Lesson 11: Ratios of Fractions and Their Unit Rates

Student Outcomes

- Students use ratio tables and ratio reasoning to compute unit rates associated with ratios of fractions in the context of measured quantities such as recipes, lengths, areas, and speed.
- Students work collaboratively to solve a problem while sharing their thinking processes, strategies, and solutions with the class.

Classwork

Example 1 (20 minutes): Who is Faster?

MP.1

Introduce the problem statement. Allow students to use any approach to find the solution. If one (or more) of the approaches was not used or if a student took a different approach, go through all the possible ways this problem can be solved as a class. Possible approaches are shown below, including bar models, equations, number lines, and clocks. Each approach reviews and teaches different concepts that are needed for the "big" picture. Starting with tables will not only reinforce all of the previous material, but will also review and address concepts required for the other possible approaches.

Note: Time can be represented in either hours or minutes; the solutions show both.

Scaffolding:

- It may be helpful to draw a clock or continually refer to a clock. Many students have difficulty telling time with the new technology available to them.
- Also, it may be helpful to do an example similar to the first example using whole numbers.

Example 1: Who is Faster?

During their last workout, Izzy ran $2\frac{1}{4}$ miles in 15 minutes, and her friend Julia ran $3\frac{3}{4}$ miles in 25 minutes. Each girl thought she was the faster runner. Based on their last run, which girl is correct? Use any approach to find the solution.

<u>Tables</u>:

IZZY

Time (minutes)	Time (hours)	Distance (miles)
15	$\frac{15}{60}=\frac{1}{4}$	$2\frac{1}{4}$
30	$\frac{30}{60}=\frac{1}{2}$	$4\frac{1}{2}$
45	$\frac{45}{60}=\frac{3}{4}$	$6\frac{3}{4}$
60	$\frac{60}{60}=1$	9
75	$\frac{75}{60}=1\frac{1}{4}$	$11\frac{1}{4}$

JULIA

Time (minutes)	Time (hours)	Distance (miles)
25	$\frac{25}{60}=\frac{5}{12}$	$3\frac{3}{4}$
50	$\frac{50}{60}=\frac{5}{6}$	$7\frac{1}{2}$
75	$\frac{75}{60}=1\frac{1}{4}$	$11\frac{1}{4}$
100	$\frac{100}{60}=1\frac{2}{3}$	15

- When looking and comparing the tables, it appears that Julia went further, so this would mean she ran faster. Is that assumption correct? Explain your reasoning.
    - *By creating a table of equivalent ratios for each runner showing the elapsed time and corresponding distance ran, it may be possible to find a time or a distance that is common to both tables. It can then be determined if one girl had a greater distance for a given time or if one girl had less time for a given distance. In this case, at 75 minutes, both girls ran $11\frac{1}{4}$ mile, assuming they both ran at a constant speed.*
- How can we use the tables to determine the unit rate?
    - *Since we assumed distance is proportional to time, the unit rate or constant of proportionality can be determined by dividing the distance by the time. When the time is in hours, then the unit rate will be calculated in miles per hour, which is 9. If the time is in minutes, then the unit rate is calculated in miles per minute, which is $\frac{3}{20}$.*
- Discuss: Some students may have chosen to calculate the unit rates for each of the girls. To calculate the unit rate for Izzy, students divided the distance ran, $2\frac{1}{4}$, by the elapsed time, $\frac{15}{60}$, which has a unit rate of 9. To find the unit rate for Julia, students divided $3\frac{3}{4}$ by $\frac{25}{60}$ and arrived at a unit rate of 9, as well, leading the students to conclude that neither girl was faster.
- We all agree that the girls ran at the same rate; however, some members of the class identified the unit rate as 9 while others gave a unit rate of $\frac{3}{20}$. How can both groups of students be correct?
    - *Time can be represented in minutes; however, in real-world contexts, most people are comfortable with distance measured by hours. It is easier for a person to visualize 9 miles per hour compared to $\frac{3}{20}$ miles per minute, although it is an acceptable answer.*

Scaffolding:

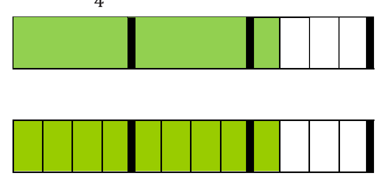
Review how to divide fractions using a bar model.

- How can you divide fractions with a picture, using a bar model?
    - *Make 2 whole unit and a third whole unit broken into fourths. Then, divide the whole units into fourths and count how many fourths there are in the original $2\frac{1}{4}$ units. The answer would be 9.*

Example 1:

$$2\frac{1}{4} \div \frac{1}{4}$$

1. Green shading represents the original $2\frac{1}{4}$ units (1st diagram).
2. Divide the whole units into $\frac{1}{4}$ units (2nd diagram).
3. How many $\frac{1}{4}$ units are there? 9

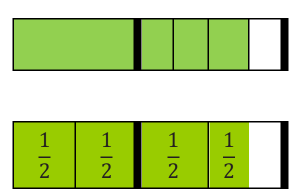


Example 2:

More practice with bar models, if needed:

$$1\frac{3}{4} \div \frac{1}{2}$$

1. Represent $1\frac{3}{4}$ units (represented, here, by green shading).
2. Divide the units into groups of $\frac{1}{2}$.
3. The number of $\frac{1}{2}$ units that are shaded are $3\frac{1}{2}$.



Equations:

Izzy

$d = rt$

$2\frac{1}{4} = r \cdot \frac{1}{4}$

$\frac{4}{1}\left[2\frac{1}{4}\right] = \left[r \cdot \frac{1}{4}\right]\frac{4}{1}$

$9 = r$

9 miles/hour

Julia

$d = rt$

$3\frac{3}{4} = r \cdot \frac{25}{60}$

$\frac{60}{25}\left[3\frac{3}{4}\right] = \left[r \cdot \frac{25}{60}\right]\frac{60}{25}$

$9 = r$

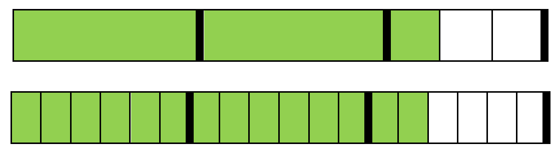
9 miles/hour

Scaffolding:

Example 3:

$$2\frac{1}{3} \div \frac{1}{6}$$

4. Represent $2\frac{1}{3}$ units.
5. Divide into groups of $\frac{1}{6}$.
6. The number of $\frac{1}{6}$ units that are shaded is 14.



- What assumptions are made when using the formula $d = rt$ in this problem?
  - *We are assuming the distance is proportional to time, and that Izzy and Julia ran at a constant rate. This means they ran the same speed the entire time, not slower at one point or faster at another.*

Picture:

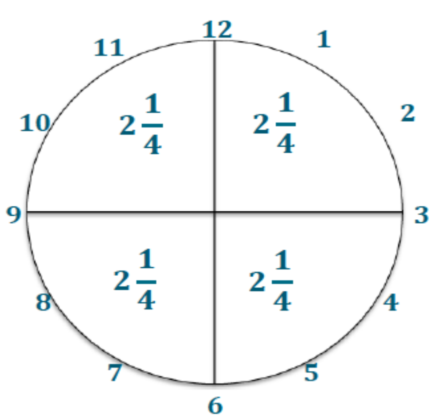
- Some students may decide to draw a clock.
  - *Possible student explanation:*

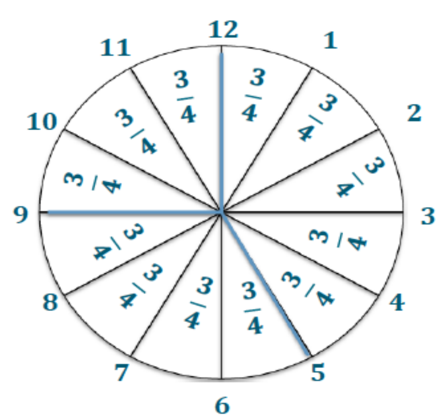
    For Izzy, every 15 minutes of running will result in a distance of $2\frac{1}{4}$ miles. Since the clock is divided into 15 minute intervals, I added the distance for each 15 minute interval until I reached 60 minutes. Julia's rate is $3\frac{3}{4}$ miles in 25 minutes, so I divided the clock into 25 minute intervals. Each of those 25 minute intervals represents $3\frac{3}{4}$ miles. At 50 minutes, the distance represented is two times $3\frac{3}{4}$, or $7\frac{1}{2}$ miles. To determine the distance ran in the last ten minutes, I needed to determine the distance for 5 minutes: $3\frac{3}{4} \div 5 = \frac{3}{4}$. Therefore, $3\frac{3}{4} + 3\frac{3}{4} + \frac{3}{4} + \frac{3}{4} = 9$, or 9 miles per hour.

Total Distance for 1 hour

Izzy: $2\frac{1}{4} + 2\frac{1}{4} + 2\frac{1}{4} + 2\frac{1}{4} = 9$, *9 miles per hour*

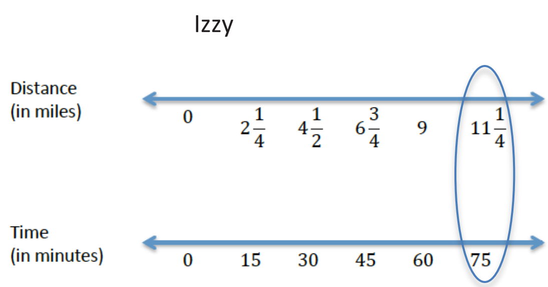
Julia: $3\frac{3}{4} + 3\frac{3}{4} + \frac{3}{4} + \frac{3}{4} = 9$, *9 miles per hour*

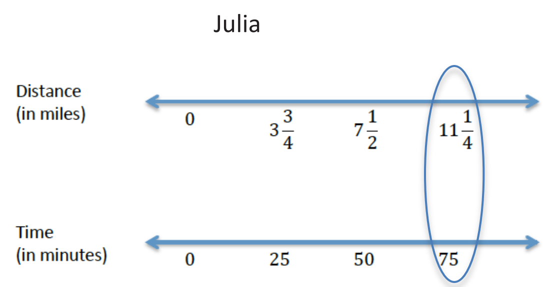




- How do you find the value of a 5-minute time increment? What are you really finding?
  - *To find the value of a* 5*-minute increment, you need to divide* $3\frac{3}{4}$ *by* 5 *since* 25 *minutes is five* 5*-minute increments. This is finding the unit rate for a* 5*-minute increment.*
- Why were 5-minute time increments chosen?
  - 5*-minute time increments were chosen for a few reasons. First, a clock can be separated into* 5*-minute intervals, so it may be easier to visualize what fractional part of an hour one has when given a* 5*-minute interval. Also,* 5 *is the greatest common factor of the two given times.*
- What if the times had been 24 and 32 minutes, or 18 and 22 minutes? How would this affect the time increments?
  - *If the times were* 24 *and* 32 *minutes, then the time increment would be* 8*-minute intervals. This is because* 8 *is the greatest common factor of* 24 *and* 32*.*
  - *If the times were* 18 *and* 22 *minutes, then the comparison should be broken into* 2*-minute intervals since the greatest common factor of* 18 *and* 22 *is* 2*.*

Double Number Line Approach:





Discuss with students the double number line approach.

- Starting with Izzy, we know for every 15 minutes she will run $2\frac{1}{4}$ miles. Therefore, we will set up a number line to represent time and a second number line with the corresponding distance. The number line representing time will be broken into 15 minute intervals. The distance number line will be broken into intervals representing the distance at the corresponding time. For example, at 15 minutes, the distance run will be $2\frac{1}{4}$ miles. At 30 minutes, the distance run will be $4\frac{1}{2}$ miles. Continue to complete both number lines for Izzy.
- Following the same procedure as we did for Izzy, set up a double number line for Julia. What is different for Julia?
  - *She travels in* 25 *minute intervals, and for each* 25 *minute interval, she will run* $3\frac{3}{4}$ *miles.*
- After both number lines are drawn for each runner, then compare the number lines and determine at what time will the distance ran by each runner be the same?
  - 75 *minutes*
- What if they did not run the same distance? How can we use the number lines to determine who is the faster runner?
  - *The faster runner will run further in the same amount of time. Therefore, you can compare the distance ran by each runner at a common time interval.*

- For this specific example, how do we know they ran the same speed?
  - *Looking at the number lines representing time, we can see both runners distance at a common time of* 75 *minutes. At* 75 *minutes, we compare the two runners distance ran and see they both ran* $11\frac{1}{4}$ *miles. Since the assumption is made that they both ran a constant, steady rate the entire time, then we can conclude they both run* 9 *miles per hour by finding the unit rate.*

Example 2 (5 minutes): Is Meredith Correct?

This will be the students' first experience evaluating complex fractions. Be sure to relate the process of evaluating complex fractions to division of fractions. Please note that the solutions shown are not the only way to solve these problems. Accept all valid solutions.

- The next example asks Meredith to determine the unit rate, expressed in miles per hour, when a turtle walks $\frac{7}{8}$ of a mile in 50 minutes. In order to determine the unit rate, we will again divide the distance by the amount of time. We see that Meredith represented her calculation with the fraction $\frac{\frac{7}{8}}{\frac{5}{6}}$. This is called a complex fraction.
- A complex fraction is simply a fraction whose numerator or denominator (or both) are fractions. Who can recall what operation the fraction bar separating the numerator and denominator represents?
  - *Division*
- Therefore, Meredith is actually dividing $\frac{7}{8}$, the distance the turtle walked in miles, by $\frac{5}{6}$, the amount of time. The complex fraction represents the division problem using fewer symbols, but the operation will always remain division.

Example 2: Is Meredith Correct?

A turtle walks $\frac{7}{8}$ of a mile in 50 minutes. What is the unit rate expressed in miles per hour?

a. To find the turtle's unit rate, Meredith wrote the following complex fraction. Explain how the fraction $\frac{5}{6}$ was obtained.

$$\frac{\left(\frac{7}{8}\right)}{\left(\frac{5}{6}\right)}$$

To determine the unit rate, Meredith divided the distance walked by the amount of time it took the turtle. Since the unit rate is expressed in miles per hour, 50 ***minutes needs to be converted to hours. Since*** 60 ***minutes is equal to*** 1 ***hour,*** 50 ***minutes can be written as*** $\frac{50}{60}$ ***hours, or*** $\frac{5}{6}$ ***hours.***

- How can we determine the unit rate? We need a denominator of 1 hour. Right now, the denominator is $\frac{5}{6}$ hours.
  - *We can multiply* $\frac{5}{6}$ *by its multiplicative inverse* $\frac{6}{5}$ *to determine a denominator of* 1 *hour.*

- Using this information, determine the unit rate in miles per hour.

b. **Determine the unit rate, expressed in miles per hour.**

$$\frac{\frac{7}{8}}{\frac{5}{6}} \cdot \frac{\frac{6}{5}}{\frac{6}{5}} = \frac{\frac{42}{40}}{1} = \frac{42}{40} = \frac{21}{20} \textit{ mph}$$

Exercises (10 minutes)

Scaffolding:

- For advanced learners:

  Ask students to calculate the number of cupcakes his mother would be able to make with 1 cup of flour? Remind students that there are 12 items in a dozen.

Exercises

1. **For Anthony's birthday, his mother is making cupcakes for his 12 friends at his daycare. The recipe calls for $3\frac{1}{3}$ cups of flour. This recipe makes $2\frac{1}{2}$ dozen cupcakes. Anthony's mother has only 1 cup of flour. Is there enough flour for each of his friends to get a cupcake? Explain and show your work.**

$$\frac{\textit{cups}}{\textit{dozen}} \quad \frac{3\frac{1}{3}}{2\frac{1}{2}} = \frac{\frac{10}{3} \times \frac{2}{5}}{\frac{5}{2} \times \frac{2}{5}} = \frac{\frac{20}{15}}{1} = 1\frac{5}{15} = 1\frac{1}{3} \textit{ cups/dozen}$$

No, since Anthony has 12 friends, he would need 1 dozen cupcakes. This means you need to find the unit rate. Finding the unit rate will tell us how much flour his mother needs for 1 dozen cupcakes. Upon finding the unit rate, Anthony's mother would need $1\frac{1}{3}$ cups of flour; therefore, she does not have enough flour to make cupcakes for all of his friends.

2. **Sally is making a painting for which she is mixing red paint and blue paint. The table below shows the different mixtures being used.**

Red Paint (Quarts)	Blue Paint (Quarts)
$1\frac{1}{2}$	$2\frac{1}{2}$
$2\frac{2}{5}$	4
$3\frac{3}{4}$	$6\frac{1}{4}$
4	$6\frac{2}{3}$
1.2	2
1.8	3

a. **What is the unit rate for the values of the amount of blue paint to the amount of red paint?**

$\frac{5}{3} = 1\frac{2}{3}$

b. **Is the amount of blue paint proportional to the amount of red paint?**

Yes. Blue paint is proportional to red paint because there exists a constant, $\frac{5}{3} = 1\frac{2}{3}$, such that when each amount of red paint is multiplied by the constant, the corresponding amount of blue paint is obtained.

c. **Describe, in words, what the unit rate means in the context of this problem.**

For every $1\frac{2}{3}$ quarts of blue paint, Sally must use 1 quart of red paint.

Closing (5 minutes)

- Can you give an example of when you might have to use a complex fraction?
- How is the unit rate calculated? Can we calculate unit rates when both values in the ratio are fractions?
- How is finding the unit rate useful?

> **Lesson Summary**
>
> **A fraction whose numerator or denominator is itself a fraction is called a complex fraction.**
>
> **Recall: A unit rate is a rate, which is expressed as $\frac{A}{B}$ units of the first quantity per 1 unit of the second quantity for two quantities A and B.**
>
> **For example: If a person walks $2\frac{1}{2}$ miles in $1\frac{1}{4}$ hours at a constant speed, then the unit rate is**
>
> $\frac{2\frac{1}{2}}{1\frac{1}{4}} = \frac{\frac{5}{2}}{\frac{5}{4}} = \frac{5}{2} \cdot \frac{4}{5} = 2$. **The person walks 2 mph.**

Exit Ticket (5 minutes)

Name ____ Date____________________

Lesson 11: Ratios of Fractions and Their Unit Rate

Exit Ticket

Which is the better buy? Show your work and explain your reasoning.

$3\frac{1}{3}$ lb. of turkey for $\$10.50$

$2\frac{1}{2}$ lb. of turkey for $\$6.25$

Exit Ticket Sample Solutions

Which is the better buy? Show your work and explain your reasoning.

$3\frac{1}{3}$ lb. of turkey for $\$10.50$

$10\frac{1}{2} \div 3\frac{1}{3} = \$3.15$

$2\frac{1}{2}$ lb. of turkey for $\$6.25$

$6\frac{1}{4} \div 2\frac{1}{2} = \$2.50$

$2\frac{1}{2}$ lb. is the better buy because the price per pound is cheaper.

Problem Set Sample Solutions

1. Determine the quotient: $2\frac{4}{7} \div 1\frac{3}{6}$

   $1\frac{5}{7}$

2. One lap around a dirt track is $\frac{1}{3}$ mile. It takes Bryce $\frac{1}{9}$ hour to ride one lap. What is Bryce's unit rate, in miles, around the track?

   3

3. Mr. Gengel wants to make a shelf with boards that are $1\frac{1}{3}$ feet long. If he has an 18-foot board, how many pieces can he cut from the big board?

   $13\frac{1}{2}$ *boards*

4. The local bakery uses 1.75 cups of flour in each batch of cookies. The bakery used 5.25 cups of flour this morning.

   a. How many batches of cookies did the bakery make?

      3 *batches*

   b. If there are 5 dozen cookies in each batch, how many cookies did the bakery make?

      $5(12) = 60$ *cookies per batch*

      $60(3) = 180$ *cookies in* 3 *batches*

5. Jason eats 10 ounces of candy in 5 days.

   a. How many pounds will he eat per day? (Recall: 16 ounces $= 1$ pound)

      $\frac{1}{8}$ *lb. each day*

   b. How long will it take Jason to eat 1 pound of candy?

      8 *days*