Lesson 22: Congruence Criteria for Triangles—SAS

Classwork

Opening Exercise

Answer the following question. Then discuss your answer with a partner.

Do you think it is possible to know that there is a rigid motion that takes one triangle to another without actually showing the particular rigid motion? Why or why not?

_____

_____

_____

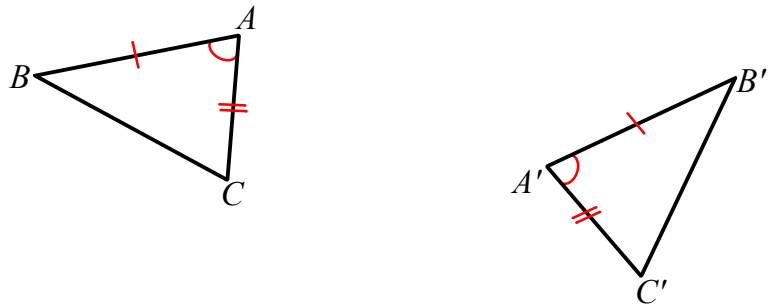
Discussion

It is true that we will not need to show the rigid motion to be able to know that there is one. We are going to show that there are criteria that refer to a few parts of the two triangles and a correspondence between them that guarantee congruency (i.e., existence of rigid motion). We start with the Side-Angle-Side (SAS) criteria.

Side-Angle-Side Triangle Congruence Criteria (SAS): Given two triangles $\triangle ABC$ and $\triangle A'B'C'$ so that $AB = A'B'$ (Side), $\mathrm{m}\angle A = \mathrm{m}\angle A'$ (Angle), $AC = A'C'$ (Side). Then the triangles are congruent.

The steps below show the most general case for determining a congruence between two triangles that satisfy the SAS criteria. Note that not all steps are needed for every pair of triangles. For example, sometimes the triangles will already share a vertex. Sometimes a reflection will be needed, sometimes not. It is important to understand that we can always use the steps below—some or all of them—to determine a congruence between the two triangles that satisfies the SAS criteria

Proof: Provided the two distinct triangles below, assume $AB = A'B'$ (Side), $\mathrm{m}\angle A = \mathrm{m}\angle A'$ (Angle), $AC = A'C'$ (Side).



By our definition of congruence, we will have to find a composition of rigid motions will map $\triangle A'B'C'$ to $\triangle ABC$. We must find a congruence F so that $F(\triangle A'B'C') = \triangle ABC$. First, use a translation T to map a common vertex.

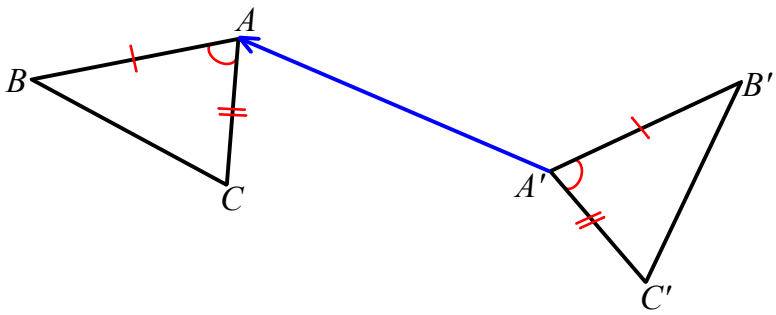
Which two points determine the appropriate vector?

Can any other pair of points be used? _____ Why or why not?

_____

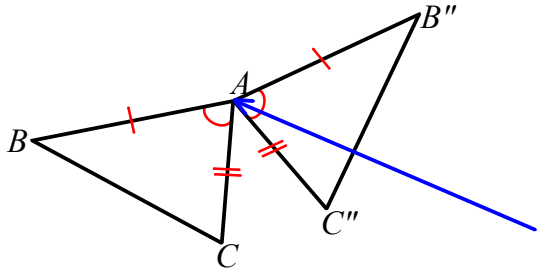
State the vector in the picture below that can be used to translate $\triangle A'B'C'$: _____

Using a dotted line, draw an intermediate position of $\triangle A'B'C'$ as it moves along the vector:

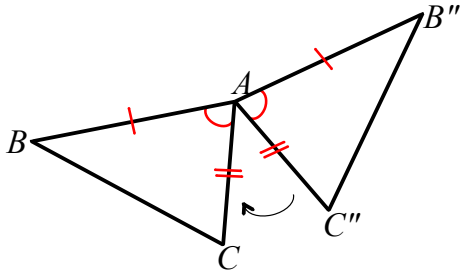


After the translation (below), $T_{\overline{A'A}}(\triangle A'B'C')$ shares one vertex with $\triangle ABC$, A. In fact, we can say

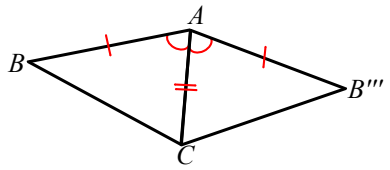
T_____($\triangle$ _____) = $\triangle$ _____.



Next, use a clockwise rotation $R_{\angle CAC''}$ to bring the sides $\overline{AC''}$ to $\overline{AC}$ (or counterclockwise rotation to bring $\overline{AB''}$to $\overline{AB}$).

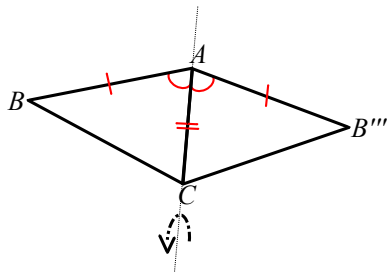


A rotation of appropriate measure will map $\overrightarrow{AC''}$ to $\overrightarrow{AC}$, but how can we be sure that vertex C'' maps to C? Recall that part of our assumption is that the lengths of sides in question are equal, ensuring that the rotation maps C'' to C. ($AC = AC''$; the translation performed is a rigid motion, and thereby did not alter the length when $\overline{AC'}$ became $\overline{AC''}$.)



After the rotation $R_{\angle CAC''}(\triangle AB''C'')$, a total of two vertices are shared with $\triangle ABC$, A and C. Therefore,

Finally, if B''' and B are on opposite sides of the line that joins AC, a reflection $r_{\overline{AC}}$ brings B''' to the same side as B.



Since a reflection is a rigid motion and it preserves angle measures, we know that $m\angle B'''AC = m\angle BAC$ and so $\overrightarrow{AB'''}$ maps to $\overrightarrow{AB}$. If, however, $\overrightarrow{AB'''}$ coincides with $\overrightarrow{AB}$, can we be certain that B''' actually maps to B? We can, because not only are we certain that the rays coincide but also by our assumption that $AB = AB'''$. (Our assumption began as $AB = A'B'$, but the translation and rotation have preserved this length now as AB'''.) Taken together, these two pieces of information ensure that the reflection over $\overline{AC}$ brings B''' to B.

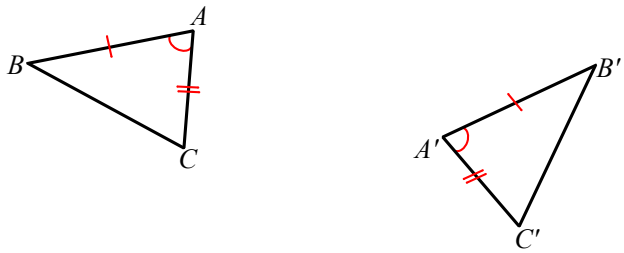
Another way to visually confirm this is to draw the marks of the _____construction for $\overline{AC}$.

Write the transformations used to correctly notate the congruence (the composition of transformations) that take $\triangle A'B'C' \cong \triangle ABC$:

F _____

G _____

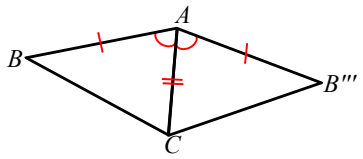
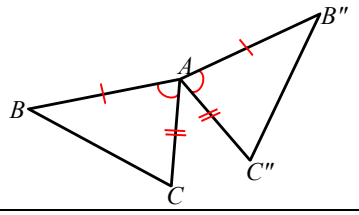
H _____



We have now shown a sequence of rigid motions that takes $\triangle A'B'C'$ to $\triangle ABC$ with the use of just three criteria from each triangle: two sides and an included angle. Given any two distinct triangles, we could perform a similar proof. There is another situation when the triangles are not distinct, where a modified proof will be needed to show that the triangles map onto each other. Examine these below. Note that when using the Side-Angle-Side triangle congruence criteria as a reason in a proof, you need only state the congruence and "SAS."

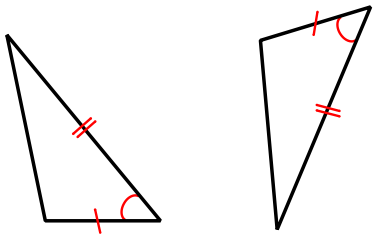
Example 1

What if we had the SAS criteria for two triangles that were not distinct? Consider the following two cases. How would the transformations needed to demonstrate congruence change?

Case	Diagram	Transformations Needed
Shared Side		
Shared Vertex		

Exercises 1–4

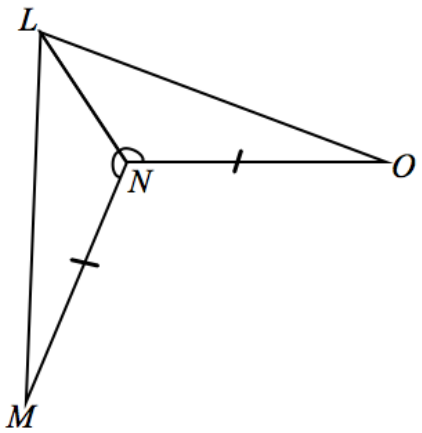
1. Given: Triangles with a pair of corresponding sides of equal length and a pair of included angles of equal measure. Sketch and label three phases of the sequence of rigid motions that prove the two triangles to be congruent.



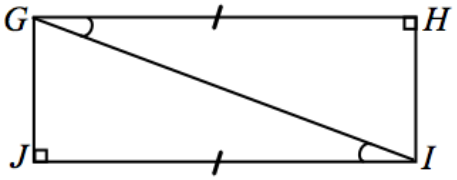
Justify whether the triangles meet the SAS congruence criteria; explicitly state which pairs of sides or angles are congruent and why. If the triangles do meet the SAS congruence criteria, describe the rigid motion(s) that would map one triangle onto the other.

Given: Triangles with a pair of corresponding sides of equal length and a pair of included angles of equal measure. Sketch and label three phases of the sequence of rigid motions that prove the two triangles to be congruent.

2. Given: $\angle LMN = \angle LNO, MN = OM$.
   Do $\triangle LMN$ and $\triangle LOM$ meet the SAS criteria?



3. Given: $\angle HGI = \angle JIG, HG = JI$.
   Do $\triangle HGI$ and $\triangle JIG$ meet the SAS criteria?

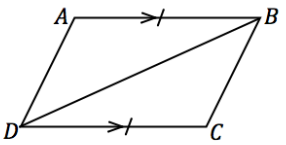


4. Is it true that we could also have proved $\triangle HGI$ and $\cong\triangle JIG$ meet the SAS criteria if we had been given that $\angle HGI \cong \angle JIG$ and $\overline{HG} \cong \overline{JI}$? Explain why or why not.

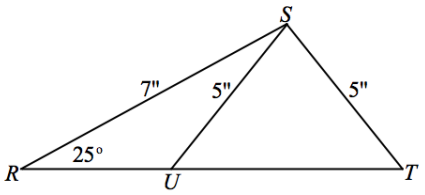
Problem Set

Justify whether the triangles meet the SAS congruence criteria; explicitly state which pairs of sides or angles are congruent and why. If the triangles do meet the SAS congruence criteria, describe the rigid motion(s) that would map one triangle onto the other.

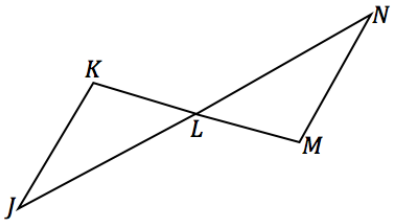
1. Given: $\overline{AB} \parallel \overline{CD},\ AB = CD$
   Do $\triangle ABD$ and $\triangle CDB$ meet the SAS criteria?



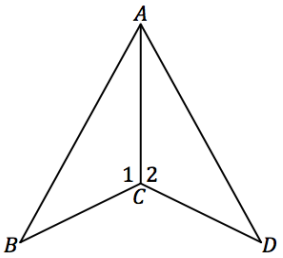
2. Given: $\mathrm{m}\angle R = 25°, RT = 7", SU = 5", ST = 5"$
   Do $\triangle RSU$ and $\triangle RST$ meet the SAS criteria?



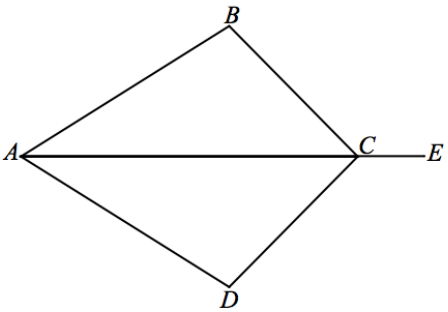
3. Given: $\overline{KM}$ and $\overline{JN}$ bisect each other.
   Do $\triangle JKL$ and $\triangle NML$ meet the SAS criteria?



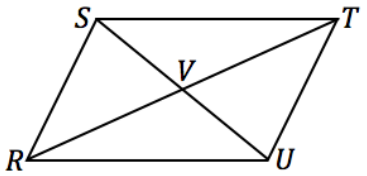
4. Given: $\mathrm{m}\angle 1 = \mathrm{m}\angle 2, BC = DC$
   Do $\triangle ABC$ and $\triangle ADC$ meet the SAS criteria?



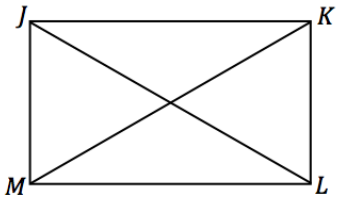
5. Given: $\overline{AE}$ bisects angle $\angle BCD, BC = DC$
   Do $\triangle CAB$ and $\triangle CAD$ meet the SAS criteria?



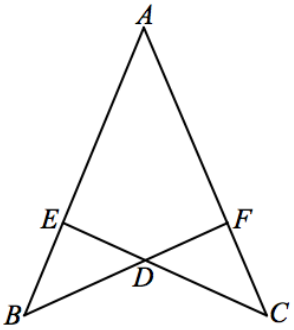
6. Given: $\overline{SU}$ and $\overline{RT}$ bisect each other
   Do $\triangle SVR$ and $\triangle UVT$ meet the SAS criteria?



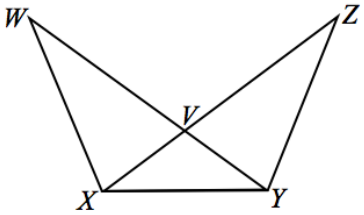
7. Given: $JM = KL$, $\overline{JM} \perp \overline{ML}$, $\overline{KL} \perp \overline{ML}$
   Do $\triangle JML$ and $\triangle KLM$ meet the SAS criteria?



8. Given: $\overline{BF} \perp \overline{AC}$, $\overline{CE} \perp \overline{AB}$
   Do $\triangle BED$ and $\triangle CFD$ meet the SAS criteria?



9. Given: $\mathrm{m}\angle VXY = \mathrm{m}\angle VYX$
   Do $\triangle VXW$ and $\triangle VYZ$ meet the SAS criteria?



10. Given: $\triangle RST$ is isosceles, $SY = TZ$.
    Do $\triangle RSY$ and $\triangle RTZ$ meet the SAS criteria?