|  |
| --- |
|  |

Lesson 18: Looking More Carefully at Parallel Lines

Student Outcomes

* Students learn to construct a line parallel to a given line through a point not on that line using a rotation by . They learn how to prove the alternate interior angles theorem using the parallel postulate and the construction.

Lesson Notes

Lesson 18 is important. It may take two days to cover all of the information in this lesson. Hiding underneath its ideas are most of the reasons why reflection and translation can be defined based upon the geometry assumptions students will learn in Lesson 33. (For example, we use only the geometry assumptions to construct a parallel line through a given point in Example 4.) While these ideas are “hiding under the surface,” do not focus on them. Instead, concentrate students’ attention on the main geometry assumption of this lesson: the parallel postulate.

We have already defined parallel lines, and in the missing angles problems we have used (and perhaps discussed casually) the following two ideas:

1. Suppose a transversal intersects a pair of lines. If a pair of alternate interior angles is equal in measure, then the pair of lines are parallel.
2. (A form of the Parallel Postulate) Suppose a transversal intersects a pair of lines. If the pair of lines are parallel, then the pair of alternate interior angles are equal in measure.

However, students have probably not made careful distinctions between these thoughts. More likely, students remember the following concept:

*Suppose a transversal intersects a pair of lines. The pair of lines are parallel if and only if a pair of alternate interior angles are equal in measure.*

Perhaps, students simply associate the ideas in this statement. When students see parallel lines cut by a transversal, they recognize the angle relations, and then use those angle relations to recognize parallel lines.

This lesson is designed to help students carefully distinguish between these two ideas. In particular, we will show why there is a need for the parallel postulate as one of our geometric assumptions:  *Through a given external point there is at most one line parallel to a given line.*

Classwork

Opening Exercise (10 minutes)

Opening Exercise

Exchange Problem Sets with a classmate. Solve the problems posed by your classmate while he or she solves yours. Compare your solutions, and then discuss and resolve any discrepancies. Why were you asked only to locate the point of rotation, rather than to rotate a pre-image to obtain the image? How did you use perpendicular bisectors in constructing your solutions?

Discussion

Discussion

We say that two lines are *parallel* if they lie in the same plane and do not intersect. Two segments or rays are parallel if the lines containing them are parallel.

Examples 1–7 (30 minutes)

Example 1

Why is the phrase *in the plane* critical to the definition of parallel lines? Explain and illustrate your reasoning.

Two lines in (3-dimensional space) are called skew lines if they do not lie in the same plane. In that case, they do not intersect (because if they did, they would share a plane together) and are not parallel. (If they were parallel, then they would both have to lie in the same plane.)

In Lesson 7, we recalled some basic facts learned in earlier grades about pairs of lines and angles created by a transversal to those lines. One of those basic facts is the following:

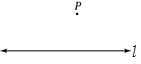
*Suppose a transversal intersects a pair of lines. The lines are parallel if and only if a pair of alternate interior angles are equal in measure.*

Our goal in this lesson is to prove this theorem using basic rigid motions, geometry assumptions, and a geometry assumption we will introduce in this lesson called the *parallel postulate.* Of all of the geometry assumptions we have given so far, the parallel postulate gets a special name because of the special role it played in the history of mathematics. (Euclid included a version of the parallel postulate in his books, and for years people tried to show that it was not a necessary assumption. Not only did it turn out that the assumption was necessary for Euclidean geometry, but study of the parallel postulate lead to the creation of non-Euclidean geometries.)

The basic fact above really has two parts, which we prove separately:

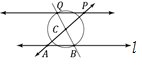
1. Suppose a transversal intersects a pair of lines. If two alternate interior angles are equal in measure, then the pair of lines are parallel.
2. Suppose a transversal intersects a pair of lines. If the lines are parallel, then the pair of alternate interior angles are equal in measure.

The second part turns out to be an equivalent form of the parallel postulate. To build up to the theorem, first we need to do a construction.

Example 2

Given a line and a point not on the line, follow the steps below to rotate by to a line that passes through :

**MP.7**

* 1. Label any point on .
  2. Find the midpoint of segment using a ruler. (Measure the length of segment , and locate the point that is distance from between and ) Label the midpoint .
  3. Perform a rotation around center . To quickly find the image of 𝒍 under this rotation by hand:
     1. Pick another point on .
     2. Draw .
     3. Draw circle: center , radius .

**MP.7**

* + 1. Label the other point where the circle intersects by .
    2. Draw .



* 1. Label the image of the rotation by of by .

How does your construction relate to the geometry assumption stated above to rotations? Complete the statement below to clarify your observations:

is a *rotation*  around . Rotations preserve *lines* , therefore maps the line to the line . What is ?

Example 3

The lines 𝒍 and in the picture above certainly look parallel, but we do not have to rely on “looks.”

Claim: In the construction above, is parallel to .

Proof: We will show that assuming they are not parallel leads to a contradiction. If they are not parallel, then they must intersect somewhere, call that point . Since is on , it must be the image of some point on under the rotation, i.e., . Since is a rotation, and must be the endpoints of a diameter of a circle that has center . In particular, must contain . Since is a point on , and is a different point on (it was the intersection of both lines), we have that because there is only one line through two points. But also contains , which means that contains . However, was constructed so that it was not on . This is absurd.

**MP.7**

There are only two possibilities for any two distinct lines and in a plane: either the lines are parallel or they are not parallel. Since assuming the lines were not parallel lead to a false conclusion, the only possibility left is that and   
were parallel to begin with.

Example 4

The construction and claim together implies the following theorem.

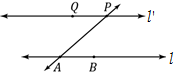
Theorem: Given a line and a point not on the line, then there exists line that contains and is parallel to .

This is a theorem we have justified before using compass and straightedge constructions, but now we see it follows directly from basic rigid motions and our geometry assumptions.

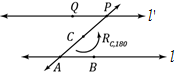
Example 5

**MP.7**

We are now ready to prove the first part of the basic fact above. We have two lines, and , and all we know is that a transversal intersects and such that a pair of alternate interior angles are equal in measure. (In the picture below we are assuming )



Let be the midpoint of . What happens if you rotate around the center ? Is there enough information to show that ?

* 1. What is the image of the segment ?
  2. In particular, what is the image of the point ?
  3. Why are the points and on the same side of ?

Sketch of the Answer: The rotation by maps to itself because contains the center . In particular, it can be shown that the rotation maps one half-plane of to the other half-plane and vice-versa. Since and are in opposite half-planes (by definition of alternate interior angles), and and are in opposite half-planes, and must be in the same half-plane.

* 1. What is the image of ? Why?

Because under the rotation, the vertex of maps to the vetex of , ray maps to ray, the point goes to the same side as . Since (by assumption), ray must map to ray . Thus, maps to .

* 1. Why is ?

Since maps to , .

We have just proved that a rotation by takes to . By the claim in Example 3, lines and must be parallel, which is summarized below.

Theorem: Suppose a transversal intersects a pair of lines. If a pair of alternate interior angles are equal in measure, then the pair of lines are parallel.

Discussion

In Example 5, suppose we had used a different rotation to construct a line parallel to that contains . Such constructions are certainly plentiful. For example, for every other point on , we can find the midpoint of segment   
 and use the construction in Example 2 to construct a different rotation around a different center such that the image of the line is a parallel line through the point . Are any of these parallel lines through different? In other words,

*Can we draw a line other than the line through that never meets ?*

The answer may surprise you; it stumped mathematicians and physicists for centuries. In nature, the answer is that it is sometimes possible and sometimes not. This is because there are places in the universe (near massive stars, for example) where the model geometry of space is not “plane-like” or flat, but is actually quite curved. To rule out these other types of “strange but beautiful” geometries, we must assume that the answer to the question above is only one line. That choice becomes one of our geometry assumptions:

*(Parallel Postulate) Through a given external point there is at most one line parallel to a given line.*

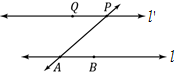
In other words, we assume that for any point in the plane not lying on a line , every line in the plane that contains intersects except at most one line—the one we call *parallel* to .

Example 6

We can use the parallel postulate to prove the second part of the basic fact.

Theorem: Suppose a transversal intersects a pair of lines. If the pair of lines are parallel, then the pair of alternate interior angles are equal in measure.

Proof: Suppose that a transversal intersects line at and at ; pick and label another point on and choose a point on on the opposite side of as . The picture might look like the figure below:



Let be the midpoint of segment , and apply a rotation by around the center . As in previous discussions, the image of is the line which is parallel to and contains point . Since and are both parallel to and contain , by the parallel postulate, they must be the same line: . In particular, . Since rotations preserve angle measures, , which was what we needed to show.

Discussion

It is important to point out that, although we only proved the alternate interior angles theorem, the same sort of proofs can be done in the exact same way to prove the corresponding angles theorem and the interior angles theorem. Thus, all of the proofs we have done so far (in class and in the Problem Sets) that use these facts are really based, in part, on our assumptions about rigid motions!

Example 7

We end this lesson with a theorem that we will just state, but can be easily proved using the parallel postulate.

Theorem: If three distinct lines , , and in the plane have the property that and , then . (In proofs, this can be written as, “If two lines are parallel to the same line, then they are parallel to each other.”)

Note that students should at least remember that in Euclidean Geometry, two lines are parallel if and only if alternate interior angles of any transversal are equal in measure, and be able to elaborate on what that means. This one statement includes both the parallel postulate and its converse. We can construct parallel lines without the parallel postulate, but in a geometry that does not satisfy the parallel postulate, there are many parallels to a given line through a point not on it. Without the parallel postulate, parallel lines are plentiful and we cannot tell much about a line if all we know is that it passes through a point and is parallel to another line.

Relevant Vocabulary

Parallel: Two lines are *parallel* if they lie in the same plane and do not intersect. Two segments or rays are parallel if the lines containing them are parallel lines.

Transversal: Given a pair of lines and in a plane, a third line is a *transversal* if it intersects at a single point and intersects at a single but different point.

The definition of transversal rules out the possibility that any two of the lines , , and are the same line.

Alternate Interior Angles: Let line be a transversal to lines and such that intersects at point and intersects at point . Let be a point on and be a point on such that the points and lie in opposite half-planes of . Then the angle and the angle are called *alternate interior angles* of the transversal with respect to and .

Corresponding Angles*:* Let line be a transversal to lines and .If and are alternate interior angles, and and are vertical angles, then and are *corresponding angles*.

Exit Ticket (5 minutes)

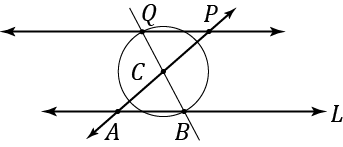
Name \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ Date\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

Lesson 18: Looking More Carefully at Parallel Lines

Exit Ticket

1. Construct a line through the point below that is parallel to the line by rotating by (using the steps outlined in Example 2).
2. Why is the parallel line you constructed the only line that contains and is parallel to ?

Exit Ticket Sample Solutions

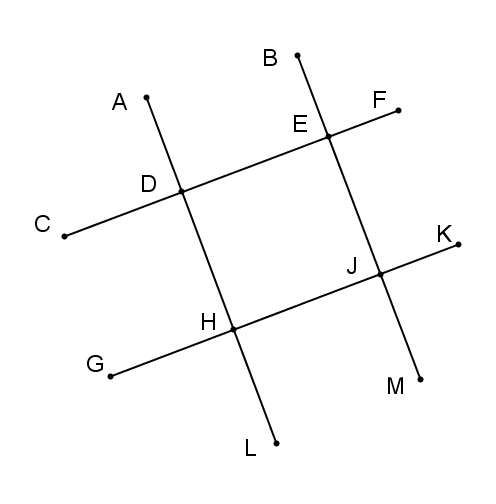
1. Construct a line through the point below that is parallel to the line by rotating by (using the steps outlined in Example 2).

Construction should look like the steps in Example 2.

1. Why is the parallel line you constructed the only line that contains and is parallel to ?

The answer should reference the parallel postulate in a meaningful way.

Problem Set Sample Solutions



1. In the figure, , ⊥ , and ⊥ . Prove that .

*⎟⎜* *Given*

*⊥* *Given*

*⊥* *Given*

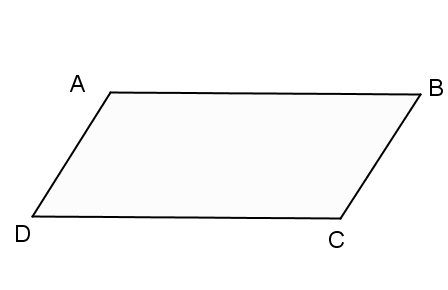
*Definition of perpendicular lines*

*Definition of perpendicular lines*

*If two parallel lines are cut by a transversal, then the corresponding angles are equal in measure*

*Substitution property of equality*

*If two lines are cut by a transversal such that the corresponding angles are equal in measure, then the lines are parallel*

1. Given that and are supplementary and , prove that ⎟⎜.

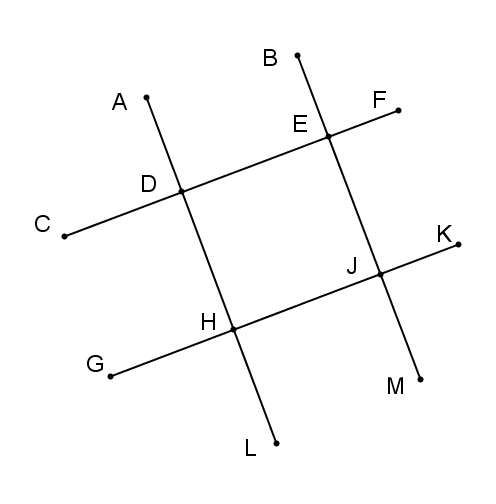
*and are supplementary* *Given*

*Given*

*Definition of supplementary angles*

*Substitution property of equality*

*If a transversal intersects two lines such that the same side interior angles are supplementary, then the lines are parallel.*

1. Mathematicians state that if a transversal to two parallel lines is perpendicular to one of the lines, then it is perpendicular to the other. Prove this statement. (Include a labeled drawing with your proof.)

*Given*

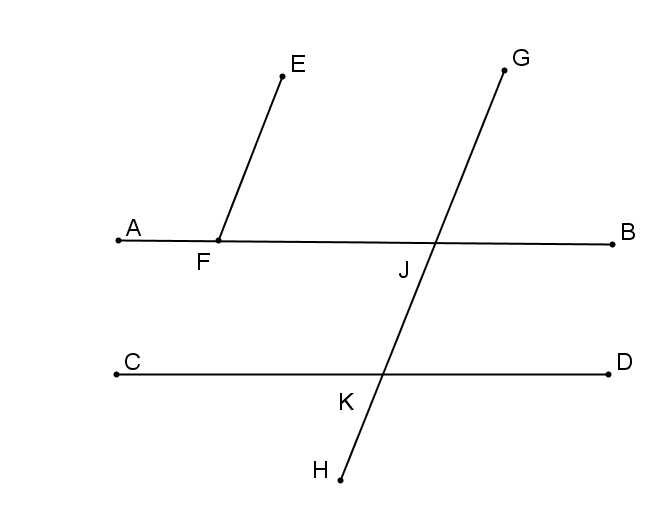
⊥  *Given*

*Definition of perpendicular lines*

*If two parallel lines are cut by a transversal, then corresponding angles are equal in measure*

*Substitution*

*Definition of perpendicular lines*

1. In the figure, and . Prove that .

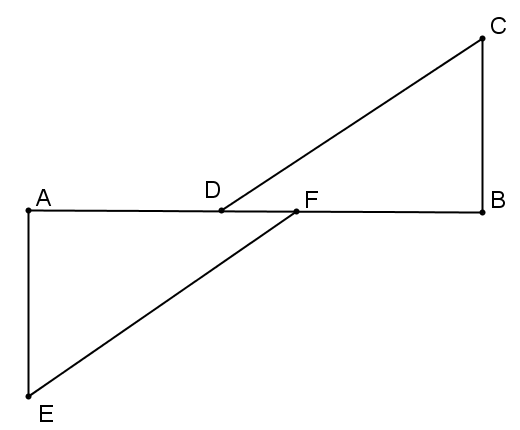
*Given*

*Given*

*If two parallel lines are cut by a trans., then the corr. angles are equal in measure*

*If two parallel lines are cut by a transversal, then the corresponding angles are equal in measure*

*Vertical angles are equal in measure*

* Transitive property*

1. In the figure, and are complementary and and are complementary. Prove that .

*and are complementary Given*

*and are complementary Given*

*Definition of comp. angles*

*Definition of comp. angles*

*Sum of the angle measures in a triangle is*

*Sum of the angle measures in a triangle is*

*; Subtraction property of equality*

*Substitution property of equality*

*If two lines are cut by a transversal such that a pair of alternate interior angles are equal in measure, then the lines are parallel.*

1. Given a line and a point not on the line, the following directions can be used to draw a line perpendicular to the line through the point based upon a rotation by :
   1. Pick and label a point on the line so that the circle (center , radius ) intersects twice.
   2. Use a protractor to draw a perpendicular line through the point (by constructing a angle).
   3. Use the directions in Example 2 to construct a parallel line through the point .

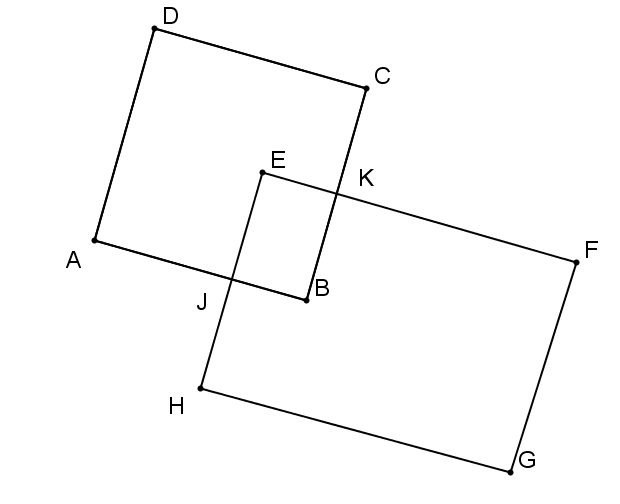
Do the construction. Why is the line perpendicular to the line in the figure you drew? Why is the line the only perpendicular line to through ?

Figures will vary, but should follow directions outlined.

Student can use Problem 3 or reprove the fact that if a transversal to two parallel lines is perpendicular to one then it is perpendicular to the other.

Suppose line contains and is perpendicular to , then by the interior angles theorem (or alternate interior angles or corresponding interior angles), is parallel to . Since and are both parallel to and contain , by the parallel postulate they must be the same line, i.e., .

A student might answer, “parallel postulate.” If so, partial credit might be awarded (depending on your tastes for partial credit).

Problems 7–10 all refer to the figure to the right. The exercises are otherwise unrelated to each other.

1. and is supplementary to . Prove that .

Given

is supplementary to Given

If a transversal intersects two lines such that the same side interior angles are supplementary, then the lines are parallel.

If two segments are parallel to the same segment, then they are parallel to

each other.

1. and . Prove that and are supplementary.

Given

Given

If two segments are parallel to the same segment, then they are parallel to each other.

*If a transversal intersects two parallel lines then the interior angles on the same side of the transversal are supplementary.*

1. and is supplementary to . Prove that .

*Given*

*is supplementary to Given*

*Definition of supplementary angles*

*Substitution property of equality*

If a transversal intersects two lines such that the same side interior angles are supplementary, then the lines are parallel.

1. , ⊥ , and is supplementary to . Prove that .

Given

⊥ Given

is supplementary to Given

*If parallel lines are cut by a transversal, then interior angles on the same side are supplementary*

*Definition of supplementary angles*

*and Definition of right angles*

*and Subtraction property of equality*

and are supplementary Definition of supplementary angles

If two lines are cut by a transversal such that a pair of interior angles on the same side are supplementary, then the lines are parallel.