Lesson 26: Recursive Challenge Problem—The Double and Add 5 Game

Student Outcomes

- Students learn the meaning and notation of recursive sequences in a modeling setting.
- Following the modeling cycle, students investigate the *double and add 5* game in a simple case in order to understand the statement of the main problem.

Lesson Notes

The *double and add 5* game is *loosely* related to the Collatz conjecture—an *unsolved* conjecture in mathematics named after Lothar Collatz, who first proposed the problem in 1937. The conjecture includes a recurrence relation, *triple and add 1,* as part of the problem statement. A worthwhile activity for you and your class is to read about the conjecture online.

Students begin by playing the *Double and Add 5* game in a simple situation. Given a number, double it and add 5. The result of round 2 is the double of the result of Round 1 plus 5, and so on. The goal of the game is to find the smallest starting whole number, a_0, that produces a number 100 or greater in three rounds or fewer (Answer: $a_0 = 9$). Students are then exposed to the more difficult challenge of finding the smallest starting whole number that produces a number $1{,}000$ or greater in three rounds or fewer. To solve this problem, the notation of recursive sequences and recursive relations are explained, and students formalize the problem in terms of an equation, solve, interpret their answer, and validate (answer: $a_0 = 121$).

Classwork

MP.4

This challenging two-day modeling lesson (see page 61 of CCLS) about recursive sequences runs through the *problem, formulate, compute, interpret, validate, report* modeling cycle. This modeling activity involves playing a game and describing the mathematical process in the game using a recurrence relation in order to solve a harder version of the game. Please read through both lessons before planning out your class time.

Example 1 (7 minutes)

This activity describes the process so students can be given the problem statement. Introduce it by stating that you want to create an interesting sequence by *doubling and adding 5.*

Work through the table below with your students on the board to explain the meaning of the following:

- starting number,
- double and add 5,
- result of round one,
- result of round two, and so on.

Here is what the table looks like at the beginning:

	Number	Double and add 5	
starting number→	1	$1 \cdot 2 + 5 = 7$	← result of round 1
	7	$7 \cdot 2 + 5 = 19$	← result of round 2

and here is the completed table:

Example 1

Fill in the *doubling and adding* 5 below:

	Number	Double and add 5	
starting number→	$\mathbf{1}$	$\mathbf{1 \cdot 2 + 5 = 7}$	**← result of round 1**
	$\mathbf{7}$	$\mathbf{7 \cdot 2 + 5 = 19}$	**← result of round 2**
	$\mathbf{19}$	$\mathbf{19 \cdot 2 + 5 = 43}$	**← result of round 3**
	$\mathbf{43}$	$\mathbf{43 \cdot 2 + 5 = 91}$	**← result of round 4**
	$\mathbf{91}$	$\mathbf{91 \cdot 2 + 5 = 187}$	**← result of round 5**

Exercise 1 (5 minutes)

Have students complete the tables in Exercise 1. Walk around the classroom to ensure they are completing the tables correctly and understand the process.

Exercise 1

Complete the tables below for the given starting number.

Number	Double and add 5
$\mathbf{2}$	$\mathbf{2 \cdot 2 + 5 = 9}$
$\mathbf{9}$	$\mathbf{9 \cdot 2 + 5 = 23}$
$\mathbf{23}$	$\mathbf{23 \cdot 2 + 5 = 51}$

Number	Double and add 5
$\mathbf{3}$	$\mathbf{3 \cdot 2 + 5 = 11}$
$\mathbf{11}$	$\mathbf{11 \cdot 2 + 5 = 27}$
$\mathbf{27}$	$\mathbf{27 \cdot 2 + 5 = 59}$

Mathematical Modeling Exercise/Exercise 2 (15 minutes)

MP.1 (Problem statement of the modeling cycle.) State the following (starter) problem to students, and let them wrestle with it until they find a solution (or at least a strategy for finding the solution):

> **Exercise 2**
>
> **Given a starting number, double it and add 5 to get the result of Round 1. Double the result of Round 1 and add 5, and so on. The goal of the game is to find the smallest starting whole number that produces a result of 100 or greater in three rounds or fewer.**

Walk around the class, observe student work, and give advice, such as:

- Does starting with 10 produce a result of 100 or greater in Round 3?
  - *Yes.*
- Why will all numbers greater than 10 work? Can you find a smaller starting number that also works?
- Do you see any patterns in the tables you have already created?
  - *As the starting number increases by 1, the result of Round 3 increases by 8.*
- Yes, 9 works. Is it the smallest?
  - *Yes. When I start with 8, I need four rounds to get past 100.*

After 8 minutes, show (or have a student show) that 9 is the correct answer by showing the tables:

Number	*Double and add 5*
$\mathbf{8}$	$\mathbf{8 \cdot 2 + 5 = 21}$
$\mathbf{21}$	$\mathbf{21 \cdot 2 + 5 = 47}$
$\mathbf{47}$	$\mathbf{47 \cdot 2 + 5 = 99}$***, no!***

Number	*Double and add 5*
$\mathbf{9}$	$\mathbf{9 \cdot 2 + 5 = 23}$
$\mathbf{23}$	$\mathbf{23 \cdot 2 + 5 = 51}$
$\mathbf{51}$	$\mathbf{51 \cdot 2 + 5 = 107}$***, yes!***

MP.1 Invite students to share other methods for finding the answer. For example, some may have *worked the problem backwards*: If 100 is reached in three rounds, then $\frac{100-5}{2} = 47.5$ must have been reached in two rounds, and $\frac{47.5-5}{2} = 21.25$ must been reached after the first round, which means the starting number is greater than $\frac{21.25-5}{2} = 8.125$, or the whole number 9.

Tell students that the next goal is to solve the same problem, but find the smallest number that results in $1{,}000$ in three rounds or fewer:

> Given a starting number, double it and add 5 to get the result of Round 1. Double the result of Round 1 and add 5, and so on. The goal of the game is to find the smallest starting whole number that produces a result of $1{,}000$ or greater in three rounds or fewer.

This problem is not as easy as the starter problem to solve by guess-and-check. To solve this problem, guide students to *formulate* an equation. But first, you will need to explain how mathematicians create and describe recursive sequences.

Let a_1 be the number of the result of Round 1. We can also label the result of Round 2 as a_2, and so on. Ask, "How could we label the starting number?" Guide them to label the starting number as a_0. Then write an equation in terms of a_0 and a_1 in the table (that is still on the board) like this:

Number	Double and add 5	Equation
$a_0 = 5$	$5 \cdot 2 + 5 = 15$	$a_0 \cdot 2 + 5 = a_1$
$a_1 = 15$	$15 \cdot 2 + 5 = 35$	
$a_2 = 35$	$35 \cdot 2 + 5 = 75$	

Ask students to help you complete and extend the table as follows:

Number	Double and add 5	Equation
$a_0 = 5$	$5 \cdot 2 + 5 = 15$	$a_0 \cdot 2 + 5 = a_1$
$a_1 = 15$	$15 \cdot 2 + 5 = 35$	$a_1 \cdot 2 + 5 = a_2$
$a_2 = 35$	$35 \cdot 2 + 5 = 75$	$a_2 \cdot 2 + 5 = a_3$
$a_3 = 75$		
a_i		$a_i \cdot 2 + 5 = a_{i+1}$
a_{i+1}		

Highlight on the board that the ordered list of *terms* $5, 15, 35, 75, \ldots$ can be described by an *initial value,* $a_0 = 5$, and a *recurrence relation*, $a_{i+1} = 2a_i + 5$, for $i \geq 0$. Written as follows:

$$a_0 = 5$$
$$a_{i+1} = 2a_i + 5, i \geq 0$$

Tell them that this is an example of a *recursively-defined sequence,* or simply, a *recursive sequence.*

- Have students mentally use the recurrence relation to find the next term after 75. Is it the *double and add* 5 *rule?*

Ask:

- What other terms have we studied so far that are defined recursively?
  - *Algebraic expressions, polynomial expressions, monomials*

Teacher note: Terms that are defined recursively often use the term itself in the statement of the definition, but the definition of the term is not considered *circular*. Circularity does not arise in recursively defined terms because they always start with a well-defined set of *base* examples, and then the definition describes how to generate new examples of the term from those base examples, which, by reiterating further, can then be used to generate all other examples of the term. The base examples prevent the definition from being circular. For recursive sequences, the base example(s) is just the initial value(s). For algebraic expressions, the well-defined base examples are numerical symbols and variable symbols.

Exercise 3 (10 minutes)

Ask students:

Exercise 3

Using a generic initial value, a_0, and the recurrence relation, $a_{i+1} = 2a_i + 5$, for $i \geq 0$, find a formula for a_1, a_2, a_3, a_4 in terms of a_0.

Let students work individually or in pairs. Visit each group and ask questions that lead students to the following:

$a_1 = 2a_0 + 5,$

$a_2 = 2a_1 + 5 = 2\ \ 2a_0 + 5\ \ + 5 = 4a_0 + 15,$

$a_3 = 2a_2 + 5 = 2\ \ 2 \cdot 2a_0 + 15\ \ + 5 = 8a_0 + 35,$

$a_4 = 2a_3 + 5 = 2\ \ 2^3 \cdot a_0 + 35\ \ + 5 = 16a_0 + 75.$

Closing (5 minutes)

Discuss the following definitions in the student materials:

Vocabulary

Sequence: A *sequence* can be thought of as an ordered list of elements. The elements of the list are called the *terms of the sequence.*

For example, (P, O, O, L) is a sequence that is different than (L, O, O, P). Usually the terms are *indexed* (and therefore ordered) by a subscript starting at either 0 or 1: $a_1, a_2, a_3, a_4, \ldots$. The "…" symbol indicates that the pattern described is regular, that is, the next term is a_5, and the next is a_6, and so on. In the first example, $a_1 = P$ is the first term, $a_2 = O$ is the second term, and so on. Both finite and infinite sequences exist everywhere in mathematics. For example, the infinite decimal expansion of $\frac{1}{3} = 0.333333333\ldots$ can be represented as the sequence, $(0.3, 0.33, 0.333, 0.3333, \ldots)$.

Recursive Sequence: An example of a *recursive sequence* is a sequence that is defined by (1) specifying the values of one or more initial terms and (2) having the property that the remaining terms satisfy a recurrence relation that describes the value of a term based upon an algebraic expression in numbers, previous terms, or the index of the term.

The sequence generated by initial term, $a_1 = 3$, and recurrence relation, $a_n = 3a_{n-1}$, is the sequence $(3, 9, 27, 81, 243, \ldots)$. Another example, given by the initial terms, $a_0 = 1$, $a_1 = 1$, and recurrence relation, $a_n = a_{n-1} + a_{n-2}$, generates the famed *Fibonacci sequence* $(1, 1, 2, 3, 5, \ldots)$.

Exit Ticket (3 minutes)

Name ____ Date____________________

Lesson 26: Recursive Challenge Problem—The Double and Add 5 Game

Exit Ticket

The following sequence was generated by an initial value a_0 and recurrence relation $a_{i+1} = 2a_i + 5$, for $i \geq 0$.

1. Fill in the blanks in the sequence:

   $(\_\_\_\_\_\_\_\_,\ 29,\ \_\_\_\_\_\_\_\_,\ \_\_\_\_\_\_\_\_,\ \_\_\_\_\_\_\_\_,\ 539,\ 1083).$

2. In the sequence above, what is a_0? What is a_5?

Exit Ticket Sample Solutions

The following sequence was generated by an initial value a_0 and recurrence relation $a_{i+1} = 2a_i + 5$, for $i \geq 0$.

1. Fill in the blanks in the sequence:

   (__12__, 29, __63__, __131__, __267__, 539, 1083)

2. In the sequence above, what is a_0? What is a_5?

   $a_0 = 12$, $a_5 = 539$

Problem Set Sample Solutions

1. Write down the first 5 terms of the recursive sequences defined by the initial values and recurrence relations below:

   a. $a_0 = 0$ and $a_{i+1} = a_i + 1$, for $i \geq 0$,

      $0, 1, 2, 3, 4$

   b. $a_1 = 1$ and $a_{i+1} = a_i + 1$, for $i \geq 1$,

      $1, 2, 3, 4, 5$

   c. $a_1 = 2$ and $a_{i+1} = a_i + 2$, for $i \geq 1$,

      $2, 4, 6, 8, 10$

   d. $a_1 = 3$ and $a_{i+1} = a_i + 3$, for $i \geq 1$,

      $3, 6, 9, 12, 15$

   e. $a_1 = 2$ and $a_{i+1} = 2a_i$, for $i \geq 1$,

      $2, 4, 8, 16, 32$

   f. $a_1 = 3$ and $a_{i+1} = 3a_i$, for $i \geq 1$,

      $3, 9, 27, 81, 243$

   g. $a_1 = 4$ and $a_{i+1} = 4a_i$, for $i \geq 1$,

      $4, 16, 64, 256, 1024$

   h. $a_1 = 1$ and $a_{i+1} = (-1)a_i$, for $i \geq 1$,

      $1, -1, 1, -1, 1$

   i. $a_1 = 64$ and $a_{i+1} = -\frac{1}{2} a_i$, for $i \geq 1$,

      $64, -32, 16, -8, 4$

2. Look at the sequences you created in Problems 1(b) through 1(d). How would you define a recursive sequence that generates multiples of 31?

   $a_1 = 31$ *and* $a_{i+1} = a_i + 31$, *for* $i \geq 1$

3. Look at the sequences you created in problems 1(e) through 1(g). How would you define a recursive sequence that generates powers of 15?

   $a_1 = 15$ *and* $a_{i+1} = 15a_i$, *for* $i \geq 1$

4. The following recursive sequence was generated starting with an initial value of a_0, and the recurrence relation $a_{i+1} = 3a_i + 1$, for $i \geq 0$. Fill in the blanks of the sequence

   (__10__, __31__, 94, __283__, 850, __2551__)

5. For the recursive sequence generated by initial value, a_0, and recurrence relation, $a_{i+1} = a_i + 2$, for $i \geq 0$, find a formula for a_1, a_2, a_3, a_4 in terms of a_0. Describe in words what this sequence is generating.

   $a_1 = a_0 + 2$,
   $a_2 = a_0 + 4$,
   $a_3 = a_0 + 6$,
   $a_4 = a_0 + 8$

   It finds the next consecutive even or odd numbers after a_0, depending on whether a_0 is even or odd.

6. For the recursive sequence generated by initial value, a_0, and recurrence relation, $a_{i+1} = 3a_i + 1$, for $i \geq 0$, find a formula for a_1, a_2, a_3, a_4 in terms of a_0.

   $a_1 = 3 \cdot a_0 + 1$,
   $a_2 = 9a_0 + 4$,
   $a_3 = 27a_0 + 13$,
   $a_4 = 81a_0 + 40$