Lesson 5: The Zero Product Property

Student Outcomes

- Students solve increasingly complex one-variable equations, some of which need algebraic manipulation, including factoring as a first step and using the zero product property.

Lesson Notes

In this lesson, we discover ways to use the factoring skills honed in the last four lessons to solve equations involving quadratic expressions. We begin with an exploration of the zero product property and a discussion how and why it is used. Students then use the property to solve basic one-variable equations, many of which are presented in the context of area (**A-CED.A.1**).

Classwork

Opening Exercise (3 minutes)

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Consider the equation $a \cdot b \cdot c \cdot d = 0$. What values of a, b, c, and d would make the equation true?

Any set of values where one of the four variables was equal to zero, and also values where two, three, or even all four of the variables were equal to zero.

Scaffolding:

- If students struggle with the concept of the zero product property, ask them to try substituting numbers into the equation until they find one or more that make the equation true. Point out that in every case, at least one factor must be 0.
- You can go back one step further by asking several students in the class for a product of any two numbers that equals 0. Then, ask what they notice about the numbers used in the products. Relate those numerical products to the equations in this example.

State or show the zero product property. Make connections between the example above and the given property.

Zero Product Property

If $ab = 0$, then $a = 0$ or $b = 0$ or $a = b = 0$.

Exercises 1–4 (8 minutes)

Exercises 1–4

Find values of c and d that satisfy each of the following equations. (There may be more than one correct answer.)

1. $cd = 0$

   Either c or d must be zero, but the other can be any number, including zero (i.e., both c and d MIGHT be equal to zero at the same time).

2. $(c-5)d = 2$

   There are an infinite number of correct combinations of c and d, but each choice of c will lead to only one choice for d and vice versa. For example, if $d = 2$, then c must be 6, and if $c = 4$, then d must be -2.

3. $(c-5)d = 0$

   Since the product must be zero, there are only two possible solution scenarios that will make the equation true, $c = 5$ (and d can be anything) or $d = 0$ (and c can be anything); specifically, one solution would be $c = 5$ and $d = 0$.

- Why can we easily pinpoint a number that must be substituted for c to make $(c-5)d = 0$ a true statement but not for $(c-5)d = 2$?

Refer to the zero product property. Discuss that if the expression is set equal to zero, we know that at least one factor must be equal to zero, limiting the number of possible solutions. Therefore, it is more convenient to set an expression equal to zero when solving.

4. $(c-5)(d+3) = 0$

   $c = 5$ or $d = -3$. Either makes the product equal zero; they could both be true, but both do not have to be true. However, at least one must be true.

Example (17 minutes)

Ask students to read the problem's prompt in their materials and to take notes as you read the questions below aloud. They should work in pairs or small groups to find the answers.

Example

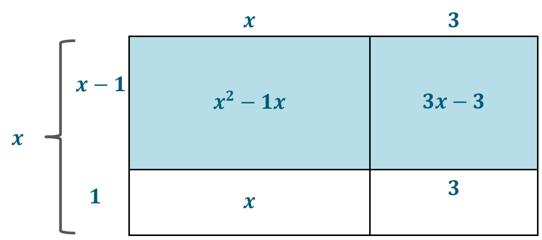
For each of the related questions below, use what you know about the zero product property to find the answers.

a. **The area of a rectangle can be represented by the expression $x^2 + 2x - 3$. Write each dimension of this rectangle as a binomial, and then write the area in terms of the product of the two binomials.**

   The factors of $x^2 + 2x - 3$ are $(x+3)(x-1)$, so the dimensions of the rectangle will be $(x-1)$ on the shorter side (width) and $(x+3)$ on the longer side (length).

b. Can we draw and label a diagram that represents the rectangle's area?

The diagram below models the problem in a way that avoids negative areas.



Let the students try using a diagram for this problem, but suggest it will not be easy to do so in a way that avoids negative areas.

Try drawing the diagram shown or ask a student who did a particularly good job to display hers for the class to examine.

It is also *very important* to point out that an area model represents the relationship between the areas and is not to be set as an actual size. In fact, every student in your class could draw a different scale diagram with the length of x looking like a different number of units in length, and all could be a correct model.

c. Suppose the rectangle's area is 21 square units. Can you find the dimensions in terms of x?

(Use the following questions to guide students through solving this part of the problem.)

- What is the problem asking us to do?
  - *Start by asking students if they can find a number to substitute for x that will make the statement true. Substituting numbers into the quadratic expression to find a way to equal 21 should prove daunting. Still, making a table of values and seeing if we can get close to 21 might be a viable strategy. Remind students that they need to find both solutions. (The correct solution is $x = 4$ or -6.)*
- Should we use the factors we already know for the quadratic expression on the left of the equation? Here is what that would look like: $(x + 3)(x - 1) = 21$. Is this easier to see the solutions?
  - *When looking at the factored form, students may find one of the correct answers right away since 21 has only two factor pairs to check. Even so, students should see that it is harder to find the solution(s) when the product is given as 21 than it is when we know the product is 0. Also, remind students that there are two correct solutions, and one may be easier to find than the other.*
- Is there a way to set the equation equal to zero rather than to 21? What is the benefit of setting an equation equal to zero? What strategies have we used previously when dealing with similar equations to solve for the variable?
  - *Yes, we can set an expression equal to any number. However, setting it equal to 0 makes our solutions easier to find. In some cases, it is more efficient to leave the number the expression equals alone. An example of this is when the structure of the expression makes it possible to simply take the square root of each side of the equation (e.g., $(x + 2)^2 = 9$).*

- Read part (d). What do you need to do first?
  - *We need to subtract* 21 *from both sides of the equation and put the quadratic in standard form (expanded form).*
- Now solve for x in the new form of the equation.

> **d. Rewrite the equation so that it is equal to zero and solve.**
>
> $x^2 + 2x - 3 - 21 = 0$ ***becomes*** $x^2 + 2x - 24 = 0$
>
> $(x+6)(x-4) = 0$***, which leads to*** $x = -6$ ***or*** 4.

- Once we have solved for x, how do we make sense of those numerical values? What do those values mean in the context of this problem? Why does only one value of x work in the context?
  - *A numerical solution only makes sense if it yields dimensions that are positive. If we try* $x = -6$ *as a solution and substitute it into the original expressions for each side of the rectangle, we arrive at dimensions of* -3 *and* -7.
    $(x+3)(x-1) = 21 \rightarrow (-6+3)(-6-1) = (-3)(-7) = 21.$
    However, having negative lengths is not viable for the rectangle. When we try the other solution, $x = 4$*, we arrive at dimensions of* 7 *and* 3.
    $(x+3)(x-1) = 21 \rightarrow (4+3)(4-1) = (7)(3) = 21.$
    Therefore, $x = 4$ *is the only solution that is useful in the context of the original problem.*

Discuss substituting values into the original factored form of the expression to check.

> **e. What are the actual dimensions of the rectangle?**
>
> ***The dimensions of the rectangle are*** $(x-1)$ ***by*** $(x+3)$***, so the dimensions would be*** $(4-1) = 3$ ft. ***by*** $(4+3) = 7$ ft.
>
> **f. A smaller rectangle can fit inside the first rectangle, and it has an area that can be expressed by the equation** $x^2 - 4x - 5$**. What are the dimensions of the smaller rectangle in terms of** x**?**
>
> ***Factoring the quadratic expression, we get*** $(x-5)(x+1)$.
>
> **g. What value for** x **would make the smaller rectangle have an area of** $\frac{1}{3}$ **that of the larger?**
>
> $\frac{1}{3}$ ***of the larger area is*** $\frac{1}{3}$ ***times*** 21***, which is*** 7 ***square units.*** $x^2 - 4x - 5 = 7$ ***is easier to solve if we subtract*** 7 ***from each side to get*** $x^2 - 4x - 12 = 0$. ***Factoring the left side gives us*** $(x-6)(x+2) = 0$. ***So,*** $x = 6$ ***or*** -2. ***Discuss which answer is correct based on this context.***

Emphasize the importance of striking or crossing out the rejected solution.

Exercises 5–8 (10 minutes)

The following exercises might be modeled by the teacher, used as guided practice, or assigned to be solved independently based on the needs of your students.

Exercises 5–8

Solve. Show your work.

5. $x^2 - 11x + 19 = -5$

$x^2 - 11x + 19 = -5$
$x^2 - 11x + 24 = 0$
$(x-3)(x-8) = 0$
$x = 3$ *or* 8

6. $7x^2 + x = 0$

$7x^2 + x = 0$
$x(7x+1) = 0$
$x = 0$ *or* $-\frac{1}{7}$

(This problem has two points to remember: All terms have a factor of 1*, and sometimes the solution is a fraction.)*

7. $7r^2 - 14r = -7$

$7r^2 - 14r = -7$
$7r^2 - 14r + 7 = 0$
$7(r^2 - 2r + 1) = 0$
$7(r-1)(r-1) = 0$ *or* $7(r-1)^2 = 0$
$r = 1$

(There is only one solution; rather, both solutions are 1 *in this case.)*

8. $2d^2 + 5d - 12 = 0$

$2d^2 + 5d - 12 = 0$

Two numbers for which the product is -24 *and the sum is* $+5$*:* -3 *and* $+8$.

So, we split the linear term: $2d^2 - 3d + 8d - 12 = 0$.

And group by pairs: $d(2d-3) + 4(2d-3) = 0$.

Then factor: $(2d-3)(d+4) = 0$.

So, $d = -4$ *or* $\frac{3}{2}$.

Closing (1 minute)

- The *zero product property* tells us that if a product equals zero, then at least one factor must be zero. Thus, the product of zero and any monomial, polynomial, or constant is always equal to zero.

Lesson Summary

Zero Product Property

If $ab = 0$, then $a = 0$ or $b = 0$ or $a = b = 0$.

When solving for the variable in a quadratic equation, rewrite the equation as a factored quadratic set equal to zero. Using the zero product property, you know that if one factor is equal to zero, then the product of all factors is equal to zero.

Going one step further, when you have set each binomial factor equal to zero and have solved for the variable, all of the possible solutions for the equation have been found. Given the context, some solutions may not be viable, so be sure to determine if each possible solution is appropriate for the problem.

Exit Ticket (6 minutes)

Name ____ Date____________________

Lesson 5: The Zero Product Property

Exit Ticket

1. Factor completely: $3d^2 + d - 10$.

2. Solve for d: $3d^2 + d - 10 = 0$.

3. In what ways are Problems 1 and 2 similar? In what ways are they different?

Exit Ticket Sample Solutions

1. Factor completely: $3d^2 + d - 10$.

   $(3d - 5)(d + 2)$

2. Solve for d: $3d^2 + d - 10 = 0$.

   $(3d - 5)(d + 2) = 0$, *so* $d = \frac{5}{3}$ *or* -2.

3. In what ways are Problems 1 and 2 similar? In what ways are they different?

   Both involve the same quadratic expression. The first is just an expression (no equal sign), so it cannot be solved, but there are two factors that are irreducible over the integers. The second is an equation in d and has two solutions that are related to those factors.

Problem Set Sample Solutions

Solve the following equations.

1. $x^2 + 15x + 40 = 4$

   $$x^2 + 15x + 36 = 0$$
   $$(x + 12)(x + 3) = 0$$
   $$x = -12 \text{ or } -3$$

2. $7x^2 + 2x = 0$

   $$x(7x + 2) = 0$$
   $$x = 0 \text{ or } -\frac{2}{7}$$

3. $b^2 + 5b - 35 = 3b$

   $$b^2 + 5b - 35 = 3b$$
   $$b^2 + 2b - 35 = 0$$
   $$(b + 7)(b - 5) = 0$$
   $$b = -7 \text{ or } 5$$

4. $6r^2 - 12r = -6$

   $$6r^2 - 12r + 6 = 0$$
   $$6(r^2 - 2r + 1) = 0$$
   $$6(r - 1)(r - 1) = 0$$
   $$r = 1$$