Lesson 26: Recursive Challenge Problem—The Double and Add 5 Game

The *double and add 5* game is *loosely* related to the Collatz conjecture—an *unsolved* conjecture in mathematics named after Lothar Collatz, who first proposed the problem in 1937. The conjecture includes a recurrence relation, *triple and add 1,* as part of the problem statement. A worthwhile activity for you and your class is to read about the conjecture online.

Classwork

**Example 1**

Fill in the *doubling and adding* $5$ below:

|  |  |
| --- | --- |
| Number | Double and add 5 |
| $$1$$ | $$1∙2+5=7$$ |
| $$7$$ | \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ |
| \_\_\_\_\_\_ | \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ |
| \_\_\_\_\_\_ | \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ |
| \_\_\_\_\_\_ | \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ |

Exercise 1

Complete the tables below for the given starting number.

|  |  |
| --- | --- |
| Number | Double and add 5 |
| $$2$$ | \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ |
| \_\_\_\_\_\_ | \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ |
| \_\_\_\_\_\_ | \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ |

|  |  |
| --- | --- |
| Number | Double and add 5 |
| $$3$$ | \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ |
| \_\_\_\_\_\_ | \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ |
| \_\_\_\_\_\_ | \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ |

Exercise 2

Given a starting number, double it and add 5 to get the result of round 1. Double the result of Round 1 and add 5, and so on. The goal of the game is to find the smallest starting whole number that produces a result of 100 or greater in three rounds or fewer.

Exercise 3

Using a generic initial value, $a\_{0}$, and the recurrence relation, $a\_{i+1}=2a\_{i}+5$, for $i\geq 0$, find a formula for $a\_{1}, a\_{2}, a\_{3}, a\_{4}$ in terms of $a\_{0}$.

**Vocabulary**

**Sequence:** A *sequence* can be thought of as an ordered list of elements. The elements of the list are called the *terms of the sequence.*

For example, (P, O, O, L) is a sequence that is different than (L, O, O, P). Usually the terms are *indexed* (and therefore ordered) by a subscript starting at either $0$ or $1$: $a\_{1}$,$ a\_{2}$, $a\_{3}$, $a\_{4}$, …. The “…” symbol indicates that the pattern described is regular, that is, the next term is $a\_{5}$, and the next is $a\_{6}$, and so on. In the first example, $a\_{1}=P$ is the first term, $a\_{2}=O$ is the second term, and so on. Both finite and infinite sequences exist everywhere in mathematics. For example, the infinite decimal expansion of $\frac{1}{3}=0.333333333…$ can be represented as the sequence, ($0.3$, $0.33$, $0.333$, $0.3333$, …).

**Recursive Sequence:** An example of a *recursive sequence* is a sequence that is defined by (1) specifying the values of one or more initial terms and (2) having the property that the remaining terms satisfy a recurrence relation that describes the value of a term based upon an algebraic expression in numbers, previous terms, or the index of the term.

The sequence generated by initial term, $a\_{1}=3$, and recurrence relation, $a\_{n}=3a\_{n-1}$, is the sequence ($3$,$ 9$,$ 27$,$ 81$,$ 243$, …). Another example, given by the initial terms, $a\_{0}=1$, $a\_{1}=1,$ and recurrence relation, $a\_{n}=a\_{n-1}+a\_{n-2}$, generates the famed *Fibonacci sequence* ($1$, $1$,$ 2$,$ 3$, $5$, …).

Problem Set

1. Write down the first 5 terms of the recursive sequences defined by the initial values and recurrence relations below:
	1. $a\_{0}=0$ and $a\_{i+1}=a\_{i}+1$, for $i\geq 0$,
	2. $a\_{1}=1$ and $a\_{i+1}=a\_{i}+1$, for $i\geq 1$,
	3. $a\_{1}=2$ and $a\_{i+1}=a\_{i}+2$, for $i\geq 1$,
	4. $a\_{1}=3$ and $a\_{i+1}=a\_{i}+3$, for $i\geq 1$,
	5. $a\_{1}=2$ and $a\_{i+1}=2a\_{i}$, for $i\geq 1$,
	6. $a\_{1}=3$ and $a\_{i+1}=3a\_{i}$, for $i\geq 1$,
	7. $a\_{1}=4$ and $a\_{i+1}=4a\_{i}$, for $i\geq 1$,
	8. $a\_{1}=1$ and $a\_{i+1}=(-1)a\_{i}$, for $i\geq 1$,
	9. $a\_{1}=64$ and $a\_{i+1}=(-\frac{1}{2})a\_{i}$, for $i\geq 1$,
2. Look at the sequences you created in Problems 1(b) through 1(d). How would you define a recursive sequence that generates multiples of $31$?
3. Look at the sequences you created in problems 1(e) through 1(g). How would you define a recursive sequence that generates powers of $15$?
4. The following recursive sequence was generated starting with an initial value of $a\_{0}$, and the recurrence relation $a\_{i+1}=3a\_{i}+1$, for$ i\geq 0$. Fill in the blanks of the sequence

$(\\_\\_\\_\\_\\_\\_\\_, \\_\\_\\_\\_\\_\\_\\_, 94, \\_\\_\\_\\_\\_\\_\\_, 850, \\_\\_\\_\\_\\_\\_)$.

1. For the recursive sequence generated by initial value, $a\_{0}$, and recurrence relation, $a\_{i+1}=a\_{i}+2$, for $i\geq 0$, find a formula for $a\_{1}$,$ a\_{2}$,$ a\_{3}$,$ a\_{4}$ in terms of $a\_{0}$. Describe in words what this sequence is generating.
2. For the recursive sequence generated by initial value, $a\_{0}$, and recurrence relation, $a\_{i+1}=3a\_{i}+1$, for $i\geq 0$, find a formula for $a\_{1}$,$ a\_{2}$,$ a\_{3}$,$ a\_{4}$ in terms of $a\_{0}$.