Lesson 11: Completing the Square

Student Outcomes

- Students rewrite quadratic expressions given in standard form, $ax^2 + bx + c$ (with $a = 1$), in the equivalent completed-square form, $a(x-h)^2 + k$, and recognize cases for which factored or completed-square form is most efficient to use.

MP.7 Throughout this lesson, students use the structure of quadratic equations to rewrite them in completed-square form.

Lesson Notes

In the Opening Exercise, students look for patterns in the structure of perfect square quadratic expressions. They recognize that when the leading coefficient is 1, the coefficient of the linear term, b, from the standard form, $x^2 + bx + c$, is always twice the constant term of the perfect square binomial $\left(x + \frac{1}{2}b\right)^2$ and that the constant term, c, must equal $\left(\frac{1}{2}b\right)^2$ for the quadratic to be rewritten as a perfect square. This leads directly to the method of completing the square. When given a quadratic that is *not* factorable as a perfect square, students can write an equivalent expression that includes a perfect square of a binomial. The importance and elegance of efficient methods is then reinforced. When is factoring the most expedient or useful method for rewriting an expression? When is it more efficient or more useful to complete the square? The examples and exercises support these ideas and introduce students to relevant business applications. Note that all quadratic expressions in this lesson have a leading coefficient of $a = 1$; other leading coefficients are addressed in Lesson 12. It may be helpful to hint to students that in later lessons we will use this method of rewriting expressions to solve quadratic equations that are not factorable.

Classwork

Opening Exercise (5 minutes)

Project or draw the table on the board or screen. Demonstrate with the whole class by filling in the first row. Then, have students work in pairs to continue filling in the blanks in the next four rows and writing down their observations.

Opening Exercise

Rewrite the following perfect square quadratic expressions in standard form. Look for patterns in the coefficients, and write two sentences describing what you notice.

FACTORED FORM	WRITE THE FACTORS	DISTRIBUTE	STANDARD FORM
Example: $(x+1)^2$	$(x+1)(x+1)$	$x \cdot x + 1x + 1x + 1 \cdot 1$	$x^2 + 2x + 1$
$(x+2)^2$	$(x+2)(x+2)$	$x \cdot x + 2x + 2x + 2 \cdot 2$	$x^2 + 4x + 4$
$(x+3)^2$	$(x+3)(x+3)$	$x \cdot x + 3x + 3x + 3 \cdot 3$	$x^2 + 6x + 9$
$(x+4)^2$	$(x+4)(x+4)$	$x \cdot x + 4x + 4x + 4 \cdot 4$	$x^2 + 8x + 16$
$(x+5)^2$	$(x+5)(x+5)$	$x \cdot x + 5x + 5x + 5 \cdot 5$	$x^2 + 10x + 25$
$(x+20)^2$	$(x+20)(x+20)$	$x \cdot x + 20x + 20x + 20 \cdot 20$	$x^2 + 40x + 400$

A, the constant in the factored form is always half of b, the coefficient of the linear term in the standard form of the quadratic equation. The constant term in the standard form of the equation, c, is always the square of the constant in the factored form, A.

- Do you see any patterns in the numbers in the first column and those in the last?
  - *$(x + A)^2 = x^2 + bx + c$. A, the constant in factored form of the equation is always half of b, the coefficient of the linear term in the standard form. c, the constant term in the standard form of the quadratic equation is always the square of the constant in the factored form, A.*
- Can you generalize this pattern so that you can square *any* binomial in the form $(x + A)^2$?
  - *$(x + A)^2 = x^2 + 2Ax + A^2$. The coefficient of the linear term of the equation in standard form, b, is always twice the constant in the binomial, A. The constant term in the standard form of the equation, c, is always the square of the constant in the binomial, A.*

Example (5 minutes)

Have students continue to work with their partner to complete this table. Encourage them to use the patterns discussed above to find the factored form efficiently. Ideally, there should be no need to guess-and-check to factor these expressions. Students should progress through the first five examples but will likely get stuck on the last expression.

Example

Now try working backwards. Rewrite the following standard form quadratic expressions as perfect squares.

STANDARD FORM	FACTORED FORM
$x^2 + 12x + 36$	$(x + 6)^2$
$x^2 - 12x + 36$	$(x - 6)^2$
$x^2 + 20x + 100$	$(x + 10)^2$
$x^2 - 3x + \frac{9}{4}$	$\left(x - \frac{3}{2}\right)^2$
$x^2 + 100x + 2,500$	$(x + 50)^2$
$x^2 + 8x + 3$	*n/a*

- What is different about $x^2 + 8x + 3$? Why is it impossible to factor this expression as a perfect square binomial?
  - *It is not a perfect square.*
- If you could change something about the last expression to make it a perfect square, what would you change?
  - *If the constant term were a 16, it would be a perfect square (4^2 as the constant term and $4(2)$ as the linear term coefficient).*

Exploratory Challenge (8 minutes)

Exploratory Challenge

Find an expression equivalent to $x^2 + 8x + 3$ that includes a perfect square binomial.

$(x + 4)^2 - 13$

Show students that when an expression is not a perfect square, they can use the tabular method learned in Lesson 2 to rewrite this expression as an equivalent perfect square binomial. Write the expression $x^2 + 8x + 3$ on the board or screen and lead them through the following process.

Under the quadratic expression, draw a 2×2 table to use as a tool. First, put the x^2 into the upper-left box; next, write an x above and to the left. Then, follow the steps below.

	x	
x	x^2	

- We are looking for a perfect square binomial that matches our quadratic expression as closely as possible. How do we know there must an x-term in our binomial?
    - *We know this because $x \cdot x$ is the only way to get x^2 (given that we are looking for polynomials which require whole number exponents).*
- The quadratic expression in standard form has a linear term of $+8x$. What constant term must the perfect square binomial have if the linear term coefficient is positive 8? Fill in the missing cells both outside and inside the square.
    - *We want the same two numbers to add to $+8x$, so that would be $+4x$ and $+4x$. Therefore, each binomial must have $+4$ as its constant term since the b coefficient in this example is 8.*

If students have not already come to this conclusion, point out that the constant term in a perfect square binomial is always half of the b coefficient.

	x	$+4$
x	x^2	$+4x$
$+4$	$+4x$	

- If the binomial to be squared is $(x+4)$, what must the constant term be when this perfect square is expanded? Fill in the final lower right box.
  - *$(x+4)^2$ has a constant term of 16 when expanded.*

	x	$+4$
x	x^2	$+4x$
$+4$	$+4x$	$+16$

- So, we know that to factor this binomial as the perfect square $(x+4)^2$, we would need $x^2+8x+16$ instead of the x^2+8x+3 that we actually have. Looks like a dead end…but wait!

$$x^2+8x+16 = x^2+8x+3+13$$

  So, all we have to do is add $+13$ to our expression, right?
  - *No, that would not be an equivalent expression.*

Give students a chance to catch the mistake here. They should see that adding 13 to the expression changes its value. Encourage them to find a way to balance the expression; ultimately lead them to write the following:

$$x^2+8x+3 \rightarrow x^2+8x+3+13-13 \rightarrow x^2+8x+(3+13)-13 \rightarrow (x^2+8x+16)-13 \rightarrow (x+4)^2-13.$$

To verify the result, you may want to undo completing the square by multiplying and combining like terms to prove that the expressions are now equivalent.

MP.8 Students notice repetition and recognize a pattern through the example above and exercises below. They use this repeated reasoning to generalize the pattern in Exercise 10.

Exercises 1–10 (20 minutes)

Exercises 1–10

Rewrite each expression by completing the square.

1. $\boldsymbol{a^2-4a+15}$

   $(a-2)^2+11$ ***(Note: Since the constant term required to complete the square is less than the constant term, $+15$, students may notice that they just need to split the $+15$ strategically.)***

2. $\boldsymbol{n^2-2n-15}$

   $(n-1)^2-16$

3. $\boldsymbol{c^2+20c-40}$

   $(c+10)^2-140$

4. $x^2 - 1{,}000x + 60{,}000$

$(x - 500)^2 - 190{,}000$

5. $y^2 - 3y + 10$

$\left(y - \frac{3}{2}\right)^2 + \frac{31}{4}$

6. $k^2 + 7k + 6$

$\left(k + \frac{7}{2}\right)^2 - \frac{25}{4}$

7. $z^2 - 0.2z + 1.5$

$(z - 0.1)^2 + 1.49$

8. $p^2 + 0.5p + 0.1$

$(p + 0.25)^2 + 0.0375$

9. $j^2 - \frac{3}{4}j + \frac{3}{4}$

$\left(j - \frac{3}{8}\right)^2 + \frac{39}{64}$

10. $x^2 - bx + c$

$\left(x - \frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2$

Closing (2 minutes)

Have students take a look at the expressions in Exercises 2 and 6.

- Is there anything you notice about these two expressions? Although we can (and did) complete the square for each, how else might they be rewritten?
  - *Both of these expressions are easy to write in factored form.*
- Note that in some circumstances, the easiest form may not be the most useful form. Even if an expression is easy to factor, we may still want to write it as a completed square.

Lesson Summary

Just as factoring a quadratic expression can be useful for solving a quadratic equation, completing the square also provides a form that facilitates solving a quadratic equation.

Exit Ticket (5 minutes)

Name ____ Date____________________

Lesson 11: Completing the Square

Exit Ticket

Rewrite the expression $r^2 + 4r + 3$, first by factoring, and then by completing the square. Which way is easier? Explain why you think so.

Exit Ticket Sample Solutions

Rewrite the expression $r^2 + 4r + 3$, first by factoring, and then by completing the square. Which way is easier? Explain why you think so.

By factoring: $(r+3)(r+1)$

By completing the square: $(r+2)^2 - 1$

Both options are fairly simple, and students may select either for their preference. The important thing is that they are thinking about efficiency in their methods and the various options available for rewriting quadratic expressions.

Problem Set Sample Solutions

Scaffolding:

If students need more practice with completing the square, use these problems or some like them.

1. $q^2 + 12q + 32$

   $(q+6)^2 - 4$

2. $m^2 - 4m - 5$

   $(m-2)^2 - 9$

3. $x^2 - 7x + 6.5$

   $\left(x - \frac{7}{2}\right)^2 - 5.75$

4. $a^2 + 70a + 1{,}225$

   $(a+35)^2$

5. $z^2 - 0.3z + 0.1$

   $(z - 0.15)^2 + 0.0775$

6. $y^2 - 6by + 20$

   $(y - 3b)^2 + 20 - 9b^2$

7. **Which of these expressions would be most easily rewritten by factoring? Justify your answer.**

   Students may respond with either Problem 1, 2, or 4, and justifications may range from a demonstration of the factoring process to a written explanation where students show that the product-sum rule can be applied to either of these expressions.