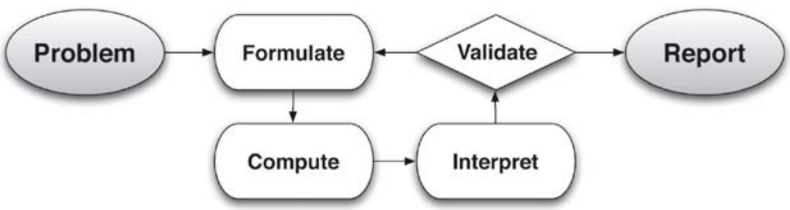
Lesson 5: Modeling from a Sequence

Student Outcomes

- Students recognize when a table of values represents an arithmetic or geometric sequence. Patterns are present in tables of values. They choose and define the parameter values for a function that represents a sequence.

Lesson Notes

This lesson takes students through the first steps of the modeling cycle, using functions that emerge from sequences. Refer to the modeling cycle below when abstracting and contextualizing. (See page 61 of the CCLS or page 72 of the CCSS.)



MP.7 & MP.4

Throughout the lesson, students will look for and make use of structure. Students will look closely to discern a pattern or structure of a sequence of numbers and be able to determine if a sequence is arithmetic, geometric, or neither. Students will also use the modeling cycle to solve problems that occur in everyday life.

Classwork

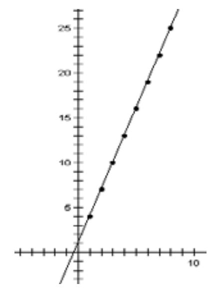
Remind your students that in Module 3, Lessons 1, 2, and 3, they learned about the linear relationship in arithmetic sequences and the exponential relationship in geometric sequences. If your students could benefit from a review of sequences, use the review that follows. If they are ready to begin, then skip to the Opening on page 73. You may choose to use the subscript notation for the expressions/equations when dealing with sequences.

Scaffolding:

- A *sequence* is a list of numbers or objects in a special order. But first we are going to learn about *geometric and arithmetic sequences.* There are two sequences in this problem. Let's review how both types are defined using the following examples:

Arithmetic Sequence: An arithmetic sequence goes from one term to the next by adding (or subtracting) the same value.

(Note: Be sure to pronounce *arithmetic* correctly. In this case, it is used as an adjective and has the emphasis on the third syllable [*adj.* ar-ith-*met*-ik] rather than on the second, as students are used to hearing it.)



Example: Start with 1, add 3 to find the next term:

n	1	2	3	4	5	6	…	n
$f(n)$	4	7	10	13	16	19		
	4	$4+3$	$4+2(3)$	$4+3(3)$	$4+4(3)$	$4+5(3)$		$4+(n-1)(3)$

$$f(n) = 4 + 3(n-1) = 4 + 3n - 3 = 3n + 1$$

Generally, we have: $f(n) = (\text{starting number}) + (n-1)(\text{common difference})$.

Note: This form is called the *explicit formula* and may also be written using subscripts, $a_n = 3n + 1$.

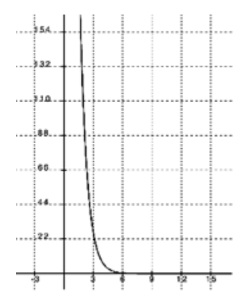
For some sequences, it is appropriate to use a *recursive formula*, which defines the terms of the sequence based on the term before. In this case, $f(n+1) = f(n) + 3$, $f(1) = 4$*; or using subscript notation,* $a_{n+1} = a_n + 3$. If we agree to call the initial value A and the common difference d, we can write the expression more simply as:

$$f(n) = A + (n-1)d$$

- Can you see from the graph that the relationship between n and the corresponding number in the sequence is a line? And that the slope of the line is 3? How is that related to the way the sequence is expanded?
  - *Yes, since n is the list of consecutive counting numbers, we add 1 to find the next. To find the next $f(n)$ value, we add three. Those two numbers, 3 and 1, are the rise and the run, respectively, for the points on the graph and so will determine the slope of the line.*

Geometric Sequence: A geometric sequence goes from one term to the next by multiplying (or dividing) by the same value.

Example: Start with 400, multiply by $\frac{1}{4}$ to find the next term:



n	1	2	3	4	…	n
$f(n)$	400	100	25	6.25		
	400	$400\left(\frac{1}{4}\right)^1$	$400\left(\frac{1}{4}\right)^2$	$400\left(\frac{1}{4}\right)^3$		$400\left(\frac{1}{4}\right)^{(n-1)}$

We have 400 as the first term and every term after than is multiplied by $\frac{1}{4}$.

This makes the exponent one less than the term number.

For this sequence, we have: $f(n) = 400\left(\frac{1}{4}\right)^{(n-1)}$

And in general terms, we have: $f(n) = (\text{starting number})(\text{common ratio})^{(n-1)}$

If we agree to call the initial value A and the common ratio r, we can write the expression more simply as:

$$f(n) = Ar^{(n-1)}$$

- Do you see from the graph that the relationship between n and the corresponding number in the sequence is exponential?

Opening Exercise (5 minutes)

Have students read the following problem and then, with a partner or small group, brainstorm the possible entry points to the solutions. Make sure they notice that there are two sequences involved, one for the length of time on each exercise, and one for the length of rest time. After they work for a while on the three questions, ask, "Why would sequences be appropriate to model this situation?" Hopefully, they will have discovered that the first column is arithmetic (linear), and the second is geometric (exponential). Also note that sequences are a good idea when the domain being modeled is whole numbers.

Opening Exercise

A soccer coach is getting her students ready for the season by introducing them to High Intensity Interval Training (HIIT). She presents the table below with a list of exercises for an HIIT training circuit and the length of time that must be spent on each exercise before the athlete gets a short time to rest. The rest times increase as the students complete more exercises in the circuit. Study the chart and answer the questions below. How long would the tenth exercise be? If a player had 30 minutes of actual gym time during a period, how many exercises could she get done? Explain your answers.

Exercise #	Length of Exercise Time	Length of Rest Time
Exercise 1	0.5 minute	0.25 minute
Exercise 2	0.75 minute	0.5 minute
Exercise 3	1 minute	1 minute
Exercise 4	1.25 minutes	2 minutes
Exercise 5	1.5 minutes	4 minutes

Discussion (15 minutes)

Display each of the two sequences from the Opening Exercise separately, showing the relationship between the number of the term, n, and the term in the sequence. Have students discuss the relationships with a partner or small group. Then use the guiding questions to inspire the discussion. (Remind students that they have studied sequences in earlier modules.) Use the tables to show that the relationship is linear for the sequence that is found by adding a constant value (the common difference) to find the next term. And then show that the relationship is exponential for the sequence that is found by multiplying by a constant value (the common ratio) to find the next term.

- What are the quantities and variables we need to define to solve this problem?
  - *Units are minutes of exercise time and minutes of rest time. Since we have two different sequences and both are related to the number of the exercise, it is likely that students will let some variable, say, n, represent the exercise number and the other two columns be $E(n)$ and $R(n)$.*
- How can we tell what type of sequence represents the completion times for the exercises?
  - *We can check to see if there is a common difference by subtracting any two consecutive terms. If the difference is constant, then it is arithmetic.*

Exercise	Exercise Time	Difference
Exercise 1	0.5 minute	
Exercise 2	0.75 minute	0.25
Exercise 3	1 minute	0.25
Exercise 4	1.25 minutes	0.25
Exercise 5	1.5 minutes	0.25

- *In this case, the sequence representing completion time for an exercise is arithmetic, so there is no need to look for a common ratio.*

- How can we tell what type of sequence the rest time is?
  - *We can subtract each term in the sequence to see if the increase is by the same number. If it is, then the sequence is arithmetic. If it is not, then the sequence might be geometric, but we would still need to make sure.*

Exercise	Rest Time	Common Difference	Common Ratio
Exercise 1	0.25 minute		
Exercise 2	0.5 minute	0.25	$\frac{0.5}{0.25} = 2$
Exercise 3	1 minute	0.5	$\frac{1}{0.5} = 2$
Exercise 4	2 minutes	1	$\frac{2}{1} = 2$
Exercise 5	4 minutes	2	$\frac{4}{2} = 2$

- It is not arithmetic, but it could be geometric. We have to analyze the ratio by dividing one term by the previous term. Looking at the column to the far right, we can see that the sequence for Rest Time is geometric.
- What is the symbolic form of the two functions?
  - *Sample responses:*

    Exercise time, E:

    $E(n) = 0.5 + 0.25(n - 1)$ *where n stands for exercise number, $n \geq 1$.*

    Rest time, R:

    $R(n)(0.25)(2)^{(n-1)}$ *where n stands for exercise number, $n \geq 1$.*
- Now let's look at those two questions above the original table. How long would the 10^{th} exercise be?
  - *Sample response:* $E(10) = 0.5 + 0.25(10 - 1) = 2.75$ *minutes*

- If a student has up to 30 minutes of actual gym time during a soccer practice, how many exercises in the circuit would she be able to complete? Explain your answer.
  - *We would have to add the Exercise and Rest Times for each row starting with Exercise 1 and see when the cumulated time exceeds 30 minutes. Here are the numbers for Exercises 1–5:*

    $$0.75 + 1.25 + 2 + 3.25 + 5.5 = 12.75.$$

    If we continue the table, we can see that we then have to add: $12.75 + 9.75 = 22.5$. *Then, for Exercise 7 we are up to more than* 40 *minutes.*

Exercise	Exercise Time	Rest Time	Total Circuit Time
Exercise 5	1.5 minutes	4 minutes	12.75
Exercise 6	1.75 minutes	8 minutes	22.5
Exercise 7	2 minutes	16 minutes	40.5 minutes

  - *So, Exercise 6 can be completed (including rest/recovery times) in* 22.5 *minutes. There is just enough time to complete Exercise 7, for a total workout time of* 24.5 *minutes before the player heads to the locker room. Students may say that there is time to finish six exercises with the rest time to follow. This would be correct if we had to include the rest time at the end of the last exercise.*

Example 1 (5 minutes)

Write this sequence on the board or screen. Have students work with a partner or small group.

Example 1

Determine whether the sequence below is arithmetic or geometric, and find the function that will produce any given term in the sequence:

$$\mathbf{16, 24, 36, 54, 81, \ldots}$$

You might use this example as an independent or guided task, depending on the needs of your students.

Is this sequence arithmetic?

The differences are 8, 12, 18, 27, … so right away we can tell this is not arithmetic—there is no common difference.

Is the sequence geometric?

The ratios are all 1.5, so this is geometric.

What is the analytical representation of the sequence?

Since the first term is 16 and the common ratio is 1.5, we have $f(n) = 16(1.5)^{n-1}$.

Exercises (15 minutes)

Have students look at the sequence for each table and then determine the analytical representation of the sequence. You might decide to use these exercises as independent practice, guided practice, or for small group work, depending on the needs of your students. Try to move them toward independence as soon as possible.

Exercises

Look at the sequence and determine the analytical representation of the sequence. Show your work and reasoning.

1. **A decorating consultant charges $\$50$ for the first hour and $\$2$ for each additional whole hour. How much would $1,000$ hours of consultation cost?**

n	1	2	3	4	5	…	n
$f(n)$	50	52	54	56	58		?

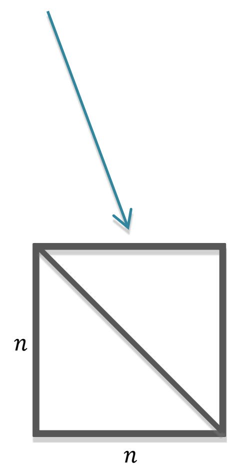
By subtracting, we see that this is an arithmetic sequence where we are adding 2 but starting at 50.

$$f(n) = 50 + 2(n-1) = 48 + 2n$$

$$f(1000) = 48 + 2(1000) = \$2048$$

2. **The sequence below represents the area of a square whose side length is the diagonal of a square with integer side length n. What would be the area for the 100^{th} square? Hint: You can use the square below to find the function model, but you can also just use the terms of the sequence.**

n	1	2	3	4	5	…	n
$f(n)$	2	8	18	32	50		?



Looking at first differences, we see that they are not the same (no common difference): $6, 10, 14, 18, \ldots$.

When we look for a common ratio, we find that the quotients of any two consecutive terms in the sequence are not the same: $\frac{8}{2} \neq \frac{18}{8} \neq \frac{32}{18} \neq \cdots$

However, I noticed that the first difference increases by 4. This is an indication of a quadratic sequence, and the function equation must have an n^2. But since for $n = 1$ we would have $n^2 = 1$, we must need to multiply that by 2 to get the first term. Now, check to see if $2n^2$ will work for the other terms.

$f(n) = 2n^2$

Checking: $f(2) = 2(2^2) = 8$
$f(3) = 2(3^2) = 18$

Yes! It works. So, $f(100) = 2(100)^2 = 20,000$.

Therefore, the area of the 100^{th} square is $20,000$ square units.

3. What would the tenth term in the sequence be?

n	1	2	3	4	…	n
$f(n)$	3	6	12	24		?

There is no common difference. But the ratios are as follows: $\frac{6}{3}=2$, $\frac{12}{6}=2$, $\frac{24}{12}=2$, ….

This is a geometric sequence with a common ratio of 2*. And the terms can be written as shown below.*

n	1	2	3	4	…	10
$f(n)$	3	$3(2)$	$3(4)$	$3(8)$		$3(2^9)$

The 10^{th} *term in the sequence is* $3(512)=1536$.

Scaffolding:

Ask students who enjoy a challenge to see if they can find another way to define the n^{th} term for Exercise 3. (They may discover that they can use 2^n in the expression if they change the number they multiply by to 1.5. Then, the function would be $f(n)=1.5(2^n)$. Check it out.)

Closing (1 minute)

- What is a convenient method for identifying whether a sequence of numbers could be modeled using a linear, exponential, or quadratic function?
  - *Look at the differences of consecutive terms. If the first differences are constant, the sequence could be modeled with a linear function. If the differences of the first differences are constant, the sequence could be modeled by a quadratic function. If neither of these is the case, look at the ratios of consecutive terms. If these are constant, the sequence could be modeled by an exponential function.*

Lesson Summary

- **A sequence is a list of numbers or objects in a special order.**
- **An arithmetic sequence goes from one term to the next by adding (or subtracting) the same value.**
- **A geometric sequence goes from one term to the next by multiplying (or dividing) by the same value.**
- **Looking at the difference of differences can be a quick way to determine if a sequence can be represented as a quadratic expression.**

Exit Ticket (4 minutes)

Name ____ Date____________________

Lesson 5: Modeling From a Sequence

Exit Ticket

A culture of bacteria doubles every 2 hours.

a. Explain how this situation can be modeled with a sequence.

b. If there are 500 bacteria at the beginning, how many bacteria will there be after 24 hours?

Exit Ticket Sample Solutions

A culture of bacteria doubles every 2 hours.

a. Explain how this situation can be modeled with a sequence.

To find the next number of bacteria, you multiply the previous number by 2. This situation can be represented by a geometric sequence. There will be a common ratio between each term of the sequence.

b. If there are 500 bacteria at the beginning, how many bacteria will there be after 24 hours?

Using the exponential function $f(n) = a \cdot b^n$, where n represents the number of times the bacteria culture doubles, a represents the amount when n = 0, so $a = 500$. b is the growth rate, so $b = 2$.

$f(n) = 500 \cdot 2^n$

$f(n) = 500 \cdot 2^{12} = 2,048,000$ *bacteria*

Problem Set Sample Solutions

Solve the following problems by finding the function/formula that represents the n^{th} term of the sequence.

1. After a knee injury, a jogger is told he can jog 10 minutes every day and that he can increase his jogging time by 2 minutes every two weeks. How long will it take for him to be able to jog one hour a day?

This is an arithmetic sequence where the minutes increase by 2 every two weeks. (Note: We can either let $2n$ represent the number of weeks or let n represent a two-week period. Either way, we will end up having to compensate after we solve.) Let's try it with n representing a two-week period:

$f(n) = \text{initial time} + (n-1)(\text{common difference})$

$60 = 10 + (n-1)(2) \rightarrow 60 = 10 + 2n - 2 \rightarrow 60 = 2n + 8$
$2n + 8 = 60 \rightarrow 2n = 52 \rightarrow n = 26$

At the beginning of the 26^{th} 2-week period, the jogger will be able to jog for 60 minutes. This will occur after $25 \cdot 2 = 50$ weeks or at the beginning of the 51^{st} week.

Week #	Daily Jog Time
1	10
2	10
3	12
4	12
5	14
6	14

2. A ball is dropped from a height of 10 feet. The ball then bounces to 80% of its previous height with each subsequent bounce.

a. Explain how this situation can be modeled with a sequence.

According to the problem, to find the next height you multiply the current height by 0.8. This means the sequence is geometric.

b. How high (*to the nearest tenth of a foot*) does the ball bounce on the fifth bounce?

$f(n) = (\text{initial height})(\text{common ratio})^n$ *for n bounces.*

$f(5) = 10(0.8)^5 = 3.2768$

The ball bounces approximately 3.3 feet on the fifth bounce.

3. **Consider the following sequence:**

$$8, 17, 32, 53, 80, 113, \dots$$

a. **What pattern do you see, and what does that pattern mean for the analytical representation of the function?**

Difference of the differences is 6. Since the second difference is a non-zero constant, then the pattern must be quadratic.

b. **What is the symbolic representation of the sequence?**

Sample response: $3n^2$ does not work by itself. (If $n = 1$, then $3n^2$ would be 3, but we have an 8 for the first term.) So, there must be a constant that is being added to it. Let's test that theory:

$$\begin{aligned} f(n) &= 3n^2 \quad + b \\ f(1) &= 3(1)^2 + b = 8 \\ &\quad 3 \quad\;\; + b = 8 \\ &\qquad\qquad b = 5 \end{aligned}$$

So, the terms of the sequence can be found using the number of the term, as follows:

$$f(n) = 3n^2 + 5$$

We can easily check to see if this function generates the sequence, and it does.

4. **Arnold wants to be able to complete 100 military-style pull-ups. His trainer puts him on a workout regimen designed to improve his pull-up strength. The following chart shows how many pull-ups Arnold can complete after each month of training. How many months will it take Arnold to achieve his goal if this pattern continues?**

Month	Pull-Up Count
1	2
2	5
3	10
4	17
5	26
6	37
…	
10	≥ 100

This pattern does not have a common difference or a common ratio. When we look at the first differences ($3, 5, 7, 9, 11, \dots$), we see that the second differences would be constant ($2, 2, 2, \dots$). That means this is a quadratic sequence with n^2 in the n^{th} term formula. For $n = 1$ we have $1^2 = 1$, so we need to add 1 to get the first term to be 2. So, in general, we have the function $f(n) = n^2 + 1$. Let's test that on the other terms: $2^2 + 1 = 5$, $3^2 + 1 = 10, \dots$. Yes, it works.

Now we need to find out which month (n) will produce 100 as the resulting number of pull-ups:

$$n^2 + 1 = 100 \rightarrow n = \sqrt{99} \approx 9.9$$

So, if this trend continues, at 10 months Arnold will be able to complete 100 pull-ups.