Lesson 12: Completing the Square

Student Outcomes

- Students rewrite quadratic expressions given in standard form, $ax^2 + bx + c$ (with $a \neq 1$), as equivalent expressions in completed-square form, $a(x-h)^2 + k$. They build quadratic expressions in basic business application contexts and rewrite them in equivalent forms.

Lesson Notes

Lesson 12 is similar in style and sequence to Lesson 11, except that leading coefficients for the quadratic expressions are not 1. Because so much of the groundwork was laid in the preceding lesson, there is more time in this lesson for covering business applications relevant to quadratic expressions.

Classwork

Opening Exercise (5 minutes)

Opening Exercise

Rewrite each expression by completing the square.

a. $z^2 - 5z + 8$

$\left(z - \frac{5}{2}\right)^2 + \frac{7}{4}$ *OR* $(z - 2.5)^2 + 1.75$

b. $x^2 + 0.6x + 1$

$(x + 0.3)^2 + 0.91$

Scaffolding:

Students who have difficulty seeing the structure of these expressions may benefit from a written procedure or list of steps they can follow for completing the square.

Example 1 (5 minutes)

Example 1

Now complete the square for $2x^2 + 16x + 3$.

Now that students are comfortable with rewriting expressions by completing the square, we can introduce expressions with leading coefficients other than 1. Start by writing the quadratic expression below on the board or screen. Then, walk them through the process of completing the square.

$$2x^2 + 16x + 3$$

Since students already know how to complete the square when the leading coefficient is 1, one way to deal with the leading coefficient is to group the x-terms and factor out the leading coefficient. Then, they can proceed exactly as they did in the previous lesson. Students should be careful to pay attention to the multiplier on the outside of the parentheses and also to the signs involved.

$$2(x^2 + 8x \qquad) + 3$$

Now complete the square of the quadratic expression in the parentheses, and offset the addition on the outside of the parentheses.

$$2(x^2 + 8x + 4^2) + 3 - 2(4^2)$$

Make sure all agree that the two operations will reverse each other and that the new expression is equivalent to the old.

$$2(x + 4)^2 + 3 - 32 \rightarrow 2(x + 4)^2 - 29$$

Check:

$$2(x + 4)^2 - 29 = 2(x^2 + 8x + 16) - 29 = 2x^2 + 16x + 32 - 29 = 2x^2 + 16x + 3$$

Yes, this matches our original expression.

Example 2 (15 Minutes)

If your students are not familiar with the vocabulary of business, you will need to spend some time going through the terms and definitions below, which are also included in the student materials. Spending a few extra minutes on the vocabulary may extend the time needed for this example but will be well worth it. After doing so, guide your students through the example that follows. Remind them to refer to the vocabulary reference in their materials when needed. This information will be used in future lessons in this module and may need to become part of the students' math journals or notebooks.

- The relationship between the cost of an item and the quantity sold is often linear. Once we determine a relationship between the selling price of an item and the quantity sold, we can think about how to generate the most profit, i.e., at what selling price do we make the most money?
- Quadratic expressions are sometimes used to model situations and relationships in business. A common application in business is to maximize profit, that is, the difference between the total revenue (income from sales) and the production cost. It will be important to understand the vocabulary used in business applications.
- It will be a good idea to put the following information in your math notebook to use as a reference, as business applications will be used in future lessons of this module.

Example 2

Business Application Vocabulary

UNIT PRICE (PRICE PER UNIT): The price per item a business sets to sell its product, which is sometimes represented as a linear expression.

QUANTITY: The number of items sold, sometimes represented as a linear expression.

REVENUE: The total income based on sales (but without considering the cost of doing business).

UNIT COST (COST PER UNIT) OR PRODUCTION COST: The cost of producing one item, sometimes represented as a linear expression.

PROFIT: The amount of money a business makes on the sale of its product. Profit is determined by taking the total revenue (the quantity sold multiplied by the price per unit) and subtracting the total cost to produce the items (the quantity sold multiplied by the production cost per unit): $\text{Profit} = \text{Total Revenue} - \text{Total Production Costs}$.

- We can integrate the linear relationship of selling price to quantity and the profit formula to create a quadratic equation, which we can then maximize.

Use the example below to model this.

The following business formulas will be used in this and the remaining lessons in the module:

$\text{Total Production Costs} = (\text{cost per unit})(\text{quantity of items sold})$

$\text{Total Revenue} = (\text{price per unit})(\text{quantity of items sold})$

$\text{Profit} = \text{Total Revenue} - \text{Total Production Costs}$

Have students work in pairs or small groups to solve the following problem:

Now solve the following problem:

A certain business is marketing its product and has collected data on sales and prices for the past few years. The company determined that when it raised the selling price of the product, the number of sales went down. The cost of producing a single item is $\$10$.

Selling Price (s)	Quantity Sold (q)
10	$1,000$
15	900
20	800
25	700

a. **Using the data the company collected in this table, determine a linear expression to represent the quantity sold, q.**

$$q = -20s + 1,200$$

Use any two of the points of the data to find the slope of the linear relationship, as found in the data, to be -20. Substitute the coordinates of any (s, q) ordered pair from the table into the equation $q = -20s + b$, and solve for b to find the y-intercept, which is $1,200$.

Or use a graphic method to find the linear equation. Plot the data on a coordinate plane with s on the horizontal axis and q on the vertical axis. Use any two known points to determine the slope to be -20. The y-intercept will be visible, but the student should use an algebraic method to verify rather than just estimating from the graph. Substitute these values into slope-intercept form $(y = mx + b)$.

Use student responses from above to derive the following steps as a class. Discuss the algebraic reasoning behind each step, and then follow with the questions below.

b. **Now find an expression to represent the profit function, P.**

Let q = the quantity sold, s = the selling price, P = total profit, and $P(s)$ = the Profit function, the profit to be made with respect to the selling price.

$P = (s)(q) - 10q$	*Profit formula is* $\text{P} = \text{Total Revenue} - \text{Production Costs}$ $\text{Total Revenue} = \text{price} \cdot \text{quantity sold}$ $\text{Production Costs} = \text{cost per item} \cdot \text{quantity sold}$
$P = s(-20s + 1,200) - 10(-20s + 1,200)$	*Substitute $-20s + 1,200$ for q in profit formula.* *(Note that if we factor the common factor from this form of $P(s)$, we get: $P(s) = (-20 + 1200)(s - 10)$. This could save time later when we need to factor.)*
$P(s) = -20s^2 + 1,200s + 200s - 12,000$	*Multiply the expressions and combine like terms. We now have a quadratic function relating the price per item, s, to Profit.*
$P(s) = -20s^2 + 1,400s - 12,000$	*This is the expression that represents the Profit function.*

- Find the equivalent factored form of the expression. How might we use the factored form?
  - $-20(s^2 - 70s + 600)$
    $-20(s - 60)(s - 10)$
    We can use the factored form of the expression to find the zeros of P by factoring and then setting the expression equal to zero.

> *Scaffolding:*
> Have students graph the quadratic equation to visually represent the function, $P(s)$, and the maximum profit.

- Use the zeros to find the minimum or maximum value for the profit function, $P(s)$. How might we use the vertex to interpret the profit function? (Remind students of the process used in earlier lessons in this Module.)
  - *If we set the expression above equal to 0, we get the following:*
    $-20(s - 60)(s - 10) = 0$*, which leads to $s = 10$ or 60.*
    Halfway between them is the axis of symmetry and the s-value for the vertex, which is $s = 35$.
    Then, $P(35) = \$12{,}500$.
    By finding the vertex, we can determine the selling price that will generate the maximum profit. The x-values (domain) represent selling price, so the value of the x-coordinate at the vertex represents the best price. The P-value at the vertex tells us the amount of profit made at the maximum.
- What is the equivalent completed-square form for the profit expression?
  $-20s^2 + 1{,}400s - 12{,}000$
  - $-20(s^2 - 70s + 35^2) - 12{,}000 + 20(35^2)$
    $-20(s - 35)^2 - 12{,}000 + 24{,}500$
    $-20(s - 35)^2 + 12{,}500$
- What do you notice about the values in the completed-square form?
  - *By finding the vertex of the parabola, we will find the selling price that will generate the most profit. The x-axis represents selling price, so the value of the x-coordinate at the vertex represents the best price.*
  - *The P-value at the vertex tells us the maximum amount of profit to be made.*

Hopefully, students will notice that the vertex can be seen in the parameters of the completed square form. If not, point it out to them: $(x - h)^2 + k$, where (h, k) is the vertex.

Exercises (10 minutes)

Exercises 16 include a business application but primarily focus on the procedure for completing the square. If time is short, you may want to choose two or three of these to work in class and assign the others along with the Problem Set.

Exercises

For Exercises 1–5, rewrite each expression by completing the square.

1. $3x^2 + 12x - 8$

   $3(x^2 + 4x) - 8 \rightarrow 3(x+2)^2 - 8 - 12 \rightarrow 3(x+2)^2 - 20$

2. $4p^2 - 12p + 13$

   $4(p^2 - 3p) + 13 \rightarrow 4\left(p - \frac{3}{2}\right)^2 + 13 - 9 \rightarrow 4\left(p - \frac{3}{2}\right)^2 + 4$

3. $\frac{1}{2}y^2 + 3y - 4$

   $\frac{1}{2}(y^2 + 6y) - 4 \rightarrow \frac{1}{2}(y+3)^2 - 4 - \frac{9}{2} \rightarrow \frac{1}{2}(y+3)^2 - \frac{17}{2}$

4. $1.2n^2 - 3n + 6.5$

   $1.2(n^2 - 2.5n) + 6.5 \rightarrow 1.2(n - 1.25)^2 + 6.5 - 1.875 \rightarrow 1.2(n - 1.25)^2 + 4.625$

5. $\frac{1}{3}v^2 - 4v + 10$

   $\frac{1}{3}(v^2 - 12v) + 10 \rightarrow \frac{1}{3}(v-6)^2 + 10 - 12 \rightarrow \frac{1}{3}(v-6)^2 - 2$

6. A fast food restaurant has determined that its price function is $3 - \frac{x}{20{,}000}$, where x represents the number of hamburgers sold.

   a. The cost of producing x hamburgers is determined by the expression $5,000 + 0.56x$. Write an expression representing the profit for selling x hamburgers.

   $\text{Profit} = \text{Total Revenue} - \text{Total Production Costs} = (\text{quantity})(\text{price}) - \text{cost}$

   $$= (x)\left(3 - \frac{x}{20,000}\right) - (5,000 + 0.56x)$$

   $$= 3x - \frac{x^2}{20,000} - 5,000 - 0.56x$$

   $$= -\frac{x^2}{20,000} + 2.44x - 5,000$$

b. **Complete the square for your expression in part (a) to determine the number of hamburgers that need to be sold to maximize the profit, given this price function.**

$$-\frac{1}{20,000}(x^2 - 48,800x + \quad) - 5,000 = -\frac{1}{20,000}(x^2 - 48,800x + 24,400^2) - 5,000 + \frac{24,400^2}{20,000}$$
$$= -\frac{1}{20,000}(x - 24,400)^2 - 5,000 + 29,768$$
$$= -\frac{1}{20,000}(x - 24,400)^2 + 24,768$$

So, $24,400$ hamburgers must be sold to maximize the profit using this price expression. Note: The profit will be $\$24,768$.

Lesson Summary

Here is an example of completing the square of a quadratic expression of the form $ax^2 + bx + c$.

$$3x^2 - 18x - 2$$
$$3(x^2 - 6x) - 2$$
$$3(x^2 - 6x + 9) - 3(9) - 2$$
$$3(x - 3)^2 - 3(9) - 2$$
$$3(x - 3)^2 - 29$$

Exit Ticket (10 minutes)

Name ______________________________ Date____________________

Lesson 12: Completing the Square

Exit Ticket

1. Complete the square: $ax^2 + x + 3$.

2. Write the expression for the profit, P, in terms of q, the quantity sold, and s, the selling price, based on the data collected below on sales and prices. Use the examples and your notes from class to then determine the function that represents yearly profit, P, in terms of the sales, s, given the production cost per item is $\$30$.

Selling Price, $\$$ (s)	**Quantity Sold (q)**
100	$7{,}000$
200	$6{,}000$
500	$3{,}000$
600	$2{,}000$
800	0

Exit Ticket Sample Solutions

1. **Complete the square: $ax^2 + x + 3$.**

   $a\left(x^2 + \frac{1}{a}x\right) + 3 \rightarrow a\left(x + \frac{1}{2a}\right)^2 + 3 - \frac{1}{4a}$

2. **Write the expression for the profit, P, in terms of q, the quantity sold, and s, the selling price, based on the data collected below on sales and prices. Use the examples and your notes from class to then determine the function that represents yearly profit, P, in terms of the sales, s, given the production cost per item is $\$30$.**

Selling Price, \$ (s)	Quantity Sold (q)
100	$7,000$
200	$6,000$
500	$3,000$
600	$2,000$
800	0

   The Revenue (total income from sales) is the price per item times number of items, $(s)(q)$, and the cost is $30q$, so the profit is $P = (s)(q) - 30q$.

   The graph showing the relationship between s and q is a line with slope -10 and q-intercept $8,000$, so $q = -10s + 8,000$.

   This means the Profit function is $P(s) = s(-10s + 8,000) - 30(-10s + 8,000)$.

   Factoring the common factors out gives us $P(s) = (-10s + 8,000)(s - 30) = -10(s - 800)(s - 30)$.

   Or, multiplying and combining like terms gives us $P(s) = -10s^2 + 8,300s - 240,000$.

Solution Notes:

1. Some students may give the answer as $q = -10s + 8{,}000$, which is an important part of finding the profit but is not complete. The revenue (total income from sales) is the product $(s)(q)$, and the cost is $30q$, so the profit is $P = (s)(q) - 30q$. Using the equation for q: $P = s(-10s + 8{,}000) - 30(-10s + 8{,}000)$. Multiplying and combining like terms gives $P = -10s^2 + 8{,}300s - 240{,}000$.
2. Students may also come up with $P = (s)(q) - 30q$. While it is true that $P = (s)(q) - 30q$, this expression has two variables. q needs to be substituted for its equivalent in terms of s so that the function $P(s)$ is written in terms of s.
3. The graph showing the relationship between q and s is a line with slope -10 and q-intercept $8{,}000$, so $q = -10s + 8{,}000$. This means $P = s(-10s + 8{,}000) - 30(-10s + 8{,}000)$. Multiplying and simplifying gives $P = -10s^2 + 8{,}300s - 240{,}000$.

Problem Set Sample Solutions

Rewrite each expression by completing the square.

1. $-2x^2 + 8x + 5$

   $-2(x^2 - 4x + 4) + 5 + 8 \rightarrow -2(x - 2)^2 + 13$

2. $2.5k^2 - 7.5k + 1.25$

   $2.5(k^2 - 3k + 2.25) + 1.25 - 5.625 \rightarrow 2.5(k - 1.5)^2 - 4.375$

3. $\frac{4}{3}b^2 + 6b - 5$

   $\frac{4}{3}\left(b^2 + \frac{9}{2}b + \frac{81}{16}\right) - 5 - \frac{27}{4} \rightarrow \frac{4}{3}\left(b + \frac{9}{4}\right)^2 - \frac{47}{4}$

4. $1{,}000c^2 - 1{,}250c + 695$

   $1{,}000(c^2 - 1.25c + 0.625^2) + 695 - 390.625 \rightarrow 1{,}000(c - 0.625)^2 + 304.375$

5. $8n^2 + 2n + 5$

   $8\left(n^2 + \frac{1}{4}n + \frac{1}{64}\right) + 5 - \frac{1}{8} \rightarrow 8\left(n + \frac{1}{8}\right)^2 + 4\frac{7}{8}$