Lesson 27: Recursive Challenge Problem—The Double and Add 5 Game

Student Outcomes

- Students learn the meaning and notation of recursive sequences in a modeling setting.
- Students use recursive sequences to model and answer problems.
- Students create equations and inequalities to solve a modeling problem.
- Students represent constraints by equations and inequalities and interpret solutions as viable or non-viable options in a modeling context.

Lesson Notes

The *double and add 5* game is *loosely* related to the Collatz conjecture—an *unsolved* conjecture in mathematics named after Lothar Collatz, who first proposed the problem in 1937. The conjecture includes a recurrence relation, *triple and add 1,* as part of the problem statement. A worthwhile activity for you and your class is to read about the conjecture online.

Students begin by playing the *Double and Add* 5 game in a simple situation. Given a number, double it and add 5. The result of round two is the double of the result of round one, plus 5, and so on. The goal of the game is to find the smallest starting whole number, a_0, that produces a number 100 or greater in three rounds or fewer (answer: $a_0 = 9$). Students are then exposed to the more difficult challenge of finding the smallest starting whole number that produces a number $1{,}000$ or greater in three rounds or fewer (answer: $a_0 = 121$). To solve this problem, the notation of recursive sequences and recursive relations are explained, and students formalize the problem in terms of an equation, solve, interpret their answer, and validate.

Classwork

This challenging two-day modeling lesson (see page 61 of CCLS) about recursive sequences runs through the *problem, formulate, compute, interpret, validate, report* modeling cycle. This modeling activity involves playing a game and describing the mathematical process in the game using a recurrence relation in order to solve a more difficult version of the game. This part two lesson picks up where the last lesson left off—in this lesson students formulate, compute, interpret, validate, and report on their answers to the *Double and Add 5* game problem stated in the previous lesson.

Recall the statement of the problem from the last lesson for your students:

- Given a starting number, double it and add 5 to get the result of round one. Double the result of round one and add 5, and so on. The goal of the game is to find the smallest starting whole number that produces a result of $1{,}000$ or greater in three rounds or fewer.

Example 1 (10 minutes)

The repeat of this example from the previous lesson speaks to the value and importance of students doing this work. This time require students to work individually to complete the task. Visit students as needed and ask questions that lead students to the correct formulas.

> **Example 1**
>
> **Review Exercise 3 from the previous lesson: Using a generic initial value, a_0, and the recurrence relation, $a_{i+1} = 2a_i + 5$, for $i \geq 0$, find a formula for a_1, a_2, a_3, a_4 in terms of a_0.**
>
> $a_1 = 2a_0 + 5$,
> $a_2 = 2a_1 + 5 = 2\ \ 2a_0 + 5\ \ + 5 = 2^2 \cdot a_0 + 15$,
> $a_3 = 2a_2 + 5 = 2\ \ 2 \cdot 2a_0 + 15\ \ + 5 = 2^3 \cdot a_0 + 35$,
> $a_4 = 2a_3 + 5 = 2\ \ 2^3 \cdot a_0 + 35\ \ + 5 = 2^4 \cdot a_0 + 75$.

Mathematical Modeling Exercise/Exercise 1 (15 minutes)

MP.2 & MP.4

(Formulation step of the modeling cycle) Ask students: Using one of the four formulas from Example 1, write an inequality that, if solved for a_0, will lead to finding the smallest starting whole number for the *Double and Add* 5 game that produces a result of $1{,}000$ or greater in three rounds or fewer.

> **Exercise 1**
>
> **Using one of the four formulas from Example 1, write an inequality that, if solved for a_0, will lead to finding the smallest starting whole number for the *double and add* 5 game that produces a result of $1{,}000$ or greater in 3 rounds or fewer.**

This exercise is loaded with phrases that students will need to interpret correctly in order to formulate an equation (*do not* expect this to be easy for them). Start with simple questions and build up:

- What does a_2 mean in terms of rounds?
  - *The result of round two*
- Write what the statement, "produce a result of $1{,}000$ or greater in two rounds," means using a term of the sequence.
  - *The result of round two, a_2, must be greater than or equal to $1{,}000$. Ask students to write the equation, $a_2 \geq 1000$, for that statement.*
- After replacing a_2 in the inequality, $a_2 \geq 1000$, with the expression in terms of a_0, what do the numbers a_0 that satisfy the inequality, $4a_0 + 15 \geq 1000$, mean?
  - *The numbers a_0 that satisfy the inequality are the starting numbers for the Double and Add 5 game that produce a result of $1{,}000$ or greater in two rounds or fewer. The "or fewer" in the previous sentence is important and can be understood by thinking about the question, "Do we need two rounds to reach $1{,}000$, starting with number 999? 800? 500?"*

Let students solve for a_0 in $4a_0 + 15 \geq 1000$, and let them find the smallest whole number a_0 for exactly two rounds *(Answer: 247)*. Then continue with your questioning:

- What inequality in terms of a_0 would you write down to find the smallest starting number for the *Double and Add* 5 game that produces a result of $1{,}000$ or greater in three rounds or fewer?
  - $8a_0 + 35 \geq 1000$

Exercise 2 (10 minutes)

(Compute, interpret, validate steps of the modeling cycle) Tell students:

MP.2 & MP.3

Exercise 2

Solve the inequality derived in Exercise 1. Interpret your answer, and validate that it is the solution to the problem. That is, show that the whole number you found results in $1,000$ or greater in three rounds, but the previous whole number takes four rounds to reach $1,000$.

$$8a_0 + 35 \geq 1000$$
$$8a_0 + 35 - 35 \geq 1000 - 35$$
$$8a_0 \geq 965$$
$$\frac{1}{8}\ 8a_0 \ \geq \frac{1}{8}\ 965$$
$$a_0 \geq \frac{965}{8}$$

Students should write or say something similar to the following response: I interpret $a_0 \geq \frac{965}{8}$ or $a_0 \geq 120.625$ as the set of all starting numbers that reach $1,000$ or greater in three rounds or fewer. Therefore, the smallest starting whole number is 121. To validate, I checked that starting with 121 results in $1,003$ after three rounds, whereas 120 results in 995 after three rounds.

Exercise 3 (5 minutes)

(This exercise cycles through the modeling cycle again.) Ask students:

Exercise 3

Find the smallest starting whole number for the *Double and Add* 5 game that produces a result of $1,000,000$ or greater in four rounds or fewer.

$$16 \cdot a_0 + 75 \geq 1,000,000$$
$$16a_0 + 75 - 75 \geq 1,000,000 - 75$$
$$16a_0 \geq 999,925$$
$$\frac{1}{16}\ 16a_0 \ \geq \frac{1}{16}\ 999,925$$
$$a_0 \geq \frac{999,925}{16}$$

Students should write or say something similar to the following response: I interpreted $a_0 \geq \frac{999,925}{16}$ or $a_0 \geq 62,495.3125$ as the set of all starting numbers that reach $1,000,000$ or greater in four rounds or fewer. Therefore, the smallest starting whole number is $62,496$. To validate, I checked that starting with $62,496$ results in $1,000,011$ after four rounds, whereas $62,495$ results in $999,995$ after four rounds.

Lesson Summary

The formula, $a_n = 2^n(a_0 + 5) - 5$, describes the n^{th} term of the *double and add 5* game in terms of the starting number a_0 and n. Use this formula to find the smallest starting whole number for the *double and add 5* game that produces a result of $10,000,000$ or greater in 15 rounds or fewer.

Exit Ticket (5 minutes)

Use the Exit Ticket to have students report their findings (the report step of the modeling cycle).

Name ____ Date____________________

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Exit Ticket

Write a *brief* report about the answers you found to the *Double and Add 5* game problems. Include justifications for why your starting numbers are correct.

Exit Ticket Sample Responses

MP.3

Write a *brief* report about the answers you found to the *Double and Add 5* game problems. Include justifications for why your starting numbers are correct.

Results for finding the smallest starting number in the Double and Add 5 *game:*

1. *Reaching 100 in three rounds or fewer: The starting number 9 results in 107 in round three. The starting number 8 results in 99 in round three, requiring another round to reach 100. Numbers 1–8 take more than three rounds to reach 100.*

2. *Reaching $1,000$ in three rounds or fewer: The starting number 121 results in $1,003$ in round three. The starting number 120 results in 995 in round three, requiring another round to reach $1,000$. All other whole numbers less than 120 take more than three rounds to reach $1,000$.*

3. *Reaching $1,000,000$ in four rounds or fewer: The starting number $62,496$ results in $1,000,011$ in round four. The starting number $62,495$ results in $999,995$ in round four, requiring another round to reach $1,000,000$. All other whole numbers less than $62,495$ take more than four rounds to reach $1,000,000$.*

Problem Set Sample Solutions

1. Your older sibling came home from college for the weekend and showed you the following sequences (from her homework) that she claimed were generated from initial values and recurrence relations. For each sequence, find an initial value and recurrence relation that describes the sequence. (Your sister showed you an answer to the first problem.)

   a. $(0, 2, 4, 6, 8, 10, 12, 14, 16, \dots)$

      $a_1 = 0$ *and* $a_{i+1} = a_i + 2$ *for* $i \geq 1$

   b. $(1, 3, 5, 7, 9, 11, 13, 15, 17, \dots)$

      $a_1 = 1$ *and* $a_{i+1} = a_i + 2$ *for* $i \geq 1$

   c. $(14, 16, 18, 20, 22, 24, 26, \dots)$

      $a_1 = 14$ *and* $a_{i+1} = a_i + 2$ *for* $i \geq 1$

   d. $(14, 21, 28, 35, 42, 49, \dots)$

      $a_1 = 14$ *and* $a_{i+1} = a_i + 7$ *for* $i \geq 1$

   e. $(14, 7, 0, -7, -14, -21, -28, -35, \dots)$

      $a_1 = 14$ *and* $a_{i+1} = a_i - 7$ *for* $i \geq 1$

   f. $(2, 4, 8, 16, 32, 64, 128, \dots)$

      $a_1 = 2$ *and* $a_{i+1} = 2a_i$ *for* $i \geq 1$

g. $(3, 6, 12, 24, 48, 96, \ldots)$

$a_1 = 3$ and $a_{i+1} = 2a_i$ for $i \geq 1$

h. $(1, 3, 9, 27, 81, 243, \ldots)$

$a_1 = 1$ and $a_{i+1} = 3a_i$ for $i \geq 1$

i. $(9, 27, 81, 243, \ldots)$

$a_1 = 9$ and $a_{i+1} = 3a_i$ for $i \geq 1$

2. **Answer the following questions about the recursive sequence generated by initial value, $a_1 = 4$, and recurrence relation, $a_{i+1} = 4a_i$ for $i \geq 1$.**

a. **Find a formula for a_1, a_2, a_3, a_4, a_5 in terms of powers of 4.**

$a_1 = 4^1$

$a_2 = 4^2$

$a_3 = 4^3$

$a_4 = 4^4$

$a_5 = 4^5$

b. **Your friend, Carl, says that he can describe the n^{th} term of the sequence using the formula, $a_n = 4^n$. Is Carl correct? Write one or two sentences using the recurrence relation to explain why or why not.**

Yes. The recurrence relation, $a_{i+1} = 4a_i$ for $i \geq 0$, means that the next term in the sequence is always 4 times larger than the current term, i.e., one more power of 4. Therefore, the n^{th} term will be n powers of 4, or 4^n.

3. **The expression, $2^n(a_0 + 5) - 5$, describes the n^{th} term of the *double and add 5* game in terms of the starting number a_0 and n. Verify that it does describe the n^{th} term by filling out the tables below for parts (b) through (e). (The first table is done for you.)**

a. **Table for $a_0 = 1$**

n	$2^n(a_0 + 5) - 5$
1	$2^1 \cdot 6 - 5 = 7$
2	$2^2 \cdot 6 - 5 = 19$
3	$2^3 \cdot 6 - 5 = 43$
4	$2^4 \cdot 6 - 5 = 91$

b. **Table for $a_0 = 8$**

n	$2^n(a_0 + 5) - 5$
1	$2^1 \cdot 13 - 5 = 21$
2	$2^2 \cdot 13 - 5 = 47$
3	$2^3 \cdot 13 - 5 = 99$
4	$2^4 \cdot 13 - 5 = 203$

c. **Table for $a_0 = 9$**

n	$2^n(a_0 + 5) - 5$
2	$2^2 \cdot 14 - 5 = 51$
3	$2^3 \cdot 14 - 5 = 107$

d. Table for $a_0 = 120$

n	$2^n(a_0 + 5) - 5$
3	$2^3 \cdot 125 - 5 = 995$
4	$2^4 \cdot 125 - 5 = 1995$

e. Table for $a_0 = 121$

n	$2^n(a_0 + 5) - 5$
2	$2^2 \cdot 126 - 5 = 499$
3	$2^3 \cdot 126 - 5 = 1003$

4. **Bilbo Baggins stated to Samwise Gamgee, "Today, Sam, I will give you $\$1$. Every day thereafter for the next 14 days, I will take the previous day's amount, double it and add $\$5$, and give that new amount to you for that day."**

   a. **How much did Bilbo give Sam on day 15? (Hint: You don't have to compute each term.)**

   $a_{15} = 2^{15}\ \ 1 + 5\ \ - 5 = 196,603$. *Bilbo gave Sam* $\$196,603$ *on day* 15.

   b. **Did Bilbo give Sam more than $\$350,000$ altogether?**

   Yes. He gave $\$98,299$ *on day* 14, $\$49,147$, *on day* 13, $\$24,571$ *on day* 12, *and so on.*

5. **The formula, $a_n = 2^{n-1}(a_0 + 5) - 5$, describes the n^{th} term of the *Double and Add 5* game in terms of the starting number a_0 and n. Use this formula to find the smallest starting whole number for the *Double and Add 5* game that produces a result of $10,000,000$ or greater in 15 rounds or fewer.**

   Solving $2^{14}\ \ a_0 + 5\ \ - 5 \geq 10,000,000$ *for* a_0 *results in* $a_0 \geq 300.1759 \ldots$.

   Hence, 301 *is the smallest starting whole number that will reach* $10,000,000$ *in* 15 *rounds or fewer.*