Lesson 14: Deriving the Quadratic Formula

Student Outcomes

- Students derive the quadratic formula by completing the square for a general quadratic equation in standard form, $ax^2 + bx + c = 0$, and use it to verify the solutions for equations from the previous lesson for which they have already factored or completed the square.

Lesson Notes

MP.1 & MP.7

Throughout this lesson, students use the structure of the equation to determine the best strategy for solving. If factoring is not possible, they solve by completing the square. While solving, they continue to notice the structure involved in the expressions in the steps to the solution, such as when fractional results are perfect squares, keeping in mind that the next step in the process will be to take the square root.

Classwork

Opening Exercises (5 minutes)

Students review the previous lesson: how to solve a quadratic equation by completing the square.

Opening Exercises

1. **Solve for x by completing the square: $x^2 + 2x = 8$.**

To solve by completing the square, factor out the leading coefficient, complete the square, and balance the equality.

$$(x^2 + 2x + \quad) = 8 +$$
$$(x^2 + 2x + 1) = 8 + 1$$
$$(x + 1)^2 = 9$$
$$(x + 1) = \pm\sqrt{9}$$
$$x = -1 \pm \sqrt{9}$$
$$x = -1 + 3 \text{ or } -1 - 3$$
$$x = 2 \text{ or } -4$$

Note: This equation can also be solved by factoring.

2. **Solve for p by completing the square: $7p^2 - 12p + 4 = 0$.**

To solve by completing the square, first gather the variable terms to one side of the equation and factor out the leading coefficient.

$$7p^2 - 12p = -4$$
$$7\left(p^2 - \frac{12}{7}p + \left(\frac{6}{7}\right)^2\right) = -4 + \frac{7(36)}{49}$$
$$7\left(p - \frac{6}{7}\right)^2 = -4 + \frac{36}{7}$$
$$7\left(p - \frac{6}{7}\right)^2 = \frac{8}{7}$$
$$\left(p - \frac{6}{7}\right)^2 = \frac{8}{49}$$
$$\left(p - \frac{6}{7}\right) = \pm\frac{\sqrt{8}}{7}$$
$$p = \frac{6}{7} \pm \frac{\sqrt{8}}{7} \text{ OR approximately } 1.26 \text{ or } 0.45$$

- Which of these problems makes more sense to solve by completing the square? Which makes more sense to solve by factoring? How could you tell early in the problem solving process which strategy to use?
  - *The best strategy for solving the second equation is by completing the square, while the first equation can be easily factored using product-sum. You can see this from the solutions since the first set of solutions is rational. To tell right away which of the expressions can be factored easily, test the sum and the product.*

Discussion (12 minutes)

Have students start with solving the general form of a one-variable linear equation, $ax + b = 0$, for x.

- How would you solve this equation for x: $ax + b = 0$, where a and b could be replaced with any numbers?
  - *Isolate the x-term and then divide by the leading coefficient:*

$$ax = -b \to x = -\frac{b}{a}.$$

- Can we say that $-\frac{b}{a}$ is a "formula" for solving any equation in the form $ax + b = 0$?
  - *Yes, it will always give us the value for x, based on the values of a and b.*

As students discuss the following questions, guide them to realize that the parameters of a quadratic are the key to determining the best entry point for solving an equation. Then, follow up by having students work in pairs or small groups to try to solve the standard form of a one-variable quadratic equation, $ax^2 + bx + c = 0$, for x.

- What factors determine how we solve an equation and what the solutions are? What makes the first Opening Exercise different from the second?
  - *The coefficients of the quadratic and linear terms, a and b, along with the constant, c, of the quadratic equation in standard form are what make the solution unique.*

- What would happen if we tried to come up with a way to use just the values of a, b, and c to solve a quadratic equation? Can we solve the general quadratic equation $ax^2 + bx + c = 0$? Is this even possible? Which method would make more sense to use, factoring or completing the square?
- Encourage students to play around with this idea for a few minutes. You may even point them in the right direction by asking questions such as the following: Can we factor if we don't know the coefficients—using a, b, and c, rather than numbers? Can we complete the square using only a, b, and c?

Ultimately, show students how to derive the quadratic formula by completing the square and have them record this in the space provided in the student handout.

Discussion

Solve $\boldsymbol{ax^2 + bx + c = 0}$.

$ax^2 + bx + c = 0$	The steps:
$a\left(x^2 + \frac{b}{a}x + \_\_\right) = -c$	Gather the variable terms, and factor out the leading coefficient.
$a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) = -c + a\left(\frac{b}{2a}\right)^2$	Complete the square inside the parentheses, and balance the equality.
$\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) = -\frac{c}{a} + \left(\frac{b^2}{4a^2}\right)$	Now, simplify the fraction on the right and multiply both sides by $\frac{1}{a}$. Remember that $a \neq 0$ since that would make the original equation not quadratic.
$\left(x + \left(\frac{b}{2a}\right)\right)^2 = \frac{-4a(c)}{4a(a)} + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}$	Factor the perfect square on the left and clean up the right. Be careful. You need to combine the two fractions on the right by finding the common denominator. Use the commutative property to reverse the two fractions on the right.
$\left(x + \frac{b}{2a}\right) = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$	Now, combine the two right-side fractions, and take the square root of both sides. Notice that the denominator is a perfect square, and do not forget the $\pm$.
$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Add $\frac{-b}{2a}$ to both sides to finally solve for x. The final step is combining the two fractions, which have the same denominator since we took the square root of $4a^2$.

Then discuss by asking the following:

- How can we verify that this formula is correct?

Encourage students to review problems from the Opening Exercises and see if they get the same answers using the quadratic formula that they got by completing the square for the standard form of a quadratic equation.

Make sure students put the equation in standard form before solving; otherwise, the quadratic formula will not work.

- Now try solving the equations in the Opening Exercises using the quadratic formula. Check your answers to make sure they are correct.
  - $x^2 + 2x = 8 \rightarrow x^2 + 2x - 8 = 0 \rightarrow a = 1, b = 2, c = -8$ (The negative is important here!)

    $$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-8)}}{2(1)} = \frac{-2 \pm \sqrt{4+32}}{2} = \frac{-2 \pm \sqrt{36}}{2} = \frac{-2 \pm 6}{2} = 2 \text{ or } -4$$

    Now check: Substitute 2 *and* -4 *into the original equation. Is* $(2)^2 + 2(2) = 8? \rightarrow 4 + 4 = 8.$

    Is $(-4)^2 + 2(-4) = 8? \rightarrow 16 - 8 = 8.$ *Or, look back at the first example to compare answers.*
  - $7p^2 - 12p + 4 = 0 \rightarrow a = 7, b = -12, c = 4$

    $$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(7)(4)}}{2(7)} = \frac{12 \pm \sqrt{144 - 112}}{14} = \frac{12 \pm \sqrt{32}}{14} = \frac{12 \pm 4\sqrt{2}}{14} = \frac{6 \pm 2\sqrt{2}}{7},$$

    which is approximately 1.26 *or* 0.45.

- Have students check whether this matches the answers from the Opening Exercises.
  - *Yes.*

Note: The simplification of the square root would not be required to get the same decimal approximations.

- Now, take a minute and have your students look closely at the quadratic formula. Point out that the whole expression can be split into two separate expressions as follows:

  $$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$

  Notice that the first part of the expression (in red) represents the axis of symmetry (the x-coordinate for the vertex). Now we step to the right and the left by the amount represented in blue to find the x-intercepts (i.e., zeros or roots). While it is never a good idea to offer memorization tricks as problem solving strategies, do not be shy about using $\frac{-b}{2a}$, as it is presented here to quickly find the axis of symmetry and the vertex for a quadratic function in standard form.

Exercises (12 minutes)

Have students work independently and check against their work from the previous lesson. If time is short, you might select two of these exercises and move on to Exercise 5. Some of these are more challenging than others. If you shorten this set, take the needs of your students into account.

After verifying that the Opening Exercises can be solved using the quadratic formula, return to the in-class exercises for Lesson 13, and solve them using the quadratic formula. Check to make sure your answers are the same.

Exercises

Use the quadratic formula to solve each equation.

1. $x^2 - 2x = 12 \rightarrow a = 1, b = -2, c = -12$ **(Watch the negatives.)**

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-12)}}{2(1)} = \frac{2 \pm \sqrt{52}}{2} = \frac{2 \pm \sqrt{4(13)}}{2} = \frac{2 \pm 2\sqrt{13}}{2} = 1 \pm \sqrt{13}$$

2. $\frac{1}{2}r^2 - 6r = 2 \rightarrow a = \frac{1}{2}, b = -6, c = -2$ **(Did you remember the negative?)**

$$r = \frac{6 \pm \sqrt{(-6)^2 - 4\left(\frac{1}{2}\right)(-2)}}{2\left(\frac{1}{2}\right)} = \frac{6 \pm \sqrt{36+4}}{1} = 6 \pm \sqrt{4(10)} = 6 \pm 2\sqrt{10}$$

3. $2p^2 + 8p = 7 \rightarrow a = 2,\ b = 8,\ c = -7$

$$p = \frac{-8 \pm \sqrt{8^2 - 4(2)(-7)}}{2(2)} = \frac{-8 \pm \sqrt{64+56}}{4} = \frac{-8 \pm \sqrt{4(30)}}{4} = \frac{-8 \pm 2\sqrt{30}}{4} = \frac{-4 \pm \sqrt{30}}{2}$$

$=$ *approximately* 0.739 *or* -4.739.

Note: In the Lesson 13 problem, the radical in the final answer was $\sqrt{\frac{15}{2}}$*, which is equivalent to* $\frac{\sqrt{30}}{2}$*.*

4. $2y^2 + 3y - 5 = 4 \rightarrow a = 2,\ b = 3,\ c = -9$

$$y = \frac{-3 \pm \sqrt{3^2 - 4(2)(-9)}}{2(2)} = \frac{-3 \pm \sqrt{9+72}}{4} = \frac{-3 \pm \sqrt{81}}{4} = \frac{-3 \pm 9}{4} = \frac{3}{2} \text{ or } -3$$

- Based on the original equation, which of the problems in the exercises would be best solved using the quadratic formula?
  - *Exercises 1–3 would need to be solved by using either the quadratic formula or by completing the square. Exercise 4 could be factored.*

Exercise 5 (7 minutes)

Have students work with a partner or in small groups to solve the following equations, using a different method for each: solve by factoring, solve by completing the square, and solve using the quadratic formula. Before they begin, ask them to consider the method they will use.

Exercise 5

Solve these quadratic equations, using a different method for each: solve by factoring, solve by completing the square, and solve using the quadratic formula. Before starting, indicate which method you will use for each.

Method ___*by factoring*___ **Method** ___*by quadratic formula*___ **Method** ___*by completing the square*___

$2x^2 + 5x - 3 = 0$ $\quad$ $x^2 + 3x - 5 = 0$ $\quad$ $\frac{1}{2}x^2 - x - 4 = 0$

(Students may choose any method to solve. However, the first and third are factorable, but the middle expression is not.)

$(2x-1)(x+3) = 0$

$x = \frac{1}{2}$ *or* -3

$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(-5)}}{2(1)}$

$x = \frac{-3 \pm \sqrt{29}}{2}$

$x = \frac{-3 + \sqrt{29}}{2}$ *or* $\frac{-3 - \sqrt{29}}{2}$

$x \approx 1.19$ *or* -4.19

$\frac{1}{2}(x^2 - 2x \quad) = 4$

$\frac{1}{2}(x^2 - 2x + 1) = 4 + \frac{1}{2}$

$\frac{1}{2}(x-1)^2 = 4.5$

$(x-1)^2 = 9$

$(x-1) = \pm 3$

$x = 1 \pm 3$

$x = 4$ *or* -2

Closing (4 minutes)

- When is completing the square the most efficient method to use for solving a quadratic equation?
  - *When it is not possible (or it is very difficult) to factor the quadratic expression, and when the leading coefficient and linear term coefficient are easy to deal with (i.e., when the leading coefficient is easily factored out resulting in an even linear term coefficient).*
- When is the quadratic formula best?
  - *When it is not possible (or it is very difficult) to factor the quadratic expression, and when the leading coefficient and/or linear term coefficient are fractions that are not easily eliminated.*
- When is factoring the most efficient method to use for solving a quadratic equation?
  - *When the factors of the equation are obvious or fairly easy to find. When factoring out the GCF or eliminating any fractional coefficients is possible.*
- Is using the quadratic formula really just completing the square? Why or why not?
  - *Yes, using the quadratic formula is really solving by completing the square because the formula is derived by completing the square for a quadratic equation in standard form. When we use the formula, we are substituting values into the expression derived by completing the square.*

Lesson Summary

The quadratic formula, $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$, is derived by completing the square on the general form of a quadratic equation: $ax^2 + bx + c = 0$, where $a \neq 0$. The formula can be used to solve any quadratic equation and is especially useful for those that are not easily solved by using any other method (i.e., by factoring or completing the square).

Exit Ticket (5 minutes)

Name ____ Date____________________

Lesson 14: Deriving the Quadratic Formula

Exit Ticket

Solve for R using any method. Show your work.

$$\frac{3}{2}R^2 - 2R = 2$$

Exit Ticket Sample Solutions

Solve for R using any method. Show your work.

$$\frac{3}{2}R^2 - 2R = 2$$

Let's start each method by multiplying both sides of the equation by 2 to eliminate the fraction.

$$3R^2 - 4R = 4$$

By factoring:	*By completing the square:*	*By using the quadratic formula:*
$3R^2 - 4R - 4 = 0$ $(R-2)(3R+2) = 0$ $R = 2 \text{ or } -\frac{2}{3}$	$3\left(R^2 - \frac{4}{3}R \quad\right) = 4$ $3\left(R^2 - \frac{4}{3}R + \left(-\frac{2}{3}\right)^2\right) = 4 + \frac{2^2}{3}$ $R^2 - \frac{4}{3}R + \left(-\frac{2}{3}\right)^2 = \frac{12}{9} + \frac{4}{9}$ $\left(R - \frac{2}{3}\right)^2 = \frac{16}{9}$ $R - \frac{2}{3} = \pm\sqrt{\frac{16}{9}}$ $R = \frac{2}{3} \pm \frac{4}{3}$ $R = 2 \text{ or } -\frac{2}{3}$	$3R^2 - 4R - 4 = 0$ $a = 3, b = -4, \text{ and } c = -4$ $R = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(-4)}}{2(3)}$ $R = \frac{4 \pm \sqrt{16+48}}{6}$ $R = \frac{4 \pm \sqrt{64}}{6} = \frac{4 \pm 8}{6}$ $R = 2 \text{ or } -\frac{2}{3}$

Problem Set Sample Solutions

Use the quadratic formula to solve each equation.

1. Solve for z: $z^2 - 3z - 8 = 0$.

   $a = 1, b = -3, c = -8$

   $$z = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-8)}}{2(1)} = \frac{3 \pm \sqrt{41}}{2}$$

2. Solve for q: $2q^2 - 8 = 3q$.

   $a = 2,\ b = -3,\ c = -8$

   $$q = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-8)}}{2(2)} = \frac{3 \pm \sqrt{73}}{4}$$

3. Solve for m: $\frac{1}{3}m^2 + 2m + 8 = 5$.

   $a = \frac{1}{3},\ b = 2, c = 3$

   $$m = \frac{-2 \pm \sqrt{2^2 - 4\left(\frac{1}{3}\right)(3)}}{2\left(\frac{1}{3}\right)} = \frac{-2 \pm \sqrt{0}}{\frac{2}{3}} = -3$$