Lesson 28: Properties of Parallelograms

Student Outcomes

- Students complete proofs that incorporate properties of parallelograms.

Lesson Notes

Throughout this module, we have seen the theme of building new facts with the use of established ones. We see this again in Lesson 28, where triangle congruence criteria are used to demonstrate why certain properties of parallelograms hold true. We begin establishing new facts using only the definition of a parallelogram and the properties we have assumed when proving statements. Students combine the basic definition of a parallelogram with triangle congruence criteria to yield properties taken for granted in earlier grades, such as opposite sides of a parallelogram are parallel.

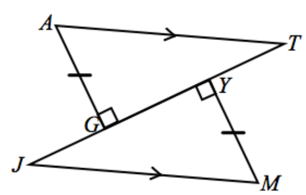
Classwork

Opening Exercise (5 minutes)

Opening Exercise

1. **If the triangles are congruent, state the congruence.**

   $\triangle AGT \cong \triangle MYJ$



2. **Which triangle congruence criterion guarantees part 1?**

   AAS

3. $\overline{TG}$ **corresponds with:**

   $\overline{JY}$

Discussion/Examples 1–7 (35 minutes)

Discussion

How can we use our knowledge of triangle congruence criteria to establish other geometry facts? For instance, what can we now prove about the properties of parallelograms?

To date, we have defined a parallelogram to be a quadrilateral in which both pairs of opposite sides are parallel. However, we have assumed other details about parallelograms to be true too. We assume that:

- **Opposite sides are congruent.**
- **Opposite angles are congruent.**
- **Diagonals bisect each other.**

Let us examine why each of these properties is true.

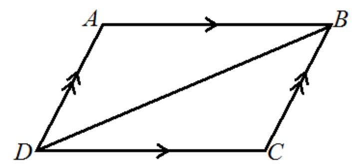
Example 1

If a quadrilateral is a parallelogram, then its opposite sides and angles are equal in measure. Complete the diagram and develop an appropriate *Given* and *Prove* for this case. Use triangle congruence criteria to demonstrate why opposite sides and angles of a parallelogram are congruent.

Given: *Parallelogram $ABCD$ ($\overline{AB} \parallel \overline{CD}$, $\overline{AD} \parallel \overline{CB}$).*

Prove: $AD = CB,\ AB = CD,\ \text{m}\angle A = \text{m}\angle C,\ \text{m}\angle B = \text{m}\angle D.$

Construction: Label the quadrilateral $ABCD$, and mark opposite sides as parallel. Draw diagonal $\overline{BD}$.

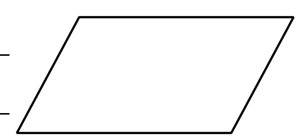


Proof:

Parallelogram $ABCD$	*Given*
$\text{m}\angle ABD = \text{m}\angle CDB$	*If parallel lines are cut by a transversal, then alternate interior angles are equal in measure*
$BD = DB$	*Reflexive property*
$\text{m}\angle CBD = \text{m}\angle ADB$	*If parallel lines are cut by a transversal, then alternate interior angles are equal in measure*
$\triangle ABD \cong \triangle CDB$	*ASA*
$AD = CB,\ AB = CD$	*Corresponding sides of congruent triangles are equal in length*
$\text{m}\angle A = \text{m}\angle C$	*Corresponding angles of congruent triangles are equal in measure*
$\text{m}\angle ABD + \text{m}\angle CBD = \text{m}\angle ABC,$	
$\text{m}\angle CDB + \text{m}\angle ADB = \text{m}\angle ADC$	*Angle addition postulate*
$\text{m}\angle ABD + \text{m}\angle CBD = \text{m}\angle CDB + \text{m}\angle ADB$	*Addition property of equality*
$\text{m}\angle B = \text{m}\angle D$	*Substitution property of equality*

Example 2

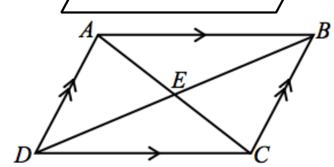
If a quadrilateral is a parallelogram, then the diagonals bisect each other. Complete the diagram and develop an appropriate *Given* and *Prove* for this case. Use triangle congruence criteria to demonstrate why diagonals of a parallelogram bisect each other. Remember, now that we have proved opposite sides and angles of a parallelogram to be congruent, we are free to use these facts as needed (i.e., $AD = CB,\ AB = CD,\ \angle A \cong \angle C,\ \angle B \cong \angle D$).



Given: *Parallelogram $ABCD$.*

Prove: *Diagonals bisect each other, $AE = CE, DE = BE$.*

Construction: Label the quadrilateral $ABCD$. Mark opposite sides as parallel. Draw diagonals AC and BD.



Proof:

Parallelogram $ABCD$	*Given*
$\text{m}\angle BAC = \text{m}\angle DCA$	*If parallel lines are cut by a transversal, then alternate interior angles are equal in measure*
$\text{m}\angle AEB = \text{m}\angle CED$	*Vertical angles are equal in measure*
$AB = CD$	*Opposite sides of a parallelogram are equal in length*
$\triangle AEB \cong \triangle CED$	*AAS*
$AE = CE,\ DE = BE$	*Corresponding sides of congruent triangles are equal in length*

Now we have established why the properties of parallelograms that we have assumed to be true are in fact true. By extension, these facts hold for any type of parallelogram, including rectangles, squares, and rhombuses. Let us look at one last fact concerning rectangles. We established that the diagonals of general parallelograms bisect each other. Let us now demonstrate that a rectangle has congruent diagonals.

Students may need a reminder that a rectangle is a parallelogram with four right angles.

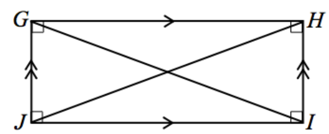
Example 3

If the parallelogram is a rectangle, then the diagonals are equal in length. Complete the diagram and develop an appropriate *Given* and *Prove* for this case. Use triangle congruence criteria to demonstrate why diagonals of a rectangle are congruent. As in the last proof, remember to use any already proven facts as needed.

Given: *Rectangle $GHIJ$.*

Prove: *Diagonals are equal in length, $GI = HJ$.*

Construction: **Label the rectangle $GHIJ$. Mark opposite sides as parallel, and add small squares at the vertices to indicate $90°$ angles. Draw diagonal GI and HJ.**



Proof:

Rectangle $GHIJ$	*Given*
$GJ = IH$	*Opposite sides of a parallelogram are equal in length*
$GH = GH$	*Reflexive property*
$\angle JGH,\ \angle IHG$ *are right angles*	*Definition of a rectangle*
$\triangle GHJ \cong \triangle IHG$	*SAS*
$GI = HJ$	*Corresponding sides of congruent triangles are equal in length*

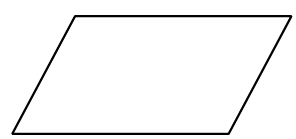
Converse Properties: Now we examine the converse of each of the properties we proved. Begin with the property and prove that the quadrilateral is in fact a parallelogram.

Example 4

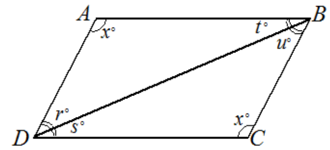
If the opposite angles of a quadrilateral are equal, then the quadrilateral is a parallelogram. Draw an appropriate diagram, and provide the relevant *Given* and *Prove* for this case.

Given: *Quadrilateral $ABCD$ with $\mathrm{m}\angle A = \mathrm{m}\angle C,\ \mathrm{m}\angle B = \mathrm{m}\angle D$.*

Prove: *Quadrilateral $ABCD$ is a parallelogram.*



Construction: Label the quadrilateral $ABCD$. Mark opposite angles as congruent. Draw diagonal BD. Label $\angle A$ and $\angle C$ as $x°$. Label the four angles created by $\overline{BD}$ as $r°$, $s°$, $t°$, and $u°$.

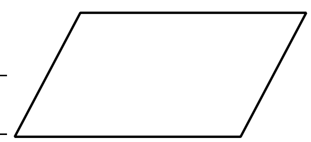


Proof:

Quadrilateral $ABCD$ with $\text{m}\angle A = \text{m}\angle C$, $\text{m}\angle B = \text{m}\angle D$	*Given*
$\text{m}\angle D = r + s,\ \text{m}\angle B = t + u$	*Angle addition*
$r + s = t + u$	*Substitution*
$x + r + t = 180$, $x + s + u = 180$	*Angles in a triangle add up to $180°$*
$r + t = s + u$	*Subtraction property of equality, substitution*
$r + t - (r + s) = s + u - (t + u)$	*Subtraction property of equality*
$t - s = s - t$	*Additive inverse property*
$t - s + (s - t) = s - t + (s - t)$	*Addition property of equality*
$0 = 2(s - t)$	*Addition and subtraction properties of equality*
$0 = s - t$	*Division property of equality*
$s = t$	*Addition property of equality*
$s = t \Rightarrow r = u$	*Substitution and subtraction properties of equality*
$\overline{AB} \parallel \overline{CD},\ \overline{AD} \parallel \overline{BC}$	*If two lines are cut by a transversal such that a pair of alternate interior angles are equal in measure, then the lines are parallel*
Quadrilateral $ABCD$ is a parallelogram	*Definition of a parallelogram*

Example 5

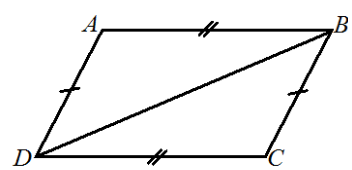
If the opposite sides of a quadrilateral are equal, then the quadrilateral is a parallelogram. Draw an appropriate diagram, and provide the relevant *Given* and *Prove* for this case.



Given: *Quadrilateral $ABCD$ with $AB = CD$, $AD = BC$.*

Prove: *Quadrilateral $ABCD$ is a parallelogram.*

Construction: Label the quadrilateral $ABCD$, and mark opposite sides as equal. Draw diagonal $\overline{BD}$.



Proof:

Quadrilateral $ABCD$ with $AB = CD$, $AD = CB$	*Given*
$BD = DB$	*Reflexive property*
$\triangle ABD \cong \triangle CDB$	*SSS*
$\angle ABD \cong \angle CDB, \angle ADB \cong \angle CBD$	*Corresponding angles of congruent triangles are congruent*
$\overline{AB} \parallel \overline{CD}, \overline{AD} \parallel \overline{CB}$	*If two lines are cut by a transversal and the alternate interior angles are congruent, then the lines are parallel*
Quadrilateral $ABCD$ is a parallelogram	*Definition of a parallelogram*

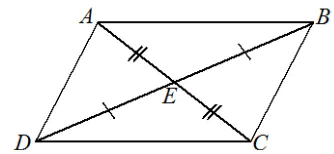
Example 6

If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram. Draw an appropriate diagram, and provide the relevant *Given* and *Prove* for this case. Use triangle congruence criteria to demonstrate why the quadrilateral is a parallelogram.

Given: *Quadrilateral $ABCD$, diagonals $\overline{AC}$ and $\overline{BD}$ bisect each other.*

Prove: *Quadrilateral $ABCD$ is a parallelogram.*

Construction: Label the quadrilateral $ABCD$, and mark opposite sides as equal. Draw diagonals $\overline{AC}$ and $\overline{BD}$.

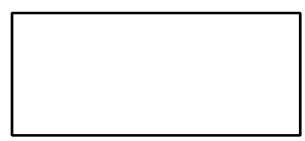


Proof:

Quadrilateral $ABCD$, diagonals $\overline{AC}$ and $\overline{BD}$ bisect each other	*Given*
$AE = CE$, $DE = BE$	*Definition of a segment bisector*
$m\angle DEC = m\angle BEA$, $m\angle AED = m\angle CEB$	*Vertical angles are equal in measure*
$\triangle DEC \cong \triangle BEA$, $\triangle AED \cong \triangle CEB$	*SAS*
$\angle ABD \cong \angle CDB$, $\angle ADB \cong \angle CBD$	*Corresponding angles of congruent triangles are congruent*
$\overline{AB} \parallel \overline{CD}$, $\overline{AD} \parallel \overline{CB}$	*If two lines are cut by a transversal such that a pair of alternate interior angles are congruent, then the lines are parallel*
Quadrilateral $ABCD$ is a parallelogram	*Definition of a parallelogram*

Example 7

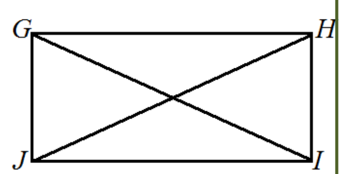
If the diagonals of a parallelogram are equal in length, then the parallelogram is a rectangle. Complete the diagram, and develop an appropriate *Given* and *Prove* for this case.



Given: *Parallelogram $GHIJ$ with diagonals of equal length, $GI = HJ$.*

Prove: *$GHIJ$ is a rectangle.*

Construction: Label the quadrilateral $GHIJ$. Draw diagonals $\overline{GI}$ and $\overline{HJ}$.



Proof:

Parallelogram $GHIJ$ with diagonals of equal length, $GI = HJ$	*Given*
$GJ = JG$, $HI = IH$	*Reflexive property*
$GH = IJ$	*Opposite sides of a parallelogram are congruent*
$\triangle HJG \cong \triangle IGJ$, $\triangle GHI \cong \triangle JIH$	*SSS*
$m\angle G = m\angle J$, $m\angle H = m\angle I$	*Corresponding angles of congruent triangles are equal in measure*
$m\angle G + m\angle J = 180°$, $m\angle H + m\angle I = 180°$	*If parallel lines are cut by a transversal, then interior angles on the same side are supplementary*
$2(m\angle G) = 180°$, $2(m\angle H) = 180°$	*Substitution property of equality*
$m\angle G = 90°$, $m\angle H = 90°$	*Division property of equality*
$m\angle G = m\angle J = m\angle H = m\angle I = 90°$	*Substitution property of equality*
$GHIJ$ is a rectangle.	*Definition of a rectangle*

Exit Ticket (5 minutes)

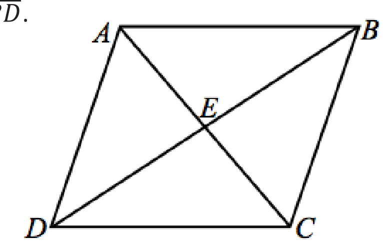
Name ____ Date____________________

Lesson 28: Properties of Parallelograms

Exit Ticket

Given: Equilateral parallelogram $ABCD$ (i.e., a rhombus) with diagonals $\overline{AC}$ and $\overline{BD}$.

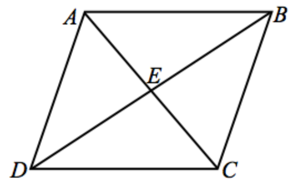
Prove: Diagonals intersect perpendicularly.



Exit Ticket Sample Solutions

Given: Equilateral parallelogram $ABCD$ (i.e., a rhombus) with diagonals $\overline{AC}$ and $\overline{BD}$.

Prove: Diagonals intersect perpendicularly.

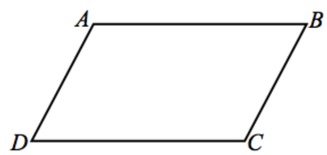


Rhombus $ABCD$	*Given*
$AE = CE,\ DE = BE$	*Diagonals of a parallelogram bisect each other*
$AB = BC + CD = DA$	*Definition of equilateral parallelogram*
$\triangle AED \cong \triangle AEB \cong \triangle CEB \cong \triangle CED$	*SSS*
$m\angle AED = m\angle AEB = m\angle BEC = m\angle CED$	*Corr. angles of congruent triangles are equal in measure*
$m\angle AED = m\angle AEB = m\angle BEC = m\angle CED = 90°$	*Angles at a point sum to $360°$. Since all four angles are congruent, each angle measures $90°$.*

Problem Set Sample Solutions

Use the facts you have established to complete exercises involving different types of parallelograms.

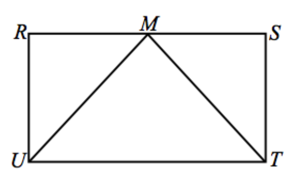
1. Given: $\overline{AB} \parallel \overline{CD}$, $AD = AB$, $CD = CB$.
   Prove: $ABCD$ is a rhombus.



Construction: Draw diagonal $\overline{AC}$.

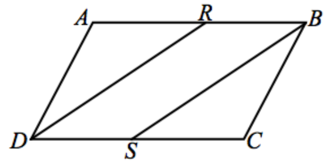
$AD = AB,\ CD = CB$	*Given*
$AC = CA$	*Reflexive property*
$\triangle ADC \cong \triangle CBA$	*SSS*
$AD = CB,\ AB = CD$	*Corresponding sides of congruent triangles are equal in length*
$AB = BC = CD = AD$	*Transitive property*
$ABCD$ is a rhombus	*Definition of a rhombus*

2. Given: Rectangle $RSTU$, M is the midpoint of $\overline{RS}$.
   Prove: $\triangle UMT$ is isosceles.



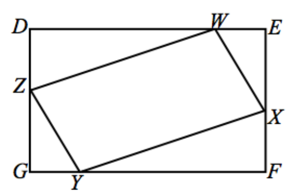
Rectangle $RSTU$	*Given*
$RU = ST$	*Opposite sides of a rectangle are congruent*
$\angle R, \angle S$ are right angles	*Definition of a rectangle*
M is the midpoint of RS	*Given*
$RM = SM$	*Definition of a midpoint*
$\triangle RMU \cong \triangle SMT$	*SAS*
$\overline{UM} \cong \overline{TM}$	*Corresponding sides of congruent triangles are congruent*
$\triangle UMT$ is isosceles	*Definition of an isosceles triangle*

3. Given: $ABCD$ is a parallelogram, $\overline{RD}$ bisects $\angle ADC$, $\overline{SB}$ bisects $\angle CBA$.
Prove: $DRBS$ is a parallelogram.



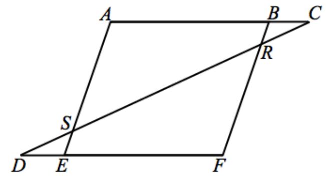
***ABCD* is a parallelogram;** $\overline{RD}$ **bisects** $\angle ADC$, $\overline{SB}$ **bisects** $\angle CBA$	***Given***
$AD = CB$	***Opposite sides of a parallelogram are congruent***
$\angle A \cong \angle C$, $\angle B \cong \angle D$	***Opposite angles of a parallelogram are congruent***
$\angle RDA \cong \angle RDS$, $\angle SBC \cong \angle SBR$	***Definition of angle bisector***
$\angle RDA + \angle RDS = \angle D, \angle SBC \cong \angle SBR = \angle B$	***Angle addition***
$\angle RDA + \angle RDA = \angle D, \angle SBC \cong \angle SBC = \angle B$	***Substitution***
$2(\angle RDA) = \angle D$, $2(\angle SBC) = \angle B$	***Addition***
$\angle RDA = \frac{1}{2}\angle D$, $\angle SBC = \frac{1}{2}\angle B$	***Division***
$\angle RDA \cong \angle SBC$	***Substitution***
$\triangle DAR \cong \triangle BCS$	***ASA***
$\angle DRA \cong \angle BSC$	***Corresponding angles of congruent triangles are congruent***
$\angle DRB \cong \angle BSD$	***Supplements of congruent angles are congruent***
***DRBS* is a parallelogram**	***Opposite angles of quadrilateral DRBS are congruent***

4. Given: $DEFG$ is a rectangle, $WE = YG$, $WX = YZ$.
Prove: $WXYZ$ is a parallelogram.



$DE = FG$, $DG = FE$	***Opposite sides of a rectangle are congruent***
***DEFG* is a rectangle;** $WE = YG$, $WX = YZ$	***Given***
$DE = DW + WE$; $FG = YG + FY$	***Segment addition***
$DW + WE = YG + FY$	***Substitution***
$DW + YG = YG + FY$	***Substitution***
$DW = FY$	***Subtraction***
$m\angle D = m\angle E = m\angle F = m\angle G = 90°$	***Definition of a rectangle***
$\triangle ZGY$, $\triangle XEW$ **are right triangles**	***Definition of right triangle***
$\triangle ZGY \cong \triangle XEW$	***HL***
$ZG = XE$	***Corresponding sides of congruent triangles are congruent***
$DG = ZG + DZ$; $FE = XE + FX$	***Partition property or segment addition***
$DZ = FX$	***Subtraction property of equality***
$\triangle DZW \cong \triangle FXY$	***SAS***
$ZW = XY$	***Corresponding sides of congruent triangles are congruent***
***WXYZ* is a parallelogram**	***Both pairs of opposite sides are congruent***

5. Given: Parallelogram $ABFE$, $CR = DS$.
Prove: $BR = SE$.



$m\angle BCR = m\angle EDS$	*If parallel lines cut by a transversal, then alternate interior angles are equal in measure*
$\angle ABF \cong \angle FEA$	*Opposite angles of a parallelogram are congruent*
$\angle CBR \cong \angle DES$	*Supplements of congruent angles are congruent*
$CR = DS$	*Given*
$\triangle CBR \cong \triangle DES$	*AAS*
$BR = SE$	*Corresponding sides of congruent triangles are equal in length*