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Lesson 11: The Decimal Expansion of Some Irrational Numbers

Student Outcomes

* Students use rational approximation to get the approximate decimal expansion of numbers like and
* Students distinguish between rational and irrational numbers based on decimal expansions.

Lesson Notes

The definition of an irrational number can finally be given and understood completely once students know that the decimal expansion of non-perfect squares like and are infinite and do not repeat. That is, square roots of non-perfect squares cannot be expressed as rational numbers and are therefore defined as irrational numbers.

Classwork

Opening Exercise (5 minutes)

Opening Exercise

Place on a number line. What decimal do **you think is equal to?** Explain your reasoning.

Lead a discussion where students share their reasoning as to the placement of on the number line. Encourage students to critique the reasoning of others while evaluating their own arguments. Consider having students vote on the placement they think is most correct.

MP.3

**Discussion (10 minutes)**

* We have studied the properties of rational numbers; today we will finally be able to characterize those numbers that are not rational.
* So far we have been able to estimate the size of a number like by stating that it is between the two perfect squares and , meaning that is between and but closer to . In our work so far we have found the decimal expansion of numbers using long division and by inspecting the denominators for products of ’s and ’s. Numbers written with a square root symbol are different and require a different method for determining their decimal expansions. The method we will learn is called rational approximation: using a sequence of rational numbers to get closer and closer to a given number to estimate the value of a number.

Example 1

Example 1

Recall the Basic Inequality:

Let andbe two positive numbers, and let be a fixed positive integer. Then if and only if .

Write the decimal expansion of .

First approximation:

* We will use the Basic Inequality that we learned in Lesson 3:

Let andbe two positive numbers, and let be a fixed positive integer. Then if and only if .

* To write the decimal expansion of we first determine between which two integers the number would lie on the number line. This is our first approximation. What are those integers?
  + *The number will be between and on the number line because and .*
* With respect to the Basic Inequality, we can verify that lies between the integers and because .
* To be more precise with our estimate of , we now look at the tenths between the numbers and . This is our second approximation.

Second approximation:



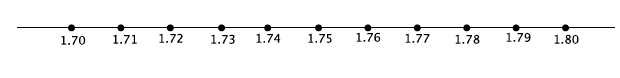
* The question becomes, where exactly would lie on this magnified version of the number line? There are 10 possibilities: , , , , or Use of the Basic Inequality can guide us to selecting the correct possibility. Specifically, we need to determine which of the inequalities shown below is correct:

, , , or.

With the help of a calculator we can see that because and ; therefore, .

* What do you think will need to be done to get an even more precise estimate of the number ?
  + *We will need to look at the interval between and more closely and repeat the process we did before.*
* Looking at the increments between and , we again have possibilities. This is our third approximation.

Third approximation:



* The Basic Inequality and the help of a calculator show that will be between and because .

Have students verify using a calculator that and and ultimately that .

* What do you think will need to be done to get an even more precise estimate of the number ?
  + *We will need to look at the interval between and more closely and repeat the process we did before.*
* At this point the pattern should be clear. Now to look more carefully at what we are actually doing. We began by looking at the sequence of integers, specifically between two positive integers and . Think of this interval as (because it equals ). Then we looked at the sequence of tenths between and ; think of this interval as (because it equals ). Then we looked at the sequence of hundredths between and ; think of this interval as (because it equals ). To determine the location of , we had to look between points that differ by for any positive integer . The intervals we investigate, i.e., , get increasingly smaller as gets larger.

*Scaffolding:*

Identifying the various forms of approximate, approximately, and approximation, may be useful to English Language Learners.

* This method of looking at successive intervals is what we call rational approximation. With each new interval we are approximating the value of the number by determining which two rational numbers it lies between.

Discussion (15 minutes)

The following discussion revisits the Opening Exercise. Before you begin, ask students to reevaluate their own reasoning, and if you had them vote, consider asking them to vote again to see if anyone wants to change their mind about the best approximation for

Example 2

Example 2

Write the decimal expansion of .

First approximation:

* We will use the method of rational approximation to estimate the location of on the number line.
* What interval of integers, i.e., an interval equal to do we examine first? Explain.
  + *We must examine the interval between and because , i.e.,*
* Now we examine the interval of tenths, i.e., , between and . Where might lie?

  
Second approximation:

* + *The number will lie between and or and or… and .*
* How do we determine which interval is correct?
  + *We must use the Basic Inequality to check each interval. For example, we need to see if the following inequality is true:*
* Before we begin checking each interval, let’s think about how we can be more methodical in our approach. We know that is between and , but which integer is it closer to?
  + *The number will be closer to than .*
* Then we should begin checking the intervals beginning with and work our way up. If the number were closer to , then we would begin checking the intervals on the right first and work our way down.

Provide students time to determine which interval the number will lie between. Ask students to share their findings, and then continue the discussion.

* Now that we know that the number lies between and , let’s check intervals of hundredths, i.e., .

Third approximation:



* Again, we should try to be methodical. Since and , where should we begin checking?
  + *We should begin with the interval between and because is closer to compared to .*

Provide students time to determine which interval the number will lie between. Ask students to share their findings, and then continue the discussion.

* Now we know that the number is between and . Let’s go one step further and examine intervals of thousandths, i.e., .

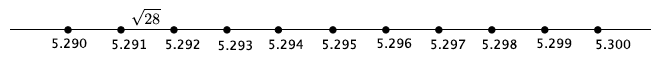
Fourth approximation:



* Since and , where should we begin our search?
  + *We should begin with the interval between and because is closer to compared to .*

Provide students time to determine which interval the number will lie between. Ask students to share their findings, and then finish the discussion.

* The number lies between and because and . At this point we have a fair approximation of the value of . It is between and on the number line:



* We could continue this process of rational approximation to see that . How is this number different from other infinite decimals we have worked with?

*Scaffolding:*

A graphic organizer may be useful. Consider the one below:

|  |  |
| --- | --- |
| Rational Numbers | Irrational Numbers |
| Definition:  Examples: | Definition:  Examples: |

* + *Other infinite decimals we have worked with have a block of digits that repeat at some point. This infinite decimal does not.*
* We know that rational numbers are those that have decimal expansions that eventually repeat. We also know that rational numbers can be expressed as a fraction in the form of a ratio of integers. In the last lesson we learned how to convert a repeating decimal to a fraction. Do you think that same process can be used with a number like ?
  + *No because the decimal expansion does not repeat.*
* Because the number cannot be expressed as a rational number, we say that it is *irrational*. Any number that cannot be expressed as a rational number is, by definition, an irrational number.
* A common irrational number is pi: . Notice that the decimal expansion of pi is infinite and does not repeat. Those qualities are what make pi an irrational number. Often for computations we give pi a rational approximation of or , but those are merely approximations, not the true value of the number pi.
* Another example of an irrational number is . What do you expect the decimal expansion of to look like?
  + *The decimal expansion of will be infinite without a repeating block.*
* The number . The decimal expansion is infinite and does not repeat.
* Is the number rational or irrational? Explain.
  + *The number . The decimal expansion of can be written as which is an infinite decimal expansion with a repeat block. Therefore, is a rational number.*
* Classify the following numbers as rational or irrational. Be prepared to explain your reasoning.

Provide students time to classify the numbers. They can do this independently or in pairs. Then select students to share their reasoning. Students should identify as irrational because it has a decimal expansion that can only be approximated by rational numbers. The number is a repeating decimal and can be expressed as a fraction and is therefore rational. The number and is therefore a rational number. The fraction by definition is a rational number because it is a ratio of integers.

Consider going back to the Opening Exercise to determine whose approximation was the closest.

Exercise 2 (5 minutes)

Students work in pairs to complete Exercise 2.

Exercise 2

Between which interval of hundredths would be located? Show your work.

The number is between integers and because . Then must be checked for the interval of tenths between and . Since is closer to , we will begin with the interval from to . The number is between and because and . Now we must look at the interval of hundredths between and . Since is closer to , we will begin with the interval to . The number is between and because and .

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

* We know that any number that cannot be expressed as a rational number is an irrational number.
* We know that to determine the approximate value of an irrational number we must determine between which two rational numbers it would lie.
* We know that the method of rational approximation uses a sequence of rational numbers, in increments of ,,, and so on, to get closer and closer to a given number.
* We have a method for determining the approximate decimal expansion of the square root of an imperfect square, which is an irrational number.

Lesson Summary

To get the decimal expansion of a square root of a non-perfect square you must use the method of rational approximation. Rational approximation is a method that uses a sequence of rational numbers to get closer and closer to a given number to estimate the value of the number. The method requires that you investigate the size of the number by examining its value for increasingly smaller powers of 10 (i.e., tenths, hundredths, thousandths, and so on). Since is not a perfect square, you would use rational approximation to determine its decimal expansion.

Example:

Begin by determining which two integers the number would lie.   
 is between the integers and because , which is equal to .

Next, determine which interval of tenths the number belongs.   
 is between and because , which is equal to .

Next, determine which interval of hundredths the number belongs.  
 is between and because , which is equal to .

A good estimate of the value of is because is closer to than it is to .

Notice that with each step we are getting closer and closer to the actual value, . This process can continue using intervals of thousandths, ten-thousandths, and so on.

Any number that cannot be expressed as a rational number is called an irrational number. Irrational numbers are those numbers with decimal expansions that are infinite and do not have a repeating block of digits.

Exit Ticket (5 minutes)

Name Date

Lesson 11: The Decimal Expansion of Some Irrational Numbers

Exit Ticket

1. Determine the decimal digit approximation of the number .
2. Classify the following numbers as rational or irrational, and explain how you know.

Exit Ticket Sample Solutions

1. Determine the decimal digit approximation of the number .

The number is between integers and because . Since is closer to , I will start checking the tenths intervals closer to . is between and since and . Checking the hundredths interval, is between and since and . Checking the thousandths interval, is between and since and   
. Since 17 is closer to than then the three decimal approximation is approximately .

1. Classify the following numbers as rational or irrational, and explain how you know.

The number , by definition, is rational because it is a ratio of integers. The number is rational because it has a repeat block. For that reason, the number can be expressed as a fraction. The number is irrational because it has a decimal expansion that can only be approximated by rational numbers. That is, the number is not equal to a rational number; therefore, it is irrational.

Problem Set Sample Solutions

1. Use the method of rational approximation to determine the decimal expansion of . Determine which interval of hundredths it would lie in.

The number is between and but closer to . Looking at the interval of tenths, beginning with to , the number lies between and because and but is closer to . In the interval of hundredths, the number lies between and because and   
.

1. Get a decimal digit approximation of the number .

The number is between and but closer to . Looking at the interval of tenths, beginning with to , the number lies between and because and and is closer to . In the interval of hundredths, the number lies between and because and   
 and is closer to . In the interval of thousandths, the number lies between and because and but is closer to . Since is closer to than then the decimal digit approximation of the number is approximately .

1. Write the decimal expansion of to at least decimal digits.

The number is between and but closer to because In the interval of tenths, the number is between and because and . In the interval of hundredths, the number is between and because and . Therefore, to decimal digits, the number is approximately but when rounded will be approximately because is closer to but not quite .

1. Write the decimal expansion of to at least decimal digits.

The number is between integers and because . Since is closer to , I will start checking the tenths intervals between and . is between and since and . Checking the hundredths interval, is between and since and . Since is closer to than , then the two decimal approximation is .

1. Explain how to improve the accuracy of decimal expansion of an irrational number.

In order to improve the accuracy of the decimal expansion of an irrational number, you must examine increasingly smaller increments on the number line. Specifically, increments of decreasing powers of . The Basic Inequality allows us to determine which interval a number will be between. We begin by determining which two integers the number lies between and then decrease the power of to look at the interval of tenths. Again using the Basic Inequality, we can narrow down the approximation to a specific interval of tenths. Then we look at the interval of hundredths and use the Basic Inequality to determine which interval of hundredths the number would lie between. Then we examine the interval of thousandths. Again the Basic Inequality allows us to narrow down the approximation to thousandths. The more intervals that are examined, the more accurate the decimal expansion of an irrational number will be.

1. Is the number rational or irrational? Explain.

The number is an irrational number because it has a decimal expansion that is infinite and does not repeat. That is, the number cannot be expressed as a rational number; therefore, it is irrational.

1. Is the number rational or irrational? Explain.

The number ; therefore, it is a rational number. Not only is the number a quotient of integers, but its decimal expansion is infinite with a repeating block of digits.

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The number is an irrational number because it has a decimal expansion that is infinite and does not repeat. That is, the number cannot be expressed as a rational number; therefore, it is irrational.

1. Challenge: Get a decimal digit approximation of the number .

The number is between integers and because . Since is closer to , I will start checking the tenths intervals between and . is between and since and . Checking the hundredths interval, is between and since and . Since is closer to than , the two decimal approximation is .