Lesson 23: Base Angles of Isosceles Triangles

Student Outcomes

- Students examine two different proof techniques via a familiar theorem.
- Students complete proofs involving properties of an isosceles triangle.

Lesson Notes

In Lesson 23, students study two proofs to demonstrate why the base angles of isosceles triangles are congruent. The first proof uses transformations, while the second uses the recently acquired understanding of the SAS triangle congruency. The demonstration of both proofs will highlight the utility of the SAS criteria. Encourage students to articulate why the SAS criteria is so useful.

The goal of this lesson is to compare two different proof techniques by investigating a familiar theorem. Be careful not to suggest that proving an established fact about isosceles triangles is somehow the significant aspect of this lesson. However, if you need to, you can help motivate the lesson by appealing to history. Point out that Euclid used SAS and that the first application he made of it was in proving that base angles of an isosceles triangle are congruent. This is a famous part of the Elements. Today, proofs using SAS and proofs using rigid motions are valued equally.

Classwork

Opening Exercise (7 minutes)

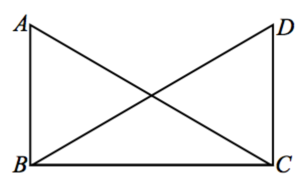
Opening Exercise

Describe the additional piece of information needed for each pair of triangles to satisfy the SAS triangle congruence criteria.

1. **Given:** $AB = DC$

   $\overline{AB} \perp \overline{BC}, \overline{DC} \perp \overline{CB}, or\ m\angle B = m\angle C$

   Prove: $\triangle ABC \cong \triangle DCB$

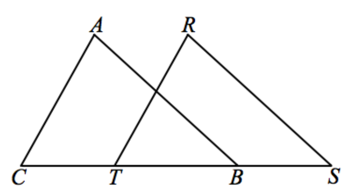


2. **Given:** $AB = RS$

   $\overline{AB} \parallel \overline{RS}$

   $CB = TS\ or\ CT = BS$

   Prove: $\triangle ABC \cong \triangle RST$



Exploratory Challenge (25 minutes)

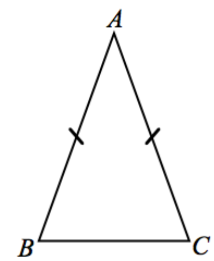
Exploratory Challenge

Today we examine a geometry fact that we already accept to be true. We are going to prove this known fact in two ways: (1) by using transformations and (2) by using SAS triangle congruence criteria.

Here is isosceles triangle ABC. We accept that an isosceles triangle, which has (at least) two congruent sides, also has congruent base angles.

Label the congruent angles in the figure.

Now we will prove that the base angles of an isosceles triangle are always congruent.



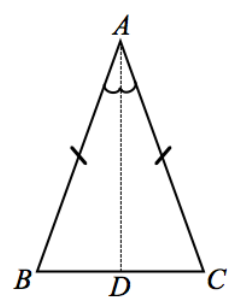
Prove Base Angles of an Isosceles are Congruent: Transformations

While simpler proofs do exist, this version is included to reinforce the idea of congruence as defined by rigid motions. Allow 15 minutes for the first proof.

Prove Base Angles of an Isosceles are Congruent: Transformations

Given: Isosceles $\triangle ABC$, with $AB = AC$

Prove: $\mathrm{m}\angle B = \mathrm{m}\angle C$



***Construction*: Draw the angle bisector $\overrightarrow{AD}$ of $\angle A$, where D is the intersection of the bisector and $\overline{BC}$. We need to show that rigid motions will map point B to point C and point C to point B.**

Let r be the reflection through $\overleftrightarrow{AD}$. Through the reflection, we want to demonstrate two pieces of information that map B to point C and vice versa: (1) $\overrightarrow{AB}$ maps to $\overrightarrow{AC}$, and (2) $AB = AC$.

Since A is on the line of reflection, $\overleftrightarrow{AD}$, $r(A) = A$. Reflections preserve angle measures, so the measure of the reflected angle $r(\angle BAD)$ equals the measure of $\angle CAD$; therefore, $r(\overrightarrow{AB}) = \overrightarrow{AC}$. Reflections also preserve lengths of segments; therefore, the reflection of $\overline{AB}$ will still have the same length as $\overline{AB}$. By hypothesis, $AB = AC$, so the length of the reflection will also be equal to AC. Then $r(B) = C$. Using similar reasoning, we can show that $r(C) = B$.

Reflections map rays to rays, so $r(\overrightarrow{BA}) = \overrightarrow{CA}$ and $r(\overrightarrow{BC}) = \overrightarrow{CB}$. Again, since reflections preserve angle measures, the measure of $r(\angle ABC)$ is equal to the measure of $\angle ACB$.

We conclude that $\mathrm{m}\angle B = \mathrm{m}\angle C$. Equivalently, we can state that $\angle B \cong \angle C$. In proofs, we can state that "base angles of an isosceles triangle are equal in measure" or that "base angles of an isosceles triangle are congruent."

Prove Base Angles of an Isosceles are Congruent: SAS

Allow 10 minutes for the second proof.

> **Prove Base Angles of an Isosceles are Congruent: SAS**
>
> **Given:** Isosceles $\triangle ABC$, with $AB = AC$
>
> **Prove:** $\angle B \cong \angle C$
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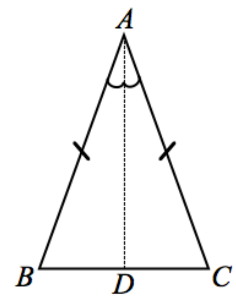
> 
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> ***Construction:*** **Draw the angle bisector $\overrightarrow{AD}$ of $\angle A$, where D is the intersection of the bisector and $\overline{BC}$. We are going to use this auxiliary line towards our SAS criteria.**

Allow students five minutes to attempt the proof on their own.

$AB = AC$	*Given*
$AD = AD$	*Reflexive property*
$m\angle BAD = m\angle CAD$	*Definition of an angle bisector*
$\triangle ABD \cong \triangle ACD$	*Side Angle Side criteria*
$\angle B \cong \angle C$	*Corresponding angles of congruent triangles are congruent*

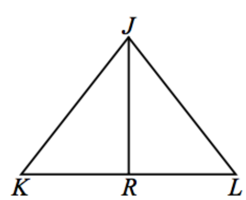
Exercises (10 minutes)

Note that in Exercise 4, students are asked to use transformations in the proof. This is intentional to reinforce the notion that congruence is defined in terms of rigid motions.

Exercises

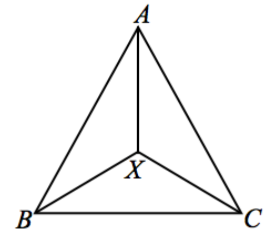
1. **Given:** $JK = JL$; $\overline{JR}$ bisects $\overline{KL}$

   Prove: $\overline{JR} \perp \overline{KL}$



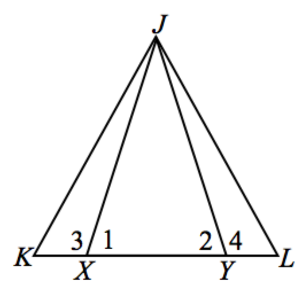
$JK = JL$	*Given*
$\angle K \cong \angle L$	*Base angles of an isosceles triangle are congruent.*
$KR = RL$	*Definition of a segment bisector*
$\therefore \triangle JRK \cong \triangle JRL$	*SAS*
$\angle JRK \cong \angle JRL$	*Corresponding angles of congruent triangles are congruent.*
$m\angle JRK + m\angle JRL = 180°$	*Linear pairs form supplementary angles.*
$2(\angle JRK) = 180°$	*Substitution property of equality*
$\angle JRK = 90°$	*Division property of equality*
$\therefore JR \perp KL$	*Definition of perpendicular lines*

2. Given: $AB = AC,\ XB = XC$
Prove: $\overline{AX}$ bisects $\angle BAC$



$AB = AC$	*Given*
$\text{m}\angle ABC = \text{m}\angle ACB$	*Base angles of an isosceles triangle are congruent.*
$XB = XC$	*Given*
$\text{m}\angle XBC = \text{m}\angle XCB$	*Base angles of an isosceles triangle are equal in measure*
$\text{m}\angle ABC = \text{m}\angle ABX + \text{m}\angle XBC,$	
$\text{m}\angle ACB = \text{m}\angle ACX + \text{m}\angle XCB$	*Angle addition postulate*
$\text{m}\angle ABX = \text{m}\angle ABC - \text{m}\angle XBC,$	
$\text{m}\angle ACX = \text{m}\angle ACB - \text{m}\angle XCB$	*Subtraction property of equality*
$\text{m}\angle ABX = \text{m}\angle ACX$	*Substitution property of equality*
$\therefore \triangle ABX \cong \triangle ACX$	*SAS*
$\angle BAX \cong \angle CAX$	*Corresponding part of congruent triangles are congruent*
$\therefore$ $\overline{AX}$ *bisects* $\angle BAC.$	*Definition of an angle bisector*

3. Given: $JX = JY, KX = LY$
Prove: $\triangle JKL$ is isosceles



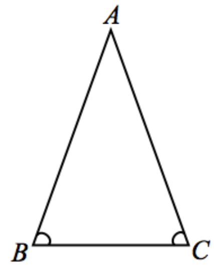
$JX = JY$	*Given*
$KX = LY$	*Given*
$\text{m}\angle 1 = m\angle 2$	*Base angles of an isosceles triangle are equal in measure*
$\text{m}\angle 1 + \text{m}\angle 3 = 180°$	*Linear pairs form supplementary angles*
$\text{m}\angle 2 + \text{m}\angle 4 = 180°$	*Linear pairs form supplementary angles*
$\text{m}\angle 3 = 180° - \text{m}\angle 1$	*Subtraction property of equality*
$\text{m}\angle 4 = 180° - \text{m}\angle 3$	*Subtraction property of equality*
$\text{m}\angle 3 = \text{m}\angle 4$	*Substitution property of equality*
$\therefore\ \triangle JKX \cong \triangle JYL$	*SAS*
$\overline{JK} \cong \overline{JL}$	*Corresponding segments of congruent triangles are congruent.*
$\therefore\ \triangle JKL$ *is isosceles*	*Definition of an isosceles triangle*

4. **Given:** $\triangle ABC$, with $\mathrm{m}\angle CBA = \mathrm{m}\angle BCA$

Prove: $BA = CA$

(Converse of base angles of isosceles triangle)

Hint: Use a transformation.



Proof:

We can prove that $AB = AC$ by using rigid motions. Construct the perpendicular bisector l to $\overline{BC}$. Note that we do not know whether the point A is on l. If we did, we would know immediately that $AB = AC$, since all points on the perpendicular bisector are equidistant from B and C. We need to show that A is on l, or equivalently, that the reflection across l takes A to A.

Let r_l be the transformation that reflects the $\triangle ABC$ across l. Since l is the perpendicular bisector, $r_l(B) = C$ and $r_l(C) = B$. We do not know where the transformation takes A; for now, let us say that $r_l(A) = A'$.

Since $\angle CBA \cong \angle BCA$ and rigid motions preserve angle measures, after applying r_l to $\angle BCA$, we get that $\angle CBA \cong \angle CBA'$. Angle $\angle CBA$ and angle $\angle CBA'$ share a ray BC, are of equal measure, and A and A' are both in the same half-plane with respect to line BC. Hence, $\overrightarrow{BA}$ and $\overrightarrow{BA'}$ are the same ray. In particular, A' is a point somewhere on $\overrightarrow{BA}$.

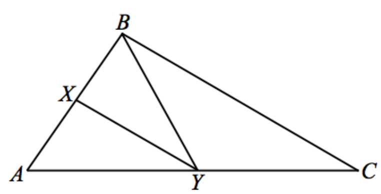
Likewise, applying r_l to $\angle CBA$ gives $\angle BCA \cong \angle BCA'$, and for the same reasons in the previous paragraph, A' must also be a point somewhere on $\overrightarrow{CA}$. Therefore, A' is common to both $\overrightarrow{BA}$ and $\overrightarrow{CA}$.

The only point common to both $\overrightarrow{BA}$ and $\overrightarrow{CA}$ is point A; thus, A and A' must be the same point, i.e., $A = A'$.

Hence, the reflection takes A to A, which means A is on the line l and $r_l(\overline{BA}) = \overline{CA'} = \overline{CA}$, or $BA = CA$.

5. **Given:** $\triangle ABC$, with $\overline{XY}$ is the angle bisector of $\angle BYA$, and $\overline{BC} \parallel \overline{XY}$

Prove: $YB = YC$



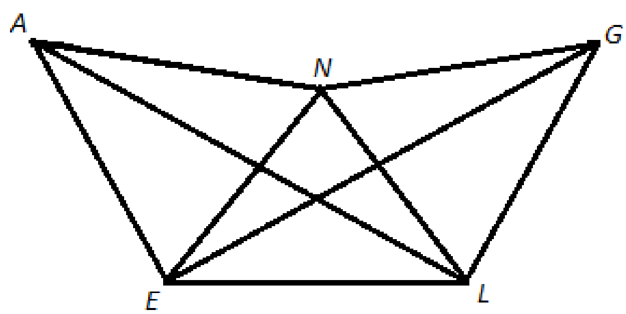
$\overline{BC} \parallel \overline{XY}$	*Given*
$\mathrm{m}\angle XYB = \mathrm{m}\angle CBY$	*When two parallel lines are cut by a transversal, the alternate interior angles are equal*
$\mathrm{m}\angle XYA = \mathrm{m}\angle BCY$	*When two parallel lines are cut by a transversal, the corresponding angles are equal*
$\mathrm{m}\angle XYA = \mathrm{m}\angle XYB$	*Definition of an angles bisector*
$\mathrm{m}\angle CBY = \mathrm{m}\angle BCY$	*Substitution property of equality*
$YB = YC$	*When the base angles of a triangle are equal in measure, the triangle is isosceles*

Exit Ticket (3 minutes)

Name ____ Date____________________

Lesson 23: Base Angles of Isosceles Triangles

Exit Ticket



For each of the following, if the given congruence exists, name the isosceles triangle and name the pair of congruent angles for the triangle based on the image above.

1. $\overline{AE} \cong \overline{EL}$

2. $\overline{LE} \cong \overline{LG}$

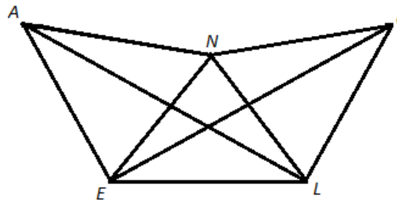
3. $\overline{AN} \cong \overline{LN}$

4. $\overline{EN} \cong \overline{NG}$

5. $\overline{NG} \cong \overline{GL}$

6. $\overline{AE} \cong \overline{EN}$

Exit Ticket Sample Solutions



For each of the following, if the given congruence exists, name the isosceles triangle and name the pair of congruent angles for the triangle based on the image above.

1. $\overline{AE} \cong \overline{EL}$

   $\triangle AEL$, $\angle EAL \cong \angle ELA$

2. $\overline{LE} \cong \overline{LG}$

   $\triangle LEG$, $\angle LEG \cong \angle LGE$

3. $\overline{AN} \cong \overline{LN}$

   $\triangle NLA$, $\angle NLA \cong \angle NAL$

4. $\overline{EN} \cong \overline{NG}$

   $\triangle NGE$, $\angle NGE \cong \angle NEG$

5. $\overline{NG} \cong \overline{GL}$

   $\triangle GLN$, $\angle GNL \cong \angle GLN$

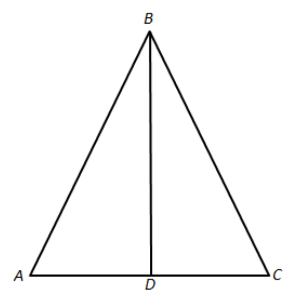
6. $\overline{AE} \cong \overline{EN}$

   $\triangle AEN$, $\angle ENA \cong \angle EAN$

Problem Set Sample Solutions

1. Given: $AB = BC$, $AD = DC$

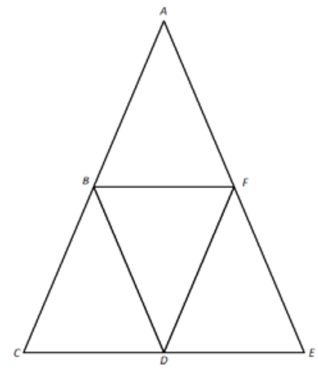
   Prove: $\triangle ADB$ and $\triangle CDB$ are right triangles



$AB = BC$	*Given*
$\triangle ABC$ *is isosceles*	*Definition of isosceles triangle*
$\mathrm{m}\angle A = \mathrm{m}\angle C$	*Base angles of an isosceles triangle are equal in measure*
$AD = DC$	*Given*
$\triangle ADB \cong \triangle CDB$	*SAS*
$\mathrm{m}\angle ADB = \mathrm{m}\angle CDB$	*Corresponding angles of congruent triangles are equal in measure*
$\mathrm{m}\angle ADB + \mathrm{m}\angle CDB = 180°$	*Linear pairs form supplementary angles*
$2(\mathrm{m}\angle ADB) = 180°$	*Substitution property of equality*
$\mathrm{m}\angle ADB = 90°$	*Division property of equality*
$\mathrm{m}\angle CDB = 90°$	*Transitive property*
$\triangle ADB$ *and* $\triangle CDB$ *are right triangles*	*Definition of a right triangle*

2. Given: $AC = AE$ and $\overline{BF} \parallel \overline{CE}$

Prove: $AB = AF$



$AC = AE$	*Given*
$\triangle$ *ACE is isosceles*	*Definition of isosceles triangle*
$\text{m}\angle ACE = \text{m}\angle AEC$	*Base angles of an isosceles triangle are equal in measure*
$\overline{BF} \parallel \overline{CE}$	*Given*
$\text{m}\angle ACE = \text{m}\angle ABF$	*If parallel lines are cut by a transversal, then corresponding angles are equal in measure*
$\text{m}\angle AFB = \text{m}\angle AEC$	*If parallel lines are cut by a transversal, then corresponding angles are equal in measure*
$m\angle AFB = m\angle ABF$	*Transitive property*
$\triangle$ *ABF is isosceles*	*If the base angles of a triangle are equal in measure, the triangle is isosceles*
$AB = AF$	*Definition of an isosceles triangle*

3. In the diagram, $\triangle ABC$ is isosceles with $\overline{AC} \cong \overline{AB}$. In your own words, describe how transformations and the properties of rigid motions can be used to show that $\angle C \cong \angle B$.

Answers will vary. A possible response would summarize the key elements of the proof from the lesson, such as the response below.

It can be shown that $\angle C \cong \angle B$ using a reflection across the bisector of $\angle A$. The two angles formed by the bisector of $\angle A$ would be mapped onto one another since they share a ray, and rigid motions preserve angle measure.

Since segment length is also preserved, B is mapped onto C and vice versa.

Finally, since rays are taken to rays and rigid motions preserve angle measure, $\overrightarrow{CA}$ is mapped onto $\overrightarrow{BA}$, $\overrightarrow{CB}$ is mapped onto $\overrightarrow{BC}$, and $\angle C$ is mapped onto $\angle B$. From this, we see that $\angle C \cong \angle B$.