Lesson 13: Solving Quadratic Equations by Completing the Square

Student Outcomes

- Students solve complex quadratic equations, including those with a leading coefficient other than 1, by completing the square. Some solutions may be irrational. Students draw conclusions about the properties of irrational numbers, including closure for the irrational number system under various operations.

Lesson Notes

MP.7

Throughout this lesson students look for and make use of the structure of expressions when completing the square. They see algebraic expressions, such as $(x - h)^2$, as single objects and the completed-square form as a single object being composed of several objects. They recognize that the graph will have a minimum or maximum point from the standard form of the quadratic equation, and they can use the structure to transform it into its vertex form. Once in that form, they use the structure of the completed-square expression to identify the coordinates of the vertex.

Classwork

Opening Exercises (4 minutes)

Students review two previous concepts relevant to this one: (1) solving a quadratic equation by factoring and (2) rewriting a quadratic expression by completing the square. In the first Opening Exercise, students gather the variable terms on one side of the equation. This is preparation for the first steps in solving an equation by completing the square. The second Opening Exercise is a reminder of the process used in completing the square for a quadratic expression.

Scaffolding:
- Remind students of the multiplicative property of zero (zero product property): If $ab = 0$, then either $a = 0$ and/or $b = 0$.

- Common error: Students may subtract 16 outside of the parentheses to balance the $+16$ inside in Opening Exercise 2. Remind them to consider the leading coefficient $\frac{1}{2}$ when balancing the expression.

Have students work with a partner or small group to solve the following exercises.

Opening Exercises

1. **Solve the equation for b: $2b^2 - 9b = 3b^2 - 4b - 14$.**

   To solve by factoring, gather all terms to one side of the equation and combine like terms so that the remaining expression is equal to zero:

   $b^2 + 5b - 14 = 0$*; then factor: $(b + 7)(b - 2) = 0$; and solve: $b = -7$ or 2.*

2. **Rewrite the expression by completing the square: $\frac{1}{2}b^2 - 4b + 13$.**

   Factor $\frac{1}{2}$ from the first two terms: $\frac{1}{2}(b^2 - 8b \qquad) + 13$, and then complete the square by adding $+16$ inside the parentheses (which is really $+8$ since there is $\frac{1}{2}$ outside the parentheses). Now, to compensate for the $+8$, we need to add -8 outside the parentheses and combine with the constant term:

   $\frac{1}{2}(b^2 - 8b + 16) + 13 - 8 = \frac{1}{2}(b - 4)^2 + 5$ *.*

Example 1 (8 minutes)

Students have learned how to solve equations by factoring. But what if an equation is not conducive to this method? Write the equation below on the board or screen, and ask students to work with a partner or small group to solve it.

Example 1

Solve for x.

$$12 = x^2 + 6x$$

Let them work at it for a few minutes, but they will find that this equation is not factorable over the integers.

- What strategies did you use to try to solve this equation? Where did you get stuck?
  - *We can subtract* 12 *from both sides, but then the product-sum method will not work because there are no pairs of factors of* −12 *that also add to* 6.

Students have learned to complete the square for an expression but not to use the technique for solving an equation. They may try to complete the square to solve this equation but will likely be confused about what to do with the equal sign.

Demonstrate that by completing the square, we can solve our quadratic equation a different way. Walk them through each step in the process. Start by gathering the variable terms on one side of the equation (usually the left side feels more *comfortable*) and the constant(s) on the other. Then, find the constant value that *completes* the square on the left, and balance the operation with the same on the right side of the equation.

> *Scaffolding:*
> - For visual learners, show students a graph of this quadratic. The axis of symmetry will be $x = -3$, and the roots are each a distance of $\sqrt{21}$ (or approximately 4.6) to the left and right of this axis.
> - Some students may need to be instructed to write the two solutions for the square root as separate numbers so that the $\pm$ is not lost. For example, in the third step of this problem, it would be $x + 3 = +\sqrt{21}$ or $-\sqrt{21}$. Then, when they add -3 to both sides of the equation, they will have two numbers.

$x^2 + 6x + 9 = 12 + 9$ Add 9 to complete the square: $\left[\frac{1}{2}(6)\right]^2$.

$(x + 3)^2 = 21$ Factor the perfect square.

$x + 3 = \pm\sqrt{21}$ Take the square root of both sides. Remind them NOT to forget the $\pm$.

$x = -3 \pm \sqrt{21}$ Add -3 to both sides to solve for x.

$x = -3 + \sqrt{21}$ or $x = -3 - \sqrt{21}$

- Remember to put the -3 to the left of the $\pm$ square root.
- Is there a simpler way to write this answer?
  - *No, since the value under the radical is not a perfect square, it cannot be written in a simpler form or combined with the* −3 *by addition or subtraction. The number cannot be expressed exactly or in any simpler form.*
  - *The solutions can also be estimated using decimals, approximately* 1.58 *or* −7.58.

Discussion (10 minutes): Rational and Irrational Numbers

▪ In the last example, we said that we couldn't combine a rational number, -3, with irrational numbers, $\sqrt{21}$ and $-\sqrt{21}$. Let's take a moment to examine what happens when we apply operations to rational and irrational numbers. For each column, describe what you observe. What do you notice?

Display the chart below, giving students an opportunity to write or share their observations with a partner and then with the class as a whole.

Column A	Column B	Column C	Column D
$4 + 9.5$	$-5(9.5)$	$-4 + \pi$	$2 \times \pi$
$\dfrac{4}{7} + \dfrac{5}{3}$	$5 \times \dfrac{1}{2}$	$3.5 + \sqrt{11}$	$4\sqrt{11}$
$\sqrt{9} - 2$	$(\sqrt{16})(\sqrt{9})$	$\sqrt{9} - \sqrt{10}$	$\sqrt{25} \times \sqrt{13}$

▪ What do you notice about Column A?

   ▫ *In Column A, all of the examples are sums or differences of rational numbers. In every case, we can name a rational number that is equal to the sum or difference. For example, $4 + 9.5 = 13.5$.*

▪ What do you notice about Column B?

   ▫ *In Column B, all of the examples are products of two rational numbers. In every case, we can name a rational number that is equal to the product. For example, $(\sqrt{16})(\sqrt{9}) = 12$.*

▪ What do you notice about Column C?

   ▫ *In Column C, all of the examples are sums or differences of a rational number and an irrational number. In every case, we cannot name a rational number that is equal to the sum or difference. For example, we can only write the sum $-4 + \pi$ as $-4 + \pi$. We could determine a decimal approximation, but the sum is irrational. The sums and differences are all equal to irrational numbers in this column.*

▪ What do you notice about Column D?

   ▫ *In Column D, all of the examples are products of a rational number and an irrational number. In every case, we cannot name a rational number that is equal to the product. For example, we can only write the product $2 \times \pi$ as 2π. We could also determine a decimal approximation here, but the product is irrational. The products are all equal to irrational numbers in this column.*

> **Rational and Irrational Numbers**
>
> The *sum or product of two rational numbers* is always a *rational number*.
>
> The *sum of a rational number and an irrational number* is always an *irrational number*.
>
> The *product of a rational number and an irrational number* is an *irrational number* as long as the rational number is not zero.

Example 2 (8 minutes)

Give students one more example, this time with a leading coefficient not equal to 1, and ask them to solve it.

Example 2

Now try this one.

$$4x^2 - 40x + 94 = 0$$

Have students explore solving by factoring. They should see that 2 is a GCF and find that $2(2x^2 - 20x + 47) = 0$ is the result. Now they may try to factor (by looking for a pair of numbers with a product of 94 and a sum of -20) the remaining quadratic factor but to no avail.

- ▪ Would it be more efficient to factor or complete the square?
  - □ *This equation is more conducive to completing the square because it cannot be factored using product-sum.*
- ▪ *Now that we have decided to complete the square, do you have any suggestions for how to start?*
  - □ *Gather the variable terms on the left and the constant on the right of the equation, or complete the square on the left, and then solve.*

There are two ways to approach this process: (1) by gathering all the terms to the left side of the equation and completing the square of the full expression before solving, or (2) by gathering only the variable terms on the left and the constants on the right, and then completing the square on the left before solving. Write the two starting positions on the board or screen.

$$4x^2 - 40x + 94 = 0 \text{ or } 4x^2 - 40x = -94$$

MP.1

Let students struggle with these two different entry points for just a minute or two and then demonstrate BOTH strategies, allowing each step to settle before moving to the next. You might even work the steps for each side by side on the board or screen. The important thing is that they see and understand the difference in the balancing step.

Gathering everything on the left:

$$4(x^2 - 10x + \underline{}) + 94 = 0$$
$$4(x^2 - 10x + 25) + 94 - 100 = 0$$
$$4(x - 5)^2 - 6 = 0$$
$$4(x - 5)^2 = 6$$
$$(x - 5)^2 = \frac{6}{4}$$
$$x - 5 = \pm\sqrt{\frac{6}{4}}$$
$$x = 5 \pm \frac{\sqrt{6}}{2}$$
$$x = 5 + \frac{\sqrt{6}}{2} \text{ or } 5 - \frac{\sqrt{6}}{2}$$

Gathering variable terms on the left and the constant on the right:

$$4(x^2 - 10x + \underline{}) = -94$$
$$4(x^2 - 10x + 25) = -94 + 100$$
$$4(x - 5)^2 = 6$$
$$(x - 5)^2 = \frac{6}{4}$$
$$x - 5 = \pm\sqrt{\frac{6}{4}}$$
$$x = 5 \pm \frac{\sqrt{6}}{2}$$
$$x = 5 + \frac{\sqrt{6}}{2} \text{ or } 5 - \frac{\sqrt{6}}{2}$$

- What is the same about these two strategies for solving by completing the square?
  - *Both complete the square for $x^2 - 10x$, but one subtracts 100 and the other adds 100 to compensate for completing the square.*

Emphasize that in the method used above and on the left, we needed to add -100 to balance the $+4(25)$ because they are both on the same side of the equation. (Adding -100 on the left side is equivalent to adding $+100$ on the right side.) In the method used above and on the right, we added $+4(25)$ on the left side of the equation to complete the square; therefore, we had to also add $+100$ on the right to balance. MAKE SURE THAT STUDENTS SEE THE REASONING BEHIND THIS DIFFERENCE. Take a few minutes to make sure all students see the importance of this difference.

- Why was the fraction $\frac{6}{4}$ not reduced in this example?
  - *Sometimes it is wise to leave the fraction unreduced if one or both parts are perfect squares, and you know that you will later be taking the square root.*

Exercises (20 minutes)

Reinforce the topic by having students individually solve the following equations by completing the square. Students can check their work by comparing their solutions to a classmate's. Some of these have fraction coefficients.

Exercises

Solve each equation by completing the square.

1. $x^2 - 2x = 12$

   $x^2 - 2x + 1 = 12 + 1$

   $(x - 1)^2 = 13$

   $x = 1 \pm \sqrt{13}$

2. $\frac{1}{2}r^2 - 6r = 2$

   $\frac{1}{2}(r^2 - 12r + 36) = 2 + 18$ *(Be careful with factoring out the rational leading coefficient.)*

   $\frac{1}{2}(r - 6)^2 = 20$

   $(r - 6)^2 = 40$

   $r - 6 = \pm\sqrt{40}$ *(The last step should be optional at this point.)*

   $r = 6 \pm \sqrt{40} = 6 \pm 2\sqrt{10}$

3.　　$2p^2 + 8p = 7$

$2(p^2 + 4p + 4) = 7 + 8$

$2(p + 2)^2 = 15$

$(p + 2)^2 = \dfrac{15}{2}$

$(p + 2) = \pm\sqrt{\dfrac{15}{2}}$

$p = -2 \pm \sqrt{\dfrac{15}{2}}\,; \ -2 + \sqrt{\dfrac{15}{2}} \ or \ -2 - \sqrt{\dfrac{15}{2}}$

4.　　$2y^2 + 3y - 5 = 4$

$2y^2 + 3y = 4 + 5$

$2\left[y^2 + \left(\dfrac{3}{2}\right)y + \dfrac{9}{16}\right] = 9 + \dfrac{9}{8}$

(Notice the square in the numerator. It is best to leave the fraction as it is in this step since we know we will eventually be taking the square root.)

$2\left(y + \dfrac{3}{4}\right)^2 = \dfrac{81}{8}$

$\left(y + \dfrac{3}{4}\right)^2 = \dfrac{81}{16}$

$\left(y + \dfrac{3}{4}\right) = \pm\sqrt{\dfrac{81}{16}}$

$y = -\dfrac{3}{4} \pm \dfrac{9}{4}$

$y = \dfrac{3}{2} \ or \ -3$

▪ Now that you know the solutions for the last exercise, could you have approached it differently?

　▫ *Since the solutions are rational, this one might have been easier to solve by factoring.*

Closing (1 minute)

▪ When a quadratic equation is not conducive to factoring, we can solve by completing the square.

▪ Completing the square is a good technique to use when the solutions are irrational and difficult to determine by factoring.

▪ If a simplified solution includes both rational and irrational components (without a perfect square under the radical), it cannot be rewritten equivalently as a single rational or irrational number. We need to learn to appreciate expressions such as $2 + 5\sqrt{3}$ and $\dfrac{\sqrt{2}}{7}$ as one single number on the real number line, which cannot be simplified any further.

> **Lesson Summary**
>
> **When a quadratic equation is not conducive to factoring, we can solve by completing the square.**
>
> **Completing the square can be used to find solutions that are irrational, something very difficult to do by factoring.**

Exit Ticket (4 minutes)

Name _____   Date_____

Lesson 13: Solving Quadratic Equations by Completing the Square

Exit Ticket

1. Solve the following quadratic equation both by factoring and by completing the square: $\frac{1}{4}x^2 - x = 3$.

2. Which method do you prefer to solve this equation? Justify your answer using algebraic reasoning.

Exit Ticket Sample Solutions

1. Solve the following quadratic equation both by factoring and by completing the square: $\frac{1}{4}x^2 - x = 3$.

   Factoring—you can eliminate the fraction by multiplying both sides by 4 to obtain an equivalent expression.

   $$x^2 - 4x = 12$$
   $$x^2 - 4x - 12 = 0$$
   $$(x - 6)(x + 2) = 0$$
   $$x = 6 \text{ or } -2$$

   Completing the Square—you can start by either multiplying both sides by 4 or by factoring out $\frac{1}{4}$ as the GCF.

   $$\frac{1}{4}(x^2 - 4x + \underline{\quad}) = 3$$
   $$\frac{1}{4}(x^2 - 4x + 4) = 3 + 1$$
   $$x^2 - 4x + 4 = 16$$
   $$(x - 2)^2 = 16$$
   $$x = 2 \pm 4$$
   $$x = 6 \text{ or } -2$$

2. Which method do you prefer to solve this equation? Justify your answer using algebraic reasoning.

   Students may have a personal preference about which way is easier; therefore, either answer is correct, but it should have mathematical reasoning to support it. They may prefer factoring since they are more familiar with it or because having a fraction as the leading coefficient can make completing the square trickier. Notice that in the factoring response, the fraction was eliminated in the first step. Remind students that this is often an option to make an equation look friendlier.

Problem Set Sample Solutions

Notice that Problem 3 has only one solution. Ask students why this happened. They should identify the point where $m = -3$ as the vertex of the graph of the original quadratic equation, making -3 a "double" solution.

Solve each equation by completing the square.

1. $p^2 - 3p = 8$

   $$p^2 - 3p + \frac{9}{4} = 8 + \frac{9}{4}$$
   $$\left(p - \frac{3}{2}\right)^2 = \frac{41}{4}$$
   $$\left(p - \frac{3}{2}\right) = \pm\sqrt{\frac{41}{4}}$$
   $$p = \frac{3}{2} \pm \frac{\sqrt{41}}{2}$$

2. $2q^2 + 8q = 3$

$2(q^2 + 4q + 4) = 3 + 8$

$2(q + 2)^2 = 11$

$(q + 2)^2 = \dfrac{11}{2}$

$(q + 2) = \pm\sqrt{\dfrac{11}{2}}$

$q = -2 \pm \sqrt{\dfrac{11}{2}}$

3. $\dfrac{1}{3}m^2 + 2m + 8 = 5$

$\dfrac{1}{3}(m^2 + 6m) + 8 - 8 = 5 - 8$

$\dfrac{1}{3}(m^2 + 6m + 9) = -3 + 3$

$\dfrac{1}{3}(m + 3)^2 = 0$

$(m + 3)^2 = 0$

$m = -3$

4. $-4x^2 = 24x + 11$

$-4x^2 - 24x = 11$ *Gather variable terms.*

$-4(x^2 + 6x + 9) = 11 - 36$ *Factor out -4; complete the square and balance the equality.*

$-4(x + 3)^2 = -25$ *Factor the perfect square.*

                    Divide both sides by -4.

$(x + 3)^2 = +\dfrac{25}{4}$

$x + 3 = \pm\dfrac{5}{2}$

$x = -3 \pm \dfrac{5}{2} = -\dfrac{1}{2} \text{ or } -5\dfrac{1}{2}$