Lesson 3: Copy and Bisect an Angle

Student Outcomes

- Students learn how to bisect an angle as well as how to copy an angle.

Note: These more advanced constructions require much more consideration in the communication of the students' steps.

Lesson Notes

In Lesson 3, students learn to copy and bisect an angle. As with Lessons 1 and 2, vocabulary and precision in language are essential to these next constructions.

Of the two constructions, the angle bisection is the simpler of the two and is the first construction in the lesson. Students watch a brief video clip to set the stage for the construction problem. Review the term *bisect*; ask if angles are the only figures that can be bisected. Discuss a method to test whether an angle bisector is really dividing an angle into two equal, adjacent angles. Help students connect the use of circles for this construction as they did for an equilateral triangle.

Next, students decide the correct order of provided steps to copy an angle. Teachers may choose to demonstrate the construction once before students attempt to the rearrange the given steps (and after if needed). Encourage students to test their arrangement before making a final decision on the order of the steps.

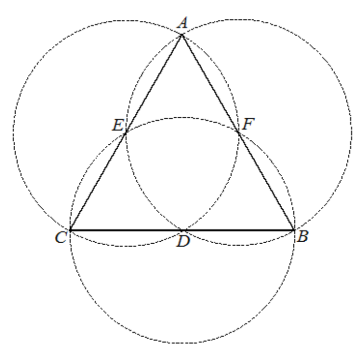
Note that while protractors are discussed in this lesson, they are not allowed on the New York State Regents Examination in Geometry. However, using protractors instructionally is helpful to develop understanding of angle measure.

Classwork

Opening Exercise (5 minutes)

Opening Exercise

In the following figure, circles have been constructed so that the endpoints of the diameter of each circle coincide with the endpoints of each segment of the equilateral triangle.



a. **What is special about points D, E, and F? Explain how this can be confirmed with the use of a compass.**

   D, E, and F are midpoints.

b. **Draw $\overline{DE}$, $\overline{EF}$, and $\overline{FD}$. What kind of triangle must $\triangle DEF$ be?**

   $\triangle DEF$ is an equilateral triangle.

c. **What is special about the four triangles within $\triangle ABC$?**

All four triangles are equilateral triangles of equal side lengths; they are congruent.

d. **How many times greater is the area of $\triangle ABC$ than the area of $\triangle CDE$?**

The area of $\triangle ABC$ is four times greater than the area of $\triangle CDE$.

Discussion (5 minutes)

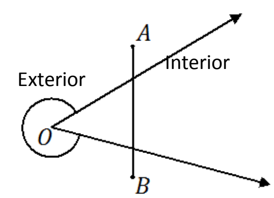
Note that an angle is defined as the union of two *non-collinear* rays with the same endpoint to make the interior of the angle unambiguous; many definitions that follow will depend on this clarity. Zero and straight angles are defined at the end of the lesson.

Discussion

Define the terms angle, interior of an angle, and angle bisector.

Angle: An *angle* is *the union of two non-collinear rays with the same endpoint.*

Interior: The *interior of angle* $\angle BAC$ is the set of points in the intersection of the half-plane of $\overleftrightarrow{AC}$ that contains B and the half-plane of $\overleftrightarrow{AB}$ that contains C. The interior is easy to identify because it is always the "smaller" region of the two regions defined by the angle (the region that is convex). The other region is called the *exterior* of the angle.



Note that every angle has two angle measurements corresponding to the interior and exterior regions of the angle: The angle measurement that corresponds to the number of degrees between $0°$ and $180°$, and the angle measurement that corresponds to the number of degrees between $180°$ and $360°$. To ensure there is absolutely no ambiguity about which angle measurement is being referred to in proofs, the angle measurement of an angle is always taken to be the number of degrees between $0°$ and $180°$. This deliberate choice is analogous to how the square root of a number is defined. Every positive number x has two square roots: $\sqrt{x}$ and $-\sqrt{x}$, so while $-\sqrt{x}$ is a square root of x, the square root of x is always taken to be $\sqrt{x}$.

For the most part, there is very little need to measure the number of degrees of an exterior region of an angle in this course. Virtually (if not all) of the angles measured in this course will either be angles of triangles or angles formed by two lines (both measurements guaranteed to be less than $180°$. The degree measure of an arc is discussed in Module 5 and can be as large as $360°$, but an arc does not have any ambiguity like an angle does. Likewise, rotations can be specified by any positive *or negative* number of degrees, a point that becomes increasingly important in Algebra II. The main thing to keep straight and to make clear to students is that degree measurements do not automatically correspond to angles; rather, a degree measurement may be referring to an angle, an arc, or a rotation in this curriculum. For example, a degree measurement of $54°$ might be referring to the measurement of an angle, but it might also be referring to the degree measure of an arc or the number of degrees of a rotation. A degree measurement of $-734°$, however, is definitely referring to the number of degrees of a rotation.

Angle Bisector: If C is in the interior of $\angle AOB$, *and $\mathrm{m}\angle AOC = \mathrm{m}\angle COB$, then $\overrightarrow{OC}$ bisects $\angle AOB$, and $\overrightarrow{OC}$ is called the bisector of $\angle AOB$.* **When we say $\mathrm{m}\angle AOC = \mathrm{m}\angle COB$, we mean that the angle measures are equal.**

Geometry Assumptions (8 minutes)

Consider accompanying this discussion with drawn visuals to illustrate the assumptions.

Geometry Assumptions

In working with lines and angles, we again make specific assumptions that need to be identified. For example, in the definition of interior of an angle above, we assumed that an angle separated the plane into two disjoint sets. This follows from the assumption: *Given a line, the points of the plane that do not lie on the line form two sets called half-planes, such that (1) each of the sets is convex, and (2) if P is a point in one of the sets, and Q is a point in the other, then the segment PQ intersects the line.*

From this assumption, another obvious fact follows about a segment that intersects the sides of an angle: *Given an angle $\angle AOB$, then for any point C in the interior of $\angle AOB$, the ray OC will always intersect the segment AB.*

In this lesson, we move from working with line segments to working with angles, specifically with bisecting angles. Before we do this, we need to clarify our assumptions about measuring angles. These assumptions are based upon what we know about a protractor that measures up to $180°$ angles:

1. **To every angle $\angle AOB$ there corresponds a quantity $m\angle AOB$ called the degree or measure of the angle so that $0° < m\angle AOB < 180°$.**

This number, of course, can be thought of as the angle measurement (in degrees) of the interior part of the angle, which is what we read off of a protractor when measuring an angle. In particular, we have also seen that we can use protractors to "add angles":

2. **If C is a point in the interior of $\angle AOB$, then $m\angle AOC + m\angle COB = m\angle AOB$.**

Two angles $\angle BAC$ and $\angle CAD$ form a *linear pair* if $\overrightarrow{AB}$ and $\overrightarrow{AD}$ are opposite rays on a line, and $\overrightarrow{AC}$ is any other ray. In earlier grades, we abbreviated this situation and the fact that the angles on a line add up to $180°$ as "$\angle s$ on a line." Now, we state it formally as one of our assumptions:

3. **If two angles $\angle BAC$ and $\angle CAD$ form a linear pair, then they are supplementary, i.e., $m\angle BAC + m\angle CAD = 180°$.**

Protractors also help us to draw angles of a specified measure:

4. **Let $\overrightarrow{OB}$ be a ray on the edge of the half-plane H. For every r such that $0° < r < 180°$, there is exactly one ray $\overrightarrow{OA}$ with A in H such that $m\angle AOB = r°$.**

Mathematical Modeling Exercise 1 (12 minutes): Investigate How to Bisect an Angle

Note to Teacher:

The speaker in the clip misspeaks by using the word *protractor* instead of *compass*. The video can even be paused, providing an opportunity to ask if the class can identify the speaker's error.

Watch the video *Angles and Trim*.

Ask students to keep the steps in the video in mind as they read the scenarios following the video and attempt the angle bisector construction on their own. (The video actually demonstrates a possible construction.)

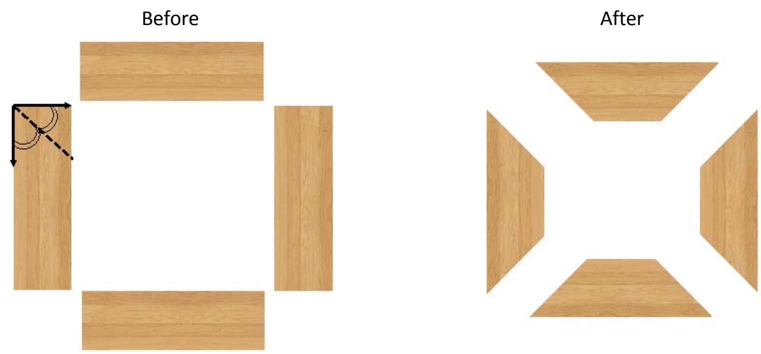
Ideas to consider:

- Are angles the only geometric figures that can be bisected?
  - *No, i.e., segments.*
- What determines whether a figure can be bisected? What kinds of figures cannot be bisected?
  - *A line of reflection must exist so that when the figure is folded along this line, each point on one side of the line maps to a corresponding point on the other side of the line. A ray cannot be bisected.*

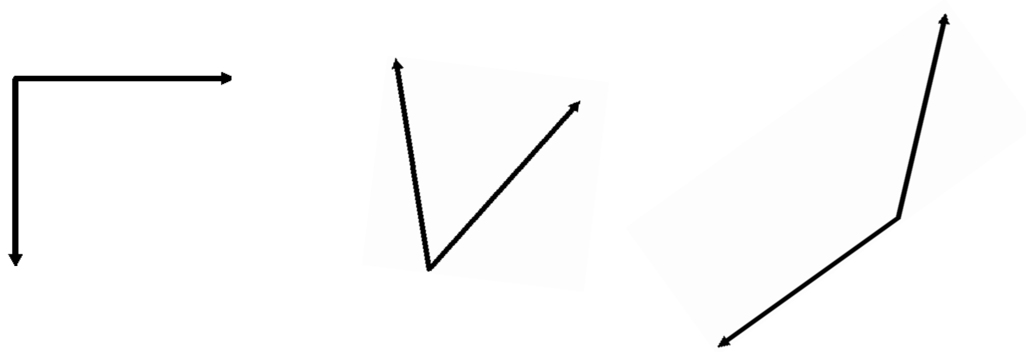
Mathematical Modeling Exercise 1: Investigate How to Bisect an Angle

You will need a compass and a straightedge.

Joey and his brother, Jimmy, are working on making a picture frame as a birthday gift for their mother. Although they have the wooden pieces for the frame, they need to find the angle bisector to accurately fit the edges of the pieces together. Using your compass and straightedge, show how the boys bisected the corner angles of the wooden pieces below to create the finished frame on the right.



Consider how the use of circles aids the construction of an angle bisector. Be sure to label the construction as it progresses and to include the labels in your steps. Experiment with the angles below to determine the correct steps for the construction.



What steps did you take to bisect an angle? List the steps below:

MP.5 & MP.6

Steps to construct an angle bisector:

1. *Label vertex of angle as A.*
2. *Draw circle A: center A, any size radius.*
3. *Label intersections of circle A with rays of angle as B and C.*
4. *Draw circle B: center B, radius BC.*
5. *Draw circle C: center C, radius CB.*
6. *At least one of the two intersection points of circle B and circle C lie in the angle. Label that intersection point D.*
7. *Draw $\overrightarrow{AD}$.*

- How does the video's method of the angle bisector construction differ from the class's method? Are there fundamental differences, or is the video's method simply an expedited form of the class's method?
  - *Yes, the video's method is an expedited version with no fundamental difference from the class's method.*

After students have completed the angle bisector construction, direct their attention to the symmetry in the construction. Note that the same procedure is done to both sides of the angle, so the line constructed bears the same relationships to each side. This foreshadows the idea of reflections and connects this exercise to the deep themes coming later. (In fact, a reflection along the bisector ray takes the angle to itself.)

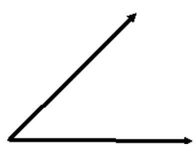
Mathematical Modeling Exercise 2 (12 minutes): Investigate How to Copy an Angle

- For Exercise 2, provide students with the Lesson 3 Supplement (Sorting Exercise) and scissors. They will cut apart the steps listed in the Supplement and arrange them until they yield the steps in correct order.

Mathematical Modeling Exercise 2: Investigate How to Copy an Angle

You will need a compass and a straightedge.

You and your partner will be provided with a list of steps (in random order) needed to copy an angle using a compass and straightedge. Your task is to place the steps in the correct order, and then follow the steps to copy the angle below.



Steps needed (in correct order):

MP.5 & MP.6

Steps to copy an angle:

1. *Label the vertex of the original angle as B.*
2. *Draw $\overrightarrow{EG}$ as one side of the angle to be drawn.*
3. *Draw circle B: center B, any radius.*
4. *Label the intersections of circle B with the sides of the angle as A and C.*
5. *Draw circle E: center E, radius BA.*
6. *Label intersection of circle E with $\overrightarrow{EG}$ as F.*
7. *Draw circle F: center F, radius CA.*
8. *Label either intersection of circle E and circle F as D.*
9. *Draw $\overrightarrow{ED}$.*

Relevant Vocabulary

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Midpoint: A point B is called a midpoint of $\overline{AC}$ if B is between A and C, and $AB = BC$.

Degree: Subdivide the length around a circle into 360 arcs of equal length. A central angle for any of these arcs is called a one-degree angle and is said to have angle measure 1 degree. An angle that turns through n one-degree angles is said to have an angle measure of n degrees.

Zero and Straight Angle: A zero angle is just a ray and measures $0°$. A straight angle is a line and measures $180°$ (the ° is a symbol for *degree*).

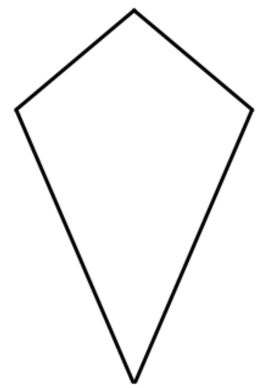
Exit Ticket (3 minutes)

Name ____ Date____________________

Lesson 3: Copy and Bisect an Angle

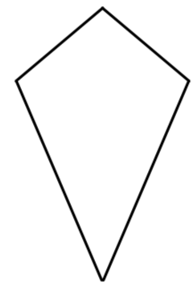
Exit Ticket

Later that day, Jimmy and Joey were working together to build a kite with sticks, newspapers, tape, and string. After they fastened the sticks together in the overall shape of the kite, Jimmy looked at the position of the sticks and said that each of the four corners of the kite is bisected; Joey said that they would only be able to bisect the top and bottom angles of the kite. Who is correct? Explain.



Exit Ticket Sample Solution

Later that day, Jimmy and Joey were working together to build a kite with sticks, newspapers, tape, and string. After they fastened the sticks together in the overall shape of the kite, Jimmy looked at the position of the sticks and said that each of the four corners of the kite is bisected; Joey said that they would only be able to bisect the top and bottom angles of the kite. Who is correct? Explain.

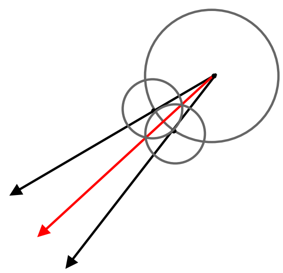


Joey is correct. The diagonal that joins the vertices of the angles between the two pairs of congruent sides of a kite also bisects those angles. The diagonal that joins the vertices of the angles created by a pair of the sides of uneven lengths does not bisect those angles.

Problem Set Sample Solutions

Bisect each angle below.

1.



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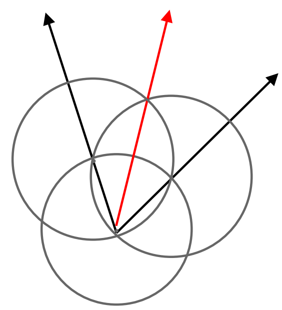
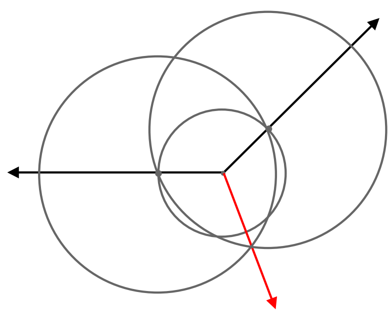
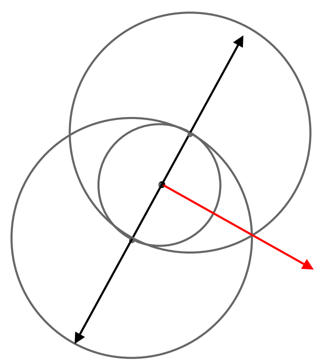


3.



4.



Copy the angle below.

5.