4) is a factor of the dividend (8). In Algebra II, students extend this idea to polynomials using the Remainder Theorem. A-APR.B.2 The Remainder Theorem essentially states that when a polynomial is divided by another polynomial, and the remainder is zero, the divisor is a factor of the dividend. The example below shows the connection between whole number division and polynomial division.

## Algebra II, Module 1, Lesson 4: Example 1

If  $x = 10$ , then the division  $1573 \div 13$  can be represented using polynomial division.

$$x+3 \overline{) x^3 + 5x^2 + 7x + 3}$$

The quotient is  $x^2 + 2x + 1$ .

The completed board work for this example should look something like this:

The image shows two handwritten division problems side-by-side. The left problem is integer division: 13 into 1573. The quotient is 121, with remainders 27, 13, and 0. The right problem is polynomial division: (x+3) into (x^3 + 5x^2 + 7x + 3). The quotient is x^2 + 2x + 1, with remainders 2x^2 + 7x, x + 3, and 0. The coefficients of the polynomial match the digits of the integer problem.

$$\begin{array}{r} 121 \\ 13 \overline{) 1573} \\ \underline{-13} \phantom{00} \\ 27 \phantom{00} \\ \underline{-26} \phantom{00} \\ 13 \phantom{00} \\ \underline{-13} \\ 0 \end{array}$$
$$\begin{array}{r} 1 \cdot x^2 + 2x + 1 \\ x+3 \overline{) x^3 + 5x^2 + 7x + 3} \\ \underline{-x^3 - 3x^2} \phantom{00} \\ 2x^2 + 7x \phantom{00} \\ \underline{-2x^2 - 6x} \phantom{00} \\ x + 3 \phantom{00} \\ \underline{-x - 3} \\ 0 \end{array}$$

Algebra II, Module 1, Lesson 4 Available from [engageny.org/resource/algebra-ii-module-1-topic-a-lesson-4](https://engageny.org/resource/algebra-ii-module-1-topic-a-lesson-4); accessed 2015-05-29. Copyright © 2015 Great Minds. UnboundEd is not affiliated with the copyright holder of this work.

Notice that in both cases, the same algorithm is used; this is a key connection point that can really help students make sense of polynomial division. Also, the numbers in the first division match the coefficients in the second. This is also a nice scaffold for students making the shift from whole number to polynomial division. Essentially, we can simplify the problem to understand the kinds of coefficients we should have in the answer. Finally, students should see that since both answers contain no remainders, the divisor is a factor of the dividend.

### Why not just use synthetic division?

As with the rest of the standards, fluency in polynomial division should follow conceptual understanding of why the process works. While synthetic division can be a useful shortcut, it doesn't clearly relate to what students have learned before, and is best reserved for a fourth-year course (if it is to be introduced at all) due to its relative abstraction. Connections between long division of integers and long division of polynomials builds understanding, while synthetic division, offered too soon, can become a distraction.

## Solving using the Remainder Theorem and long division

Once students understand polynomial long division and the relationship between remainders and factors with polynomials, you can shift from a conversation about the structure of division to reasoning about equations. Briefly, the goal is for students to find the values of  $x$  that make both sides of an equation equal 0, just as they did with quadratics. A-APR.B.3 And, just as they did with quadratic equations, students are looking for factors. The nice thing about the Remainder Theorem and long division is that it allows students to get a foothold into the solution of certain higher-order polynomials by reducing their degree through long division. (Equations used in lessons on this idea will, of course, need to be amenable to long division.) Once the degree is reduced to 2 (a quadratic), they have plenty of tools to finish solving. In later courses, students will learn about the Rational Roots Theorem, which guides them in choosing a potential root to start with. For now, though, we suggest either looking at parts of a graph to get started, or to start by checking a given potential candidate and then finding others.

Let's go back to our example above:  $x^3 - 3x^2 - x = -3$ . As we did with quadratics, we start by adding 3 to both sides, in order to set the equation equal to 0:  $x^3 - 3x^2 - x + 3 = 0$ . From here, we begin looking for binomials that will divide into the polynomial with no remainder. If we try  $(x - 1)$ , we get  $x^2 - 2x - 3$ . Since there was no remainder, we know that  $(x - 1)$  is a factor. So we have  $(x - 1)(x^2 - 2x - 3) = 0$ . From there, it is a simple process of factoring the remaining quadratic:  $(x - 3)(x + 1)$ . Based on the factor theorem, we know that the solutions to  $x^3 - 3x^2 - x = -3$  are 1, 3, and -1. The example below shows a slightly different twist:

### Algebra II, Module 1, Lesson 19: Example 4

1. Consider the polynomial  $P(x) = x^3 + kx^2 + x + 6$ .

a. Find the value of  $k$  so that  $x + 1$  is a factor of  $P$ .

*In order for  $x + 1$  to be a factor of  $P$ , the remainder must be zero. Hence, since  $x + 1 = x - (-1)$ , we must have  $P(-1) = 0$  so that  $0 = -1 + k - 1 + 6$*

*Then  $k = -4$*

b. Find the other two factors of  $P$  for the value of  $k$  found in part (a).

$$P(x) = (x + 1)(x^2 - 5x + 6) = (x + 1)(x - 2)(x - 3)$$

Algebra II, Module 1, Lesson 19 Available from [engageny.org/resource/algebra-ii-module-1-topic-b-lesson-19](http://engageny.org/resource/algebra-ii-module-1-topic-b-lesson-19); accessed 2015-05-29. Copyright © 2015 Great Minds. UnboundEd is not affiliated with the copyright holder of this work.

In this task, students need to reason about the nature of factor: namely, by the zero product property, if  $x + 1$  is a factor, then  $x = -1$  is the root, and evaluation of the function at the value of the root is 0. After that, they can put the factor to use in finding  $k$ , which requires additional application of the Remainder Theorem. Note that students seek all of the factors—not just one.

Here's another example, which is a bit more classical.

#### Using the Remainder Theorem and Long Division to Solve a Polynomial Equation

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Suppose we have the polynomial equation  $x^3 - 2x^2 - 21x = 18$ , and we want to determine whether  $x - 6$  is a factor (and thus, that 6 is a solution), and, if it is, what the other factors and solutions are.

The Remainder Theorem tells us two things. First, it tells us that if we substitute 6 in for  $x$ , then both sides of the equation should be equal.

Let's check:  $6^3 - 2 \cdot 6^2 - 21 \cdot 6 = 216 - 72 - 126 = 18$ . Great! Since  $18 = 18$ , we know that 6 is a solution.

But, how do we find the others? This is the second way in which the Remainder Theorem is helpful. Since 6 is a solution,  $x - 6$  is a factor. Knowing this, we can boldly divide our original polynomial by  $x - 6$ . We can predict that our quotient is now a very tolerable quadratic, which we can solve any number of ways.

Let's start by checking:  $\frac{x^3 - 2x^2 - 21x - 18}{x - 6}$  yields  $x^2 + 4x + 3$  as a quotient, with no remainder.

As such, we are both sure that  $x - 6$  is a factor (and 6 is a solution), and we can factor  $x^2 + 4x + 3 = 0$  quite easily.

Factoring yields  $(x + 3)(x + 1) = 0$ , leaving us with -3 and -1 as the other two solutions, when each expression is set to 0. So,  $x^3 - 2x^2 - 21x = 18$  is really  $(x - 6)(x + 3)(x + 1) = 0$ , so, by the zero product property, our solutions are 6, -3, and -1.

In previous courses, reasoning about the solution(s) to a third-degree polynomial may not have been manageable unless through technology. Now, with tools like the Remainder Theorem and long division, students can find all solutions. (It's important to note that A-APR.B.3 also mentions sketching graphs once we find the zeros. More on that in Part 2 of this guide.)

## Factoring with higher degree polynomials: Another path to solving

The Remainder Theorem and long division are not the only tools available to students in solving problems with higher degree polynomials. In fact, students should be extending their work from Algebra I around factoring quadratics and special case polynomials to Algebra II, using structure as a tool to support problem-solving.A-SSE.A.2 In the same way students come to see polynomial long division as based on the same structure of whole number long division, they come to see some special case higher order polynomial factorizations as based on many of the same structure as quadratic factorization, which allows for more tools in solving problems. The exercise below illustrates a way to bring this point home:

### Algebra II, Module 1, Lesson 13: Opening Exercise

Factor each of the following expressions. What similarities do you notice between the examples in the left column and those on the right?

a. $x^2 - 1$ $(x - 1)(x + 1)$	b. $9x^2 - 1$ $(3x - 1)(3x + 1)$
c. $x^2 + 8x + 15$ $(x + 5)(x + 3)$	d. $4x^2 + 16x + 15$ $(2x + 5)(2x + 3)$
e. $x^2 - y^2$ $(x - y)(x + y)$	f. $x^4 - y^4$ $(x^2 - y^2)(x^2 + y^2)$

Algebra II, Module 1, Lesson 13 Available from [engageny.org/resource/algebra-ii-module-1-topic-b-lesson-13](https://engageny.org/resource/algebra-ii-module-1-topic-b-lesson-13); accessed 2015-05-29. Copyright © 2015 Great Minds. UnboundEd is not affiliated with the copyright holder of this work.

The exercise serves as a launch point for being able to utilize structure to solve problems with higher order polynomials using factoring. In particular, notice how part (f) changes a difference of two quartics into two factors, which are both quadratic in nature, and where the first quadratic is factorable as the difference of two squares.

## Reasoning about rational and radical equations

After students have reasoned about polynomial equations and about constraints through quadratic equations, it's time to dive into rational and radical equations. Let's take rational equations first, starting with rational expressions. When applying the Remainder Theorem using long division, students will likely wonder how to handle situations where the remainder is not zero. Rational expressions are structured in the form  $a(x)/b(x)$ , where  $a(x)$  and  $b(x)$  are polynomials. Ideally, you'll be able to help students see that rational expressions are structured like rational numbers, and can be rewritten like fractions. For example, where  $1473/15$  can be rewritten as  $98 + 1/5$  or  $98 \frac{1}{5}$ ,  $(x^2 - 5x + 7)/(x - 2)$  can be rewritten as  $(x - 3) + 1/(x - 2)$  using long division. A-SSE.A.2 Moreover, just as rewriting fractions helps us understand them better, rewriting and reasoning about rational expressions gives additional insight into their structure and uses. A-APR.D.6 We can approach the rewriting through a few different methods, depending on the complexity of the rational expression. One way is through inspection: just as we know that  $25/3$  is  $8\frac{1}{3}$  by looking at it, we know that  $(x - 3)/(x - 3)^2$  is  $1/(x - 3)$  quickly and accurately (keeping in mind that  $x \neq 3$ ). In other words, inspection can be thought of as reasoning fluently. Here's another nice

example of what we're talking about. After getting a common denominator on the right-hand side, we have  $\frac{x+1}{x(x+1)}$ , which, by inspection, we can immediately see is the same as  $\frac{1}{x}$ .

### Egyptian Fractions II (excerpted)

b. We will see how we can use identities between rational expressions to help in our understanding of Egyptian fractions. Verify the following identity for any  $x > 0$ :

$$\frac{1}{x} = \frac{1}{x+1} + \frac{1}{x(x+1)}$$

"Egyptian Fractions II" by Illustrative Mathematics is licensed under CC BY 4.0.

Another way is through long division: Some expressions take a little more work, like our example above. Lastly, the standard (A-APR.D.6) also suggests the possible use of computer algebra systems (CAS) to solve "more complicated examples." These can be useful tools for students, but only once they have a solid understanding of simpler examples.

Once students know how to work with rational expressions, they can move into work with rational equations and reason about their solutions. Remember our mantra here: It's not about more procedural gymnastics. Let's take the following task as an example.

## Algebra II, Module 1, Lesson 26: Exercise 3

Solve the following equation:  $\frac{3}{x} = \frac{8}{x-2}$

Method 1: Convert both expressions to equivalent expressions with a common denominator. The common denominator is  $x(x-2)$ , so we use the identity property of multiplication to multiply the left side by  $\frac{x-2}{x-2}$  and the right side by  $\frac{x}{x}$ . This does not change the value of the expression on either side of the equation.

$$\begin{aligned}\left(\frac{x-2}{x-2}\right) \cdot \left(\frac{3}{x}\right) &= \frac{x}{x} \cdot \left(\frac{8}{x-2}\right) \\ \frac{3x-6}{x(x-2)} &= \frac{8x}{x(x-2)}\end{aligned}$$

Since the denominators are equal, we can see that the numerators must be equal; thus  $3x-6=8x$ . Solving for  $x$  gives a solution of  $-\frac{6}{5}$ . At the outset of this example, we noted that  $x$  cannot take on the value of  $0$  or  $2$  but there is nothing preventing  $x$  from taking on the value  $-\frac{6}{5}$ . Thus, we have found a solution. We can check our work. Substituting  $-\frac{6}{5}$  into  $\frac{3}{x}$  gives us  $\frac{3}{(-6/5)} = -\frac{5}{2}$ , and substituting  $-\frac{6}{5}$  into  $\frac{8}{x-2}$  gives us  $\frac{8}{(-6/5)-2} = -\frac{5}{2}$ . Thus, when  $x = -\frac{6}{5}$ , we have  $\frac{3}{x} = \frac{8}{x-2}$ ; therefore,  $-\frac{6}{5}$  is indeed a solution.

Algebra II, Module 1, Lesson 26 Available from [engageny.org/resource/algebra-ii-module-1-topic-c-lesson-26](http://engageny.org/resource/algebra-ii-module-1-topic-c-lesson-26); accessed 2015-05-29. Copyright © 2015 Great Minds. UnboundEd is not affiliated with the copyright holder of this work.

The example above makes wonderfully clear how reasoning looks in Algebra II. Notice how each step taken in the solution process is communicated. Notice also how students can *reason* that, “since the denominators are equal, we can see that the numerators must be equal: thus,  $3x - 6 = 8x$ . Finally, notice how extraneous solutions and constraints are accounted for in defense of the solution, including the checking of the answer.

With a more straightforward example under their belts, students are ready to consider some modeling, too:

## Canoe Trip

Jamie and Ralph take a canoe trip up a river for 1 mile and then return. The current in the river is 1 mile per hour. The total trip time is 2 hours and 24 minutes. Assuming that they are paddling at a constant rate throughout the trip, find the speed that Jamie and Ralph are paddling.

Suppose we let  $x$  denote the speed, in miles per hour, that the canoe would travel with no current. When they are traveling against the current, Jamie and Ralph's speed will be  $x - 1$  miles per hour and when they are traveling with the current their speed will be  $x + 1$  miles per hour. The trip upstream will take  $\frac{1}{x-1}$  hours and the trip downstream will take  $\frac{1}{x+1}$  hours. There are  $\frac{2}{5}$  of an hour in 24 minutes so the total trip lasts for  $2\frac{2}{5}$  hours giving us

$$\frac{1}{x-1} + \frac{1}{x+1} = \frac{12}{5}$$

Multiplying both sides of the equation by  $(x-1)(x+1) = x^2 - 1$  gives

$$(x+1) + (x-1) = \frac{12}{5}(x^2 - 1)$$

This equation simplifies to  $\frac{12}{5}x^2 - 2x - \frac{12}{5} = 0$  or, after further manipulation,

$$6x^2 - 5x - 6 = 0$$

We can use the quadratic formula to solve for  $x$ :

$$x = \frac{5 \pm \sqrt{25 + 144}}{12}$$

We have  $\sqrt{169} = 13$  so the two solutions are  $x = \frac{5 \pm 13}{12}$  or  $x = \frac{3}{2}$  and  $x = -\frac{2}{3}$ . The second solution does not make any sense in this context as the speed cannot be negative. So Jamie and Ralph are paddling at a rate of  $\frac{3}{2}$  miles per hour. Going upstream, the trip takes longer against the current and going with the current the trip is shorter.

"Canoe Trip" by Illustrative Mathematics is licensed under CC BY 4.0.

There are a number of noteworthy aspects to this problem. First, it works with a "simple rational equation," as described in A-REI.A.2. Second, it works with a mainstay concept in mathematics, namely that the distance an object travels is proportional to its rate and the time it travels ( $d = rt$ ). This should be a fairly easy one for students to remember, but is also a quick review if they've forgotten. Third, the resulting equation is our old friend—a quadratic equation. Finally, and perhaps most interesting, is the task offers another opportunity to discuss constraints. This time, it's not about the number set, but rather context and which answer is right given the context. The constraint of the context yields an extraneous solution:  $-2/3$ .

While long division is not needed here, other reasoning processes about equations are. Students use properties of equality (e.g. multiplying both sides of the equation by the same value). They also use properties of operations with fractions (i.e., finding a common denominator) learned in working with uncommon denominators in elementary school and proportional relationships in middle school to simplify the rational expressions.



From the perspective of coherence across grades, we again see Algebra II, and reasoning about rational equations in particular, as both a culmination of prior work and launching of new ideas. Radical equations follow the same suit. Students should begin by making connections between radical equations and the study of rational exponents. N-RN.A.1, N-RN.A.2 They should see, as the standard states, that “the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents.” The lesson below shows an interesting and fairly straightforward way of introducing this idea using the properties of integer exponents as a launching pad. Notice how, once again, students have an opportunity to make use of structure (in this case, the structure of the properties of integer exponents) to bring coherence to a new idea. In particular, students use the product rule for integer exponents,  $x^m x^n = x^{m+n}$ , to explain the meaning of non-integer exponents. For example, we can use the product rule to see that the following is true:  $5^{2/3} \cdot 5^{1/3} = 5$  by showing this:  $5^{2/3+1/3} = 5$  and, therefore, that  $\sqrt[3]{5^2} \cdot \sqrt[3]{5} = 5$ . This task is another example.

## Algebra II, Module 3, Lesson 3: Discussion

Assume for the moment that whatever  $2^{\frac{1}{2}}$  means, it satisfies our known rule for integer exponents  $b^m \cdot b^n = b^{m+n}$

- Working with this assumption, what is the value of  $2^{\frac{1}{2}} \cdot 2^{\frac{1}{2}}$ ?

*It would be 2 because  $2^{\frac{1}{2}} \cdot 2^{\frac{1}{2}} = 2^{\frac{1}{2} + \frac{1}{2}} = 2^1 = 2$*

- What unique positive number squares to 2? That is, what is the only positive number that when multiplied by itself is equal to 2?

*By definition, we call the unique positive number that squares to 2 the square root of 2 and we write  $\sqrt{2}$*

Write the following statements on the board, and ask students to compare them and think about what the statements must tell them about the meaning of  $2^{\frac{1}{2}}$ .

$$2^{\frac{1}{2}} \cdot 2^{\frac{1}{2}} = 2 \text{ and } \sqrt{2} \cdot \sqrt{2} = 2$$

- What do these two statements tell us about the meaning of  $2^{\frac{1}{2}}$ ?

*Since both statements involve multiplying a number by itself and getting 2 and we know that there is only one number that does that, we can conclude that  $2^{\frac{1}{2}} = \sqrt{2}$ .*

At this point, have students confirm these results by using a calculator to approximate both  $2^{\frac{1}{2}}$  and  $\sqrt{2}$  to several decimal places. In the Opening,  $2^{\frac{1}{2}}$  was approximated graphically, and now it has been shown to be an irrational number.

Next, ask students to think about the meaning of  $2^{\frac{1}{3}}$  using a similar line of reasoning

- Assume that whatever  $2^{\frac{1}{3}}$  means will satisfy  $b^m \cdot b^n = b^{m+n}$ 
  - What is the value of  $(2^{\frac{1}{3}})(2^{\frac{1}{3}})(2^{\frac{1}{3}})$ ?

*The value is 2 because  $(2^{\frac{1}{3}})(2^{\frac{1}{3}})(2^{\frac{1}{3}}) = 2^{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = 2^1 = 2$*

- What is the value of  $\sqrt[3]{2} \cdot \sqrt[3]{2} \cdot \sqrt[3]{2}$ ?

*The value is 2 because  $\sqrt[3]{2} \cdot \sqrt[3]{2} \cdot \sqrt[3]{2} = (\sqrt[3]{2})^3 = 2$*

- What appears to be the meaning of  $2^{\frac{1}{3}}$ ?

*Since both the exponent expression and the radical expression involve multiplying a number by itself three times and the result is equal to 2, we know that  $2^{\frac{1}{3}} = \sqrt[3]{2}$*

Algebra II, Module 3, Lesson 3 Available from [engageny.org/resource/algebra-ii-module-3-topic-a-lesson-3](http://engageny.org/resource/algebra-ii-module-3-topic-a-lesson-3); accessed 2015-05-29. Copyright © 2015 Great Minds. UnboundEd is not affiliated with the copyright holder of this work.

The scaffolding suggestion in the right margin is a good one. Some students may have an easier time with this line of questioning if they're able to work with perfect squares and perfect cubes. For example, in the first sequence of questions,  $9^{1/2}$  might work even better than  $2^{1/2}$ , since 9 has a well-known square root. Moreover, using a number with an integer square root would allow students to see that  $9^{1/2} \neq 9(1/2)$ , while also connecting back to the exponent rules for integer exponents. Number 2 in the following task shows a nice example rewriting such expressions:

### Properties of Exponents and Radicals

1. Find the exact value of  $9^{\frac{11}{10}} \cdot 9^{\frac{2}{5}}$  without using a calculator.
2. Justify that  $\sqrt[3]{8} \cdot \sqrt[3]{8} = \sqrt{16}$  using the properties of exponents in at least two different ways.

EngageNY, Algebra II, Module 3, Topic A, Lesson 4: <https://www.engageny.org/resource/algebra-ii-module-3-topic-a-lesson-4>

This problem is nice and simple for two main reasons. First, it asks students to prove the truth of an equation, rather than rewriting for its own sake. In other words, it involves a bit of argument, which engages students in a different way than simply evaluating an expression. Second, it asks students to do the rewriting in two ways (using radicals and using rational exponents), which helps them synthesize the relationships between the two.

Once students see both the properties and notation that rational exponents provide, they can move into reasoning about radical equations. Students should see how strategies for solving a radical equation are a consequence of the properties of exponents, as well as how the structure of radical equations often gives rise to extraneous solutions. This task offers them the opportunity to do both.

## Radical Equations

- a. Solve the following two equations by isolating the radical on one side and squaring both sides:

i.  $\sqrt{2x+1}-5 = -2$

ii.  $\sqrt{2x+1}+5 = 2$

Be sure to check your solutions.

- b. If we raise both sides of an equation a power, we sometimes obtain an equation which has more solutions than the original one. (Sometimes the extra solutions are called extraneous solutions.) Which of the following equations result in extraneous solutions when you raise both sides to the indicated power? Explain.

iii.  $\sqrt{x} = 5$ , square both sides

iv.  $\sqrt{x} = -5$ , square both sides

v.  $\sqrt[3]{x} = 5$ , cube both sides

vi.  $\sqrt[3]{x} = -5$ , cube both sides

- c. Create a square root equation similar to the one in part (a) that has an extraneous solution. Show the algebraic steps you would follow to look for a solution, and indicate where the extraneous solution arises.

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Here, we see the use of “explain” in Part (b) to engage students more deeply in reasoning—they need to be able to tell us how extraneous solutions rise—namely, that some steps are not reversible. For example, if we assume that  $x$  is a number that satisfies situation (i), then we can quickly see that  $x$  is 25. However, that is different than saying, if  $x^2 = 25$ , then  $x$  is 5 (it might be  $-5$ ).<sup>3</sup> Situation (ii) is also a great chance to dig into this with kids. Opportunities exist in Part (a) as well, should you choose to ask students to explain how they used properties of operations, rational exponents, and equality to arrive at their solutions. (Note how extraneous solutions are raised there, too.)

# The role of Mathematical Practices

The Standards don't just include knowledge and skills; they also recognize the need for students to engage in certain important practices of mathematical thinking and communication. These "mathematical practices" have their own set of standards, which contain the same basic objectives for Grades K-12.<sup>4</sup> (The idea is that students should cultivate the same habits of mind in increasingly sophisticated ways over the years.) But rather than being "just another thing" for teachers to incorporate into their classes, the practices are ways to help students arrive at the deep conceptual understandings required in each grade. The table below contains a few examples of how the practices might help students understand and work with equations in Algebra II.

Opportunities for Mathematical Practices:	Teacher actions:
The Remainder Theorem is based on the same basic concepts of division that students get in the elementary grades (for example, the idea that one number divided by another equals a quotient plus a remainder). Therefore, students can better understand the Remainder Theorem when they relate it to division with integers. In this way, they make use of structures they already know to understand new ideas. (MP.7)	Throughout work with polynomials, introduce concepts as "an extension of a structure we already know." When examining division of polynomials, for example, you can show students examples of integer long division next to examples of polynomial long division, and ask them to draw connections between the two methods (see <a href="#">here</a> for an example).
Each of the equation types in this guide, which are mainstays of Algebra II, have their own unique structures for students to discover and use. (MP.7) In particular, quadratic structures offer many opportunities to rewrite expressions in the process of solving equations. Radical and rational equations also offer rich opportunities to consider structure.	For example, students can view rational expressions in light of what they know about division and fractions from previous grades. You might begin a lesson on rewriting rational expressions by linking to what students learned earlier in the course about polynomial long division.A-APR.D.6
Equations derived from a real-world context are abstractions of relationships between quantities. When students solve equations and interpret their solutions in terms of that context, they reason abstractly and quantitatively. (MP.2) They also attend to precision (MP.6) when they use precise reasoning in their solutions, understanding the assumptions and limits involved in that reasoning.	For example, students should realize that every solution process begins with the assumption that an equation has a solution.A-REI.A.1 In the case of quadratic equations, they'll discover that there may not be a solution over the real numbers. When no solution exists, they'll have the chance to abstractly reason about larger number sets.

Podcast clip: Importance of the Mathematical Practices with Andrew Chen and Peter Coe(start 30:33, end 43:39)

## Part 2: Reasoning about graphical solutions to equations

So far in this guide, we've focused almost exclusively on symbolic solutions to equations. But graphical methods for reasoning about solutions to equations A-REI.D.11 can also be powerful. Not only does graphing provide a more concrete representation for students to consider, but it also provides additional tools to support reasoning. While this guide has addressed the symbolic and graphical approaches separately, you are encouraged to teach them simultaneously, so as to maximize their mutually reinforcing attributes.

If your instruction integrates equations and related functions (for example, you teach a unit involving polynomial equations and polynomial functions), your students are well-positioned to see the connections between symbolic and graphical approaches to solving equations.F-IF.C.7.C More broadly, they'll have the opportunity to see and explain the relationship between equations and functions. However, if your plan for the year touches first on various types of equations, and then turns to functions much later in the year, you may have to help them tie these two threads together. In either case, students will benefit from questions that highlight the commonalities between different-looking methods. The following example helps highlight the connections:

### Solving a Simple Cubic Equation

- Find all the values of  $x$  for which the equation  $9x = x^3$  is true.
- Use graphing technology to graph  $f(x) = x^3 - 9x$ . Explain where you can see the answers from part (a) in this graph, and why.
- Someone attempts to solve  $9x = x^3$  by dividing both sides by  $x$ , yielding  $9 = x^2$ , and going from there. Does this approach work? Why or why not?

"Solving a Simple Cubic Equation" by Illustrative Mathematics is licensed underCC BY 4.0.

This task explicitly draws a connection between symbolic and graphical methods: Students solve one way, then the other way, and describe the relationship between the two solutions. They also use the two solutions to discover an important takeaway regarding the reasoning involved with solving equations. The use of graphical methods to solve equations become particularly helpful when students begin modeling real-world situations with mathematics. Here are two examples—one using polynomial expressions and equations and one about rational expressions and equations—that show the power of reasoning graphically about solutions with or without tools.

## College Fund

When Marcus started high school, his grandmother opened a college savings account. On the first day of each school year she deposited money into the account: \$1000 in his freshmen year, \$600 in his sophomore year, \$1100 in his junior year and \$900 in his senior year. The account earns interest of  $r\%$  at the end of each year. When Marcus starts college after four years, he gets the balance of the savings account plus an extra \$500.

- If  $r$  is the annual interest rate of the bank account, then at the end of the year the balance in the account is multiplied by a growth factor of  $x = 1 + r$ . Find an expression for the total amount of money Marcus receives from his grandmother as a function of this annual growth factor  $x$ .
- Suppose that altogether he receives \$4400 from his grandmother. Use appropriate technology to find the interest rate that the bank account earned.
- How much total interest did the bank account earn over the four years?
- Suppose the bank account had been opened when Marcus started Kindergarten. Describe how the expression for the amount of money at the start of college would change. Give an example of what it might look like.

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Though the task does not explicitly require students to create the graph of the function, part b offers an opportunity to highlight using a graph to reason about the solutions (a computer algebra system or graphing calculator may be a useful tool here). By examining the graph, students get a better understanding of the various aspects of meaning of the variables and their relationships in the function.

## Ideal Gas Law

A certain number of Xenon gas molecules are placed in a container at room temperature. If  $V$  is the volume of the container and  $P(V)$  is the pressure exerted on the container by the Xenon molecules, a model predicts that

$$P(V) = \frac{40}{2V-1} - \left(\frac{4}{V}\right)^2$$

for all  $V > \frac{1}{2}$ . Here the units for volume are liters and the units for pressure are atmospheres.

- Sketch a graph of  $P$ .
- Using the graph, approximate the volume for which the pressure is 10 atmospheres.

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Note that the directions call for students to “sketch” a graph of  $P$  in part a. Certainly, students could be asked to solve the equation by hand, but by sketching a graph, they have a ready-made representation of the function from which they can find  $V$  when  $P(V)$  is 10. The helpfulness of the graph becomes more apparent when we attempt to solve the problem using algebraic techniques only.

In addition to reasoning about solutions by graphing, A-APR.B.3 suggests that solutions can be helpful in graphing. Namely, if we know the solutions—or zeros—to a polynomial equation, we can also sketch a graph of it. When graphing by hand in an Algebra II course, students graph the zeros, and then by find additional pairs of points through evaluations at chosen  $x$ -values. In most cases, tasks should provide fairly simple graphs when hand graphing is expected. Here’s a great example that ties all of the work with polynomials together.



## Graphing from Factors III

Mike is trying to sketch a graph of the polynomial

$$f(x) = x^3 + 4x^2 + x - 6$$

He notices that the coefficients of  $f(x)$  add up to zero ( $1 + 4 + 1 - 6 = 0$ ) and says

*This means that 1 is a root of  $f(x)$ , and I can use this to help factor  $f(x)$  and produce the graph.*

- Is Mike right that 1 is a root of  $f(x)$ ? Explain his reasoning.
- Find all roots of  $f(x)$ .
- Find all inputs  $x$  for which  $f(x) < 0$ .
- Use the information you have gathered to sketch a rough graph of  $f$ .

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Students can find the first factor by reasoning about the nature of the function and its coefficients (if all the coefficients add to zero, then 1 is a root because evaluating the function at 1 is the same as working with only the coefficients). Once they find a root of 1, long division (the Remainder Theorem) can be used to find the other factors. Students can also see where the function is negative. As the commentary to this task mentions, "to give a negative output, exactly one of the three factors (or all three factors) has to be negative, giving  $x < -3$  or  $-2 < x < 1$ ."

## Part 3: Where does reasoning with equations come from?

There's quite a bit of important content packed into an Algebra II course, but these standards are intended as a capstone of the learning that occurs in Grades K-8. If your students have been following a strong, standards-aligned program for several years, you might be reaping the benefits of their experience as you read this. But high school teachers often find themselves in a challenging position, as it can be tough to ascertain what students learned in previous grades. And sometimes students have real gaps in their learning that need to be filled before high school work can begin. In this section, we'll consider a few questions you might have as you're planning. First, what exactly were students supposed to learn in the middle school grades that supports their work with equations in Algebra II? How do I leverage what they already know to make high school concepts accessible? And if some of my students are behind, how can I meet their needs without sacrificing focus on high school content?

*Podcast clip: Importance of Coherence with Andrew Chen and Peter Coe (start 9:34, end 26:19)*

### *Algebra I: Solving quadratic equations*

In Grade 8 (**8.EE.A.2**) and in Algebra I, students begin to understand solving equations as a process of reasoning, developing successive equations with the same solutions by applying properties of operations. A-REI.A.1 Most of their early work in this arena involves linear equations, with quadratics emerging later in the year. In Algebra I, quadratics are limited to equations with real solutions, A-REI.B.4.B and the methods used to solve quadratic equations are generally factoring, completing the square, and in some cases, the quadratic formula. (While some students may not be comfortable using the quadratic formula to solve equations, they should at least be aware of how it's derived and why it's useful. A-REI.B.4) The progression of problems below illustrates the development of equation-solving throughout Algebra I and II.

## Algebra I: Solving as a process of reasoning

- Why should the equations  $(x - 1)(x + 3) = 17 + x$  and  $(x - 1)(x + 3) = x + 17$  have the same solution set?
- Why should the equations  $(x - 1)(x + 3) = 17 + x$  and  $(x + 3)(x - 1) = 17 + x$  have the same solution set?
- Do you think the equations  $(x - 1)(x + 3) = 17 + x$  and  $(x - 1)(x + 3) + 500 = 517 + x$  should have the same solution set? Why?
- Do you think the equations  $(x - 1)(x + 3) = 17 + x$  and  $3(x - 1)(x + 3) = 51 + 3x$  should have the same solution set? Explain why.

Source: [EngageNY Algebra I Module 1 Lesson 12](#)

→ This problem, taken from an early lesson on the reasoning involved in equation solving, asks students to consider how the properties of equality can be used to obtain equations with the same solutions. This conceptual foundation allows students to understand why the algebraic procedures for solving linear and quadratic equations work, and prepares them to see the limitations of these procedures when working with rational and radical equations in Algebra II.

## Algebra I: Quadratics with real solutions

$$2d^2 + 5d - 12 = 0$$

$$2d^2 + 5d - 12 = 0$$

Two numbers for which the product is—24 and the sum is +5:  $-3$  and  $+8$ .

So, we split the linear term:  $2d^2 - 3d + 8d - 12 = 0$ .

And group by pairs:  $d(2d - 3) + 4(2d - 3) = 0$ .

Then factor:  $(2d - 3)(d + 4) = 0$ .

So,  $d = -4$  or  $\frac{3}{2}$ .

Source: [EngageNY Algebra I, Module 4, Lesson 5](#)

→ This equation, typical of quadratics in Algebra I, has real number solutions and is solvable by factoring. Other equations might lend themselves to solution by completing the square, which should be viewed as an extension of factoring.

## Algebra II: Quadratics with nonreal solutions

Solve the equation  $5x^2 - 4x + 3 = 0$ .

*We have a quadratic equation with  $a = 5$ ,  $b = -4$ , and  $c = 3$ .*

$$x = \frac{-(-4) \pm 2\sqrt{-11}}{2(5)}$$

*So, the solutions are  $\frac{2}{5} + \frac{i\sqrt{11}}{5}$  and  $\frac{2}{5} - \frac{i\sqrt{11}}{5}$ .*

Source: [EngageNY Algebra II, Module 1, Lesson 38](#)

→ Equations like this one, which is similar to others in this guide, expand upon the work students do in Algebra I, requiring them to use the Quadratic Formula and understand the complex number system in order to make sense of previously unsolvable equations.

## Grade 8 and Algebra I: Exponents and radicals

Starting in Grade 8, students should be able to describe the properties of integer exponents (■ **8.EE.A.1**) and use square and cube roots to solve problems. (■ **8.EE.A.2**) In the course of solving these types of problems, they should also come to understand that there are numbers which can't be represented in rational form, (□ **8.NS.A.1**) and should recognize common irrational numbers (e.g.  $\sqrt{2}$ ). In Algebra I, students encounter exponents and radicals in the context of equation solving and performing operations with polynomials. Completing the square, for example, relies on an understanding of the relationship between a square and a square root. Moreover, the solutions to equations for which completing the square is useful may well be irrational, so students should also be able to explain the relationship between rational and irrational numbers under the operations of addition and multiplication. N-RN.B.3 Another progression of problems shows how exponents and radicals evolve over the years.

### Grade 8: Properties of exponents

$$1423 \times 148 =$$

$$a^{23} \times a^8 =$$

Let  $x$  be a positive integer. If  $(-3)^9 \times (-3)^x = (-3)^{14}$ , what is  $x$ ?

Source: [EngageNY Grade 8, Module 1, Lesson 2](#)

→ With problems like these, students begin to understand how the structure of exponential expressions gives rise to their properties. It's important that students take the time to rewrite expressions and explain these properties as they learn them, rather than memorizing a set of rules.

### Algebra I: Equations with irrational solutions

Solve for  $x$ .

$$x^2 + 6x = 12$$

$$x^2 + 6x + 9 = 12 + 9 \text{ Add 9 to complete the square: } \left[\frac{1}{2}(6)\right]^2$$

$$(x + 3)^2 = 21 \text{ Factor the perfect square.}$$

$$x + 3 = \pm \sqrt{21} \text{ Take the square root of both sides.}$$

Remind students NOT to forget the  $\pm$ .

$$x = -3 \pm \sqrt{21} \text{ Add } -3 \text{ to both sides to solve for } x.$$

$$x = -3 + \sqrt{21} \text{ or } x = -3 - \sqrt{21}$$

Source: [EngageNY Algebra I, Module 4, Lesson 13](#)

→ In Algebra I, solving equations like this one requires students to apply their basic understandings of exponents and radicals. Along with just being able to solve, students should also be able to explain why both solutions, which have a rational and an irrational component, are irrational.

## Algebra II:

Solve the equation  $6 = x + \sqrt{x}$ .

$$6 - x = \sqrt{x}$$

$$(6 - x)^2 = \sqrt{x}^2$$

$$36 - 12x + x^2 = x$$

$$x^2 - 13x + 36 = 0$$

$$(x - 9)(x - 4) = 0$$

The solutions are 9 and 4.

Check  $x = 9$ :

$$9 + \sqrt{9} = 9 + 3 = 12$$

$$6 \neq 12$$

Check  $x = 4$ :

$$4 + \sqrt{4} = 4 + 2 = 6$$

So, 9 is an extraneous solution.

The only valid solution is 4.

Source: [EngageNY Algebra II, Module 1, Lesson 29](#)

→ Equations like these require students to utilize the properties of exponents to a greater extent than ever before, and to understand why squaring a number in the solution process often entails extraneous solutions.

## *Suggestions for students who are behind*

If you know your students don't have a solid grasp of the pre-requisites to the ideas named above (or haven't encountered them at all), what can you do? It's not practical (or even desirable) to re-teach everything students should have learned in Grades 7, 8 and Algebra I, so the focus needs to be on grade-level standards. At the same time, there are strategic ways of wrapping up "unfinished learning" from prior grades and honing essential competencies within instruction focused on the content above. Here are a few ideas for adapting your instruction to bridge the gaps.

- If a significant number of students don't understand the notion of **solutions to quadratic equations**, you could plan a lesson or two on that idea before starting work with solving quadratics over the complex numbers. [This lesson](#), which introduces the Zero Product Property and includes work with visual models, could be a good place to start. And if you think an entire lesson is too much, but your students could still use some review, you could use 2-3 quadratic equation problems as "warm-ups" to start your first few lessons.
- If a significant number of students don't understand **fractions as division**, you could likewise plan a lesson or two on that before introducing polynomial division. ([This Grade 5 lesson](#) may be a good starting place.) Again, if you think that an entire lesson is too much, you could use some problems involving this idea as warm-ups for a few lessons.
- If a significant number of students don't understand the concept of **constraints on solutions to radical and rational equations** you can use a few warm-ups to review the existence of two solutions to square roots of positive integers ([This Grade 8 lesson](#) has some problems that might come in handy.)
- For students who lack **fluency with important operations**—including operations with fractions, decimals and applications of properties—consider incorporating drills involving these operations into your weekly routine. (Drills are more common in the elementary and middle grades, but with a little convincing, even high school students will engage in these sorts of activities.) These could be as simple as a set of ten problems on a certain focus skill (page 22 of [this Grade 6 lesson](#) has an example) or as involved as a timed "sprint" exercise (pages 26-29 of [this Grade 6 lesson](#) show how these might look)

If you've just finished this entire guide, congratulations! Hopefully it's been informative, and you can return to it as a reference when planning lessons, creating units, or evaluating instructional materials. For more guides in this series, please visit our [Enhance Instruction page](#). For more ideas of how you might use these guides in your daily practice, please visit our [Frequently Asked Questions page](#). And if you're interested in learning more about Algebra II, don't forget these resources:

[PARCC Model Content Frameworks](#)

[Draft High School Progression on Algebra](#)

[EngageNY: Algebra II Materials](#)

[Illustrative Mathematics Algebra Tasks](#)

# Endnotes

[1] Though many people know PARCC as an organization dedicated to assessment, much of PARCC's early work focused on helping educators interpret and implement the Standards. Its Model Content Frameworks were among those efforts: They describe the mathematics that students learn at each grade level and in each high school course, and the relative amount of time and attention that standards should be given in a grade or course. Many teachers, regardless of their state's affiliation with PARCC, have found these documents helpful. You can find them [here](#). In this guide, major clusters and standards are denoted by a green square (e.g. A-REI.A.1).

[2] If you're interested in seeing an example of a high-quality Algebra II sequence, you might want to check out the **Illustrative Mathematics Algebra II course blueprint**.

[3] For more on the issue of extraneous solutions, see the [Draft High School Progression for Algebra](#) (p. 13).

[4] You can read the full text of the Standards for Mathematical Practices [here](#).