Lesson 23: Solution Sets to Simultaneous Equations

Student Outcomes

- Students create systems of equations that have the same solution set as a given system.
- Students understand that adding a multiple of one equation to another creates a new system of two linear equations with the same solution set as the original system. This property provides a justification for a method to solve a system of two linear equations algebraically.

Lesson Notes

Students explore standard **A.REI.C.5** in great detail. They have already developed proficiency with solving a system of two linear equations. This lesson delves into why the *elimination method* works and further enhances student understanding of equivalence.

Classwork

Opening Exercise (3 minutes)

This should go very quickly. Expect students to substitute 3 for x and 4 for y into both equations. If students struggle with this piece, you may need to reinforce what it means when an ordered pair is a solution to simultaneous equations and continue to reinforce that notion throughout the lesson.

Opening Exercise

Here is a system of two linear equations. Verify that the solution to this system is $(3, 4)$.

Equation A1: $y = x + 1$

Equation A2: $y = -2x + 10$

Substitute 3 for x and 4 for y into both equations.

$4 = 3 + 1$ ***is a true equation.***

$4 = -2(3) + 10$ ***is a true equation.***

Exploratory Challenge (23 minutes)

Students should work in groups to complete parts (a)–(e) for about 5–7 minutes. Have one or more groups share their solutions with the class. These first questions get students thinking about ways to create a new system of equations with the same solution set as the Opening Exercise. Expect a variety of responses from groups as they create their new systems.

Scaffolding:

If groups are struggling to get started, remind them that $(3, 4)$ must be a solution to their new equations. Then ask them to consider how to make a line that includes that point. Encourage them to use the grid or work with graph paper.

Exploratory Challenge

a. **Write down another system of two linear equations whose solution is $(3, 4)$. This time make sure both linear equations have a positive slope.**

Equation B1: $y = x + 1$

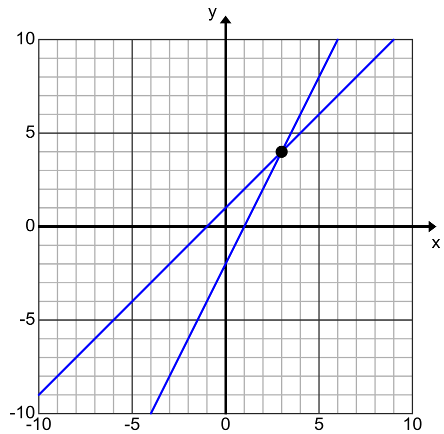
Equation B2: $y = 2\ \ x - 3\ \ + 4$

b. **Verify that the solution to this system of two linear equations is $(3, 4)$.**

$4 = 3 + 1$ *and* $4 = 2(3 - 3) + 4$

$4 = 4$ $\quad$ $4 = 4$

c. **Graph equation B1 and B2.**



d. **Are either B1 or B2 equivalent to the original A1 or A2? Explain your reasoning.**

Yes, I used the same B1. B2 is a different equation because it has a different slope. The only thing all the equations have in common is the point $(3, 4)$.

e. **Add A1 and A2 to create a new equation C1. Then, multiply A1 by 3 to create a new equation C2. Why is the solution to this system also $(3, 4)$? Explain your reasoning.**

Equation C1: $2y = -x + 11$

Equation C2: $3y = 3x + 3$

If you substitute 3 for x and 4 for y, both equations are true, so $(3, 4)$ is a solution. When A1 was multiplied by 3, it did not create a new equation. Both equations had $(3, 4)$ as a solution; therefore, when we add the equations, $(3, 4)$ will still be a solution because of the addition property of equality.

Hold a class discussion before moving on to parts (f)–(i). When you debrief and discuss, be sure to highlight the different approaches. Begin to distinguish between solutions where students created a new system by simply multiplying one equation or both by a constant factor and those who create two new equations that both contain the point $(3, 4)$.

- What different approaches did groups use to solve this problem?
  - *I used guess and check.*
  - *I started with the point $(3, 4)$ and realized I could pick any slope I wanted. I moved left 3 and down 4 until I got to the $(0, 0)$; therefore, my equation would be $y = \frac{4}{3}x$.*
- When you multiplied one equation by a constant, did it actually create a different linear equation? When you added two equations together, did it actually create a different equation?
  - *Multiplying by a constant doesn't create a different equation because the slope and y-intercept are the same. Adding two equations together does create a new equation because the slope is different.*

Move on to parts (f)–(i). These questions specifically direct students to consider creating a new system by multiplying one equation by a constant and adding it to another. Students are considering whether or not this is a valid way to generate a system with the same solution. Have each group record their answer to part (i) on the board to show that this method works regardless of the number by which you multiply.

The following system of equations was obtained from the original system by adding a multiple of equation A2 to equation A1.

Equation D1: $y = x + 1$

Equation D2: $3y = -3x + 21$

f. What multiple of A2 was added to A1 to create D2?

A2 was multiplied by 2 and then added to A1.

g. What is the solution to the system of two linear equations formed by D1 and D2?

The solution is still $(3, 4)$. I checked by substituting $(3, 4)$ into both equations.

h. Is D2 equivalent to the original A1 or A2? Explain your reasoning.

No, the slope of D2 is -1. Neither of the original equations had that slope.

i. Start with equation A1. Multiply it by a number of your choice and add the result to equation A2. This creates a new equation E2. Record E2 below to check if the solution is $(3, 4)$.

Equation E1: $y = x + 1$

Equation E2: $5y = 2x + 14$

I multiplied A1 by 4 to get $4y = 4x + 4$. Adding it to A2 gives $5y = 2x + 14$. We already know $(3, 4)$ is a solution to $y = x + 1$. Substituting into E2 gives $5\ \ 4\ \ = 2\ \ 3\ \ + 14$, which is a true equation. Therefore, $(3, 4)$ is a solution to this new system.

Wrap up the discussion by emphasizing the following:

- Will this method of creating a new system work every time? Why does it work?
  - *This method will always work because multiplying by a constant is a property of equality that keeps the point of intersection (i.e., the solution set) the same.*
- Who said you can add two equations like that? Left-hand side to left-hand side; right-hand side to right-hand side? How do you know that the solutions are not changed by that move?

Example 1 (4 minutes): Why Does the Elimination Method Work?

Students will see how **A.REI.C.5** provides a justification for solving a system by elimination. Choose a multiple that will eliminate a variable when the two equations are added together. Be sure to emphasize that this process generates a new system of two linear equations where one of the two equations contains only a single variable and is thus easy to solve.

Example 1: Why Does the Elimination Method Work?

Solve this system of linear equations algebraically.

ORIGINAL SYSTEM	NEW SYSTEM	SOLUTION
$2x + y = 6$		
$x - 3y = -11$		

ORIGINAL SYSTEM	*NEW SYSTEM*		*SOLUTION*
$2x + y = 6$	$6x + 3y = 18$		$x = 1$
$x - 3y = -11$	$x - 3y = -11$	$7x = 7$	$2(1) + y = 6$ *so* $y = 4$

Multiply the first equation by 3 *and add it to the second. Solve the new system.* $(1, 4)$

- Why did I multiply by the number 3?
  - *Multiplying by* 3 *allows one to generate* $3y$ *to eliminate the* $-3y$ *in the other equation when both equations are added together; it leads to an equation in* x *only. Selecting this number strategically created a new system where one equation had only a single variable.*
- Could I have selected a different number and created a system that was easy to solve?
  - *Yes, you could have multiplied the second equation by* -2 *to create a system that eliminated* x.

Exercises 1–2 (8 minutes)

Both of these exercises mimic the example. Students should be able to work quickly through them since they learned to solve systems by elimination in Grade 8.

Exercises 1–2

1. Explain a way to create a new system of equations with the same solution as the original that eliminates variable y from one equation. Then determine the solution.

ORIGINAL SYSTEM	NEW SYSTEM	SOLUTION
$2x + 3y = 7$		
$x - y = 1$		

Multiply the second equation by 3*, and add it to the first one.*

ORIGINAL SYSTEM	*NEW SYSTEM*	*SOLUTION*
$2x + 3y = 7$	$2x + 3y = 7$	$x = 2$
$x - y = 1$	$\underline{+(3x - 3y = 3)}$	$2(2) + 3y = 7$, *so* $y = 1$
	$5x = 10$	$(2, 1)$

2. **Explain a way to create a new system of equations with the same solution as the original that eliminates variable x from one equation. Then determine the solution.**

ORIGINAL SYSTEM	NEW SYSTEM	SOLUTION
$2x + 3y = 7$		
$x - y = 1$		

Multiply the second equation by -2, and add it to the first one.

ORIGINAL SYSTEM	*NEW SYSTEM*	*SOLUTION*
$2x + 3y = 7$	$2x + 3y = 7$	$y = 1$
$x - y = 1$	$\underline{+(-2x + 2y = -2)}$	$2x + 3(1) = 7$, *so* $x = 2$
	$5y = 5$	$(2, 1)$

Closing (2 minutes)

Close with a reminder that this lesson was about proving that a technique to solve a system of equations is valid.

- There are many ways to generate systems of equations that have the same solution set, but the technique explored in Exercises 1 and 2 is especially helpful if you are trying to solve a system algebraically.

Exit Ticket (5 minutes)

Name ____ Date____________________

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Exit Ticket

The sum of two numbers is 10 and the difference is 6. What are the numbers?

1. Create a system of two linear equations to represent this problem.

2. What is the solution to the system?

3. Create a new system of two linear equations using the methods described in part (i) of the Exploratory Challenge. Verify that the new system has the same solution.

Exit Ticket Sample Solutions

The sum of two numbers is 10 and the difference is 6. What are the numbers?

1. Create a system of two linear equations to represent this problem.

   $x + y = 10$ *and* $x - y = 6$

2. What is the solution to the system?

   $x = 8$ *and* $y = 2$

3. Create a new system of two linear equations using the methods described in part (i) of the Exploratory Challenge. Verify that the new system has the same solution.

   $4(x + y = 10) \rightarrow$

   $$\begin{array}{r} 4x + 4y = 40 \\ + \quad (x - y = 6) \\ \hline 5x + 3y = 46 \end{array}$$

   Solution to $x + y = 10$ *and* $5x + 3y = 46$ *is still* $(8, 2)$

   $8 + 2 = 10$ $\quad$ $5(8) + 3(2) = 46$

Problem Set Sample Solutions

Try to answer the following without solving for x and y first:

1. If $3x + 2y = 6$ and $x + y = 4$, then

   a. $2x + y = ?$ $\quad$ b. $4x + 3y = ?$

   a. $2x + y = 2$ $\quad$ *b.* $4x + 3y = 10$

   Answers in parts (a) and (b) are obtained by adding and subtracting the two original equations WITHOUT actually solving for x and y first.

   The solution $(-2, 6)$ satisfies all four equations.

2. You always get the same solution no matter which two of the four equations you choose from Problem 1 to form a system of two linear equations. Explain why this is true.

   The reason is that the 3rd equation is the difference of the 1st and the 2nd; the 4th equation is the sum of the 1st and the 2nd. When we add (or subtract) two equations to create a new equation, no new (or independent) information is created. The 3rd and 4th equations are thus not independent of the 1st and the 2nd. They still contain the solution common to their parent equations, the 1st and 2nd.

3. Solve the system of equations $\begin{array}{c} y = \frac{1}{4}x \\ y = -x + 5 \end{array}$ by graphing. Then, create a new system of equations that has the same solution. Show either algebraically or graphically that the systems have the same solution.

   Solution is $(4, 1)$.

   One example of a second system: $\begin{array}{c} y = \frac{2}{3}x - \frac{5}{3} \\ y = 4x - 15 \end{array}$

4. **Without solving the systems, explain why the following systems must have the same solution.**

   System (i): $4x - 5y = 13$ **System (ii):** $8x - 10y = 26$

   $3x + 6y = 11$ $x - 11y = 2$

   The first equation in system (ii) is created by multiplying the first equation in system (i) by 2. The second equation in system (ii) is creating by subtracting the two equations from system (i). Neither of these actions will change the solution to the system. Multiplying and adding equations are properties of equality that keep the point(s) of intersection (which is the solution set) the same.

Solve each system of equations by writing a new system that eliminates one of the variables.

5. $2x + y = 25$
   $4x + 3y = 9$

   $-4x - 2y = -50$
   $4x + 3y = 9$

   $y = -41$

   $(33, -41)$

6. $3x + 2y = 4$
   $4x + 7y = 1$

   $12x + 8y = 16$
   $-12x - 21y = -3$

   $y = -1$

   $(2, -1)$