Lesson 2: Formal Definition of a Function

Student Outcomes

- Students know that a function assigns to each input exactly one output.
- Students know that some functions can be expressed by a formula or rule, and when an input is used with the formula, the outcome is the output.

Lesson Notes

A *function* is defined as a rule (or formula) that assigns to each input exactly one output. Functions can be represented in a table, as a rule, as a formula or an equation, as a graph, or as a verbal description. The word *function* will be used to describe a predictive relationship. That relationship is described with a rule or formula when possible. Students should also know that frequently the word *function* is used to mean the formula or equation representation, specifically. The work in this module will lay a critical foundation for students' understanding of functions. This is the first time function is defined for students. We ask students to consider range and domain informally. High school standards **F-IF.A.1** and **F-IF.B.5** address these along with function notation.

This lesson continues the work of Example 2 from Lesson 1 leading to a formal definition of a function. Consider asking students to recap what they learned about functions from Lesson 1. The purpose would be to abstract the information in Example 2—specifically, that in order to show all possible time intervals for the stone dropping, we had to write the inequality for time t as $0 \leq t \leq 4$.

Classwork

Opening (3 minutes)

- Shown below is the table from Example 2 of the last lesson and another table of values. Make a conjecture about the differences between the two tables. What do you notice?

Number of seconds (x)	Distance traveled in feet (y)
0.5	4
1	16
1.5	36
2	64
2.5	100
3	144
3.5	196
4	256

Number of seconds (x)	Distance traveled in feet (y)
0.5	4
1	4
1	36
2	64
2.5	80
3	99
3	196
4	256

Allow students to share their conjectures about the differences between the two tables. Then proceed with the discussion that follows.

Discussion (8 minutes)

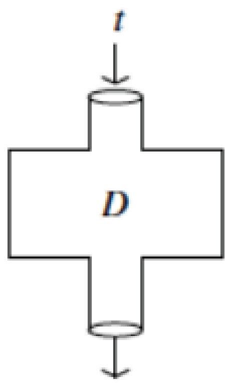
- Using the table on the left (above), state the distance the stone traveled in 1 second.
  - *After* 1 *second, the stone traveled* 16 *feet.*
- Using the table on the right (above), state the distance the stone traveled in 1 second.
  - *After* 1 *second, the stone traveled* 4 *or* 36 *feet.*
- Which of the two tables above allows us to make predictions with some accuracy? Explain.
  - *The table on the left seems like it would be more accurate. The table on the right gives two completely different distances for the stone after* 1 *second. We cannot make an accurate prediction because after* 1 *second, the stone may either be* 4 *feet from where it started or* 36 *feet.*
- We will define a *function* to describe the motion given in the table on the right. The importance of a function is that, once we define it, we can immediately point to the position of the stone at exactly t seconds after the stone's release from a height of 256 feet. It is the ability to *assign*, or *associate,* the distance the stone has traveled at each time t from 256 feet that truly matters.
- Let's formalize this idea of *assignment* or *association* with a symbol, D, where D is used to suggest the distance of the fall at time t. So, D assigns to each number t (where $0 \leq t \leq 4$) another number, which is the distance of the fall of the stone in t seconds. For example, we can rewrite the table from the last lesson as shown below:

Number of seconds (t)	Distance traveled in feet (y)
0.5	4
1	16
1.5	36
2	64
2.5	100
3	144
3.5	196
4	256

- We can also rewrite it as the following table, which emphasizes the assignment the function makes to each input.

D assigns 4 to 0.5
D assigns 16 to 1
D assigns 36 to 1.5
D assigns 64 to 2
D assigns 100 to 2.5
D assigns 144 to 3
D assigns 196 to 3.5
D assigns 256 to 4

- Think of it as an *input–output machine*. That is, we put in a number (the input) that represents the time interval, and out comes another number (the output) that tells us the distance that was traveled in feet during that particular interval.



> *Scaffolding:*
> Highlighting the components of the words *input* and *output* and exploring how the words describe related concepts would be useful.

- With the example of the falling stone, what are we inputting?
  - *The input would be the time interval.*
- What is the output?
  - *The output is the distance the stone traveled in the given time interval.*
- If we input 3 into the machine, what is the output?
  - *The output is* 144.
- If we input 1.5 into the machine, what is the output?
  - *The output is* 36.
- Of course, with this particular machine, we are limited to inputs in the range of 0 to 4 because we are inputting the time it took for the stone to fall; that is, time t where $0 \leq t \leq 4$.

  The function D can be expressed by a formula in the sense that the number assigned to each t can be calculated with a mathematical expression, which is a property that is generally not shared by other functions. Thanks to Newtonian physics (Isaac Newton—think apple falling on your head from a tree), for a distance traveled in feet for a time interval of t seconds, the function can be expressed as the following:

$$\text{distance for time interval } t = 16t^2$$

- From your work in the last lesson, recall that you recognized 16 as a factor for each of the distances in the table below.

Time of interval in seconds (t)	1	2	3	4
Distance stone fell in feet (y)	16	64	144	256

- Functions can be represented in a variety of ways. At this point, we have seen the function that describes the distance traveled by the stone pictorially (from Lesson 1, Example 2), as a table of values, and as a rule. We could also provide a verbal description of the movement of the stone.

Exercise 1 (5 minutes)

Have students verify that the function we are using to represent this situation is accurate by completing Exercise 1. To expedite the verification, consider allowing the use of calculators.

Exercise 1–5

1. **Let y be the distance traveled in time t. Use the function $y = 16t^2$ to calculate the distance the stone dropped for the given time t.**

Time of interval in seconds (t)	0.5	1	1.5	2	2.5	3	3.5	4
Distance stone fell in feet (y)	4	16	36	64	100	144	196	256

   a. **Are the distances you calculated equal to the table from Lesson 1?**

   Yes.

   b. **Does the function $y = 16t^2$ accurately represent the distance the stone fell after a given time t? In other words, does the function assign to t the correct distance? Explain.**

   Yes, the function accurately represents the distance the stone fell after the given time interval. Each computation using the function resulted in the correct distance. Therefore, the function assigns to t the correct distance.

Discussion (10 minutes)

- Being able to write a formula for the function has fantastic implications—it is predictive. That is, we can predict what will happen each time a stone is released from a height of 256 feet. The function makes it possible for us to know exactly how many feet the stone will fall for a time t as long as we select a t so that $0 \leq t \leq 4$.
- Not every function can be expressed as a formula. Imagine being able to write a formula that would allow you to predict the correct answers on a multiple-choice test.
- Now that we have a little more background on functions, we can define them formally. A *function* is a rule (formula) that assigns to each input exactly one output.
- Let's examine that definition more closely. A function is a rule that assigns to each input *exactly one output.* Can you think of why the phrase *exactly one output* must be in the definition?

MP.6

Provide time for students to consider the phrase. Allow them to talk in pairs or small groups and then share their thoughts with the class. Use the question below, if necessary. Then resume the discussion.

- Using our stone-dropping example, if D assigns 64 to 2—that is, the function assigns 64 feet to the time interval 2 seconds—would it be possible for D to assign 65 to 2 as well? Explain.
  - *It would not be possible for D to assign 64 and 65 to 2. The reason is that we are talking about a stone dropping. How could the stone drop 64 feet in 2 seconds **and** 65 feet in 2 seconds? The stone cannot be in two places at once.*
- In order for functions to be useful, the information we get from a function must be useful. That is why a function assigns to each input exactly one output. We also need to consider the situation when using a function. For example, if we use the function, distance for time interval $t = 16t^2$, for $t = -2$, then it would make no sense to explain that -2 would represent 2 seconds before the stone was dropped.

MP.6

Yet, in the function, when $t = -2$,

$$\begin{aligned}\text{distance for time interval } t &= 16t^2\\ &= 16(-2)^2\\ &= 16(4)\\ &= 64\end{aligned}$$

we could conclude that the stone dropped a distance of 64 feet 2 seconds before the stone was dropped. Of course, it makes no sense. Similarly, if we use the formula to calculate the distance when $t = 5$:

$$\begin{aligned}\text{distance for time interval } t &= 16t^2\\ &= 16(5)^2\\ &= 16(25)\\ &= 400\end{aligned}$$

- What is wrong with this statement?
  - *It would mean that the stone dropped 400 feet in 5 seconds, but the stone was dropped from a height of 256 feet. It makes no sense.*
- To summarize, a function is a rule that assigns to each input exactly one output. Additionally, we should always consider the context, if provided, when working with a function to make sure our answer makes sense. In many cases, functions are described by a formula. However, we will soon learn that the assignment of some functions cannot be described by a mathematical rule. The work in Module 5 is laying a critical foundation for students' understanding of functions in high school.

Exercises 2–5 (10 minutes)

Students work independently to complete Exercises 2–5.

Exercises 2–5

2. Can the table shown below represent values of a function? Explain.

Input (x)	1	3	5	5	9
Output (y)	7	16	19	20	28

No, the table cannot represent a function because the input of 5 has two different outputs. Functions assign only one output to each input.

3. Can the table shown below represent values of a function? Explain.

Input (x)	0.5	7	7	12	15
Output (y)	1	15	10	23	30

No, the table cannot represent a function because the input of 7 has two different outputs. Functions assign only one output to each input.

4. **Can the table shown below represent values of a function? Explain.**

Input (x)	10	20	50	75	90
Output (y)	32	32	156	240	288

Yes, the table can represent a function. Even though there are two outputs that are the same, each input has only one output.

5. **It takes Josephine 34 minutes to complete her homework assignment of 10 problems. If we assume that she works at a constant rate, we can describe the situation using a function.**

   a. **Predict how many problems Josephine can complete in 25 minutes.**

   Answers will vary.

   b. **Write the two-variable linear equation that represents Josephine's constant rate of work.**

   Let y be the number of problems she can complete in x minutes.

$$\frac{10}{34} = \frac{y}{x}$$
$$y = \frac{10}{34}x$$
$$y = \frac{5}{17}x$$

   c. **Use the equation you wrote in part (b) as the formula for the function to complete the table below. Round your answers to the hundredths place.**

Time taken to complete problems (x)	5	10	15	20	25
Number of problems completed (y)	1.47	2.94	4.41	5.88	7.35

   After 5 minutes, Josephine was able to complete 1.47 problems, which means that she was able to complete 1 problem, then get about halfway through the next problem.

   d. **Compare your prediction from part (a) to the number you found in the table above.**

   Answers will vary.

   e. **Use the formula from part (b) to compute the number of problems completed when $x = -7$. Does your answer make sense? Explain.**

$$y = \frac{5}{17}(-7)$$
$$= -2.06$$

   No, the answer does not make sense in terms of the situation. The answer means that Josephine can complete -2.06 problems in -7 minutes. This obviously does not make sense.

f. **For this problem, we assumed that Josephine worked at a constant rate. Do you think that is a reasonable assumption for this situation? Explain.**

It does not seem reasonable to assume constant rate for this situation. Just because Josephine was able to complete 10 problems in 34 minutes does not necessarily mean she spent the exact same amount of time on each problem. For example, it may have taken her 20 minutes to do 1 problem and then 14 minutes total to finish the remaining 9 problems.

Closing (4 minutes)

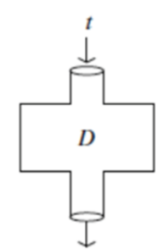
Summarize, or ask students to summarize, the main points from the lesson:

- We know that a function is a rule or formula that assigns to each input exactly one output.
- We know that not every function can be expressed by a mathematical rule or formula. The rule or formula can be a description of the assignment.
- We know that functions have limitations with respect to the situation they describe. For example, we cannot determine the distance a stone drops in -2 seconds.

Lesson Summary

A *function* is a rule that assigns to each input *exactly one output*. The phrase *exactly one output* must be part of the definition so that the function can serve its purpose of being predictive.

Functions are sometimes described as an *input–output machine*. For example, given a function D, the input is time t, and the output is the distance traveled in t seconds.



Distance traveled in t seconds

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 2: Formal Definition of a Function

Exit Ticket

1. Can the table shown below represent values of a function? Explain.

Input (x)	10	20	30	40	50
Output (y)	32	64	96	64	32

2. Kelly can tune up 4 cars in 3 hours. If we assume he works at a constant rate, we can describe the situation using a function.

   a. Write the rule that describes the function that represents Kelly's constant rate of work.

   b. Use the function you wrote in part (a) as the formula for the function to complete the table below. Round your answers to the hundredths place.

Time it takes to tune up cars (x)	2	3	4	6	7
Number of cars tuned up (y)					

c. Kelly works 8 hours per day. How many cars will he finish tuning up at the end of a shift?

d. For this problem, we assumed that Kelly worked at a constant rate. Do you think that is a reasonable assumption for this situation? Explain.

Exit Ticket Sample Solutions

1. Can the table shown below represent values of a function? Explain.

Input (x)	10	20	30	40	50
Output (y)	32	64	96	64	32

Yes, the table can represent a function. Each input has exactly one output.

2. Kelly can tune up 4 cars in 3 hours. If we assume he works at a constant rate, we can describe the situation using a function.

   a. Write the function that represents Kelly's constant rate of work.

   Let y represent the number of cars Kelly can tune up in x hours; then

$$\frac{y}{x} = \frac{4}{3}$$
$$y = \frac{4}{3}x$$

   b. Use the function you wrote in part (a) as the formula for the function to complete the table below. Round your answers to the hundredths place.

Time it takes to tune up cars (x)	2	3	4	6	7
Number of cars tuned up (y)	2.67	4	5.33	8	9.33

   c. Kelly works 8 hours per day. How many cars will he finish tuning up at the end of a shift?

   Using the function, Kelly will tune up 10.67 cars at the end of his shift. That means he will finish tuning up 10 cars and begin tuning up the 11^{th} car.

   d. For this problem, we assumed that Kelly worked at a constant rate. Do you think that is a reasonable assumption for this situation? Explain.

   No, it does not seem reasonable to assume a constant rate for this situation. Just because Kelly tuned up 4 cars in 3 hours does not mean he spent the exact same amount of time on each car. One car could have taken 1 hour, while the other three could have taken 2 hours total.

Problem Set Sample Solutions

1. The table below represents the number of minutes Francisco spends at the gym each day for a week. Does the data shown below represent values of a function? Explain.

Day (x)	1	2	3	4	5	6	7
Time in minutes (y)	35	45	30	45	35	0	0

Yes, the table can represent a function because each input has a unique output. For example, on day 1, Francisco was at the gym for 35 minutes.

2. **Can the table shown below represent values of a function? Explain.**

Input (x)	9	8	7	8	9
Output (y)	11	15	19	24	28

No, the table cannot represent a function because the input of 9 has two different outputs, and so does the input of 8. Functions assign only one output to each input.

3. **Olivia examined the table of values shown below and stated that a possible rule to describe this function could be $y = -2x + 9$. Is she correct? Explain.**

Input (x)	-4	0	4	8	12	16	20	24
Output (y)	17	9	1	-7	-15	-23	-31	-39

Yes, Olivia is correct. When the rule is used with each input, the value of the output is exactly what is shown in the table. Therefore, the rule for this function must be $y = -2x + 9$.

4. **Peter said that the set of data in part (a) describes a function, but the set of data in part (b) does not. Do you agree? Explain why or why not.**

a.

Input (x)	1	2	3	4	5	6	7	8
Output (y)	8	10	32	6	10	27	156	4

b.

Input (x)	-6	-15	-9	-3	-2	-3	8	9
Output (y)	0	-6	8	14	1	2	11	41

Peter is correct. The table in part (a) fits the definition of a function. That is, there is exactly one output for each input. The table in part (b) cannot be a function. The input -3 has two outputs, 14 and 2. This contradicts the definition of a function; therefore, it is not a function.

5. **A function can be described by the rule $y = x^2 + 4$. Determine the corresponding output for each given input.**

Input (x)	-3	-2	-1	0	1	2	3	4
Output (y)	13	8	5	4	5	8	13	20

6. **Examine the data in the table below. The inputs and outputs represent a situation where constant rate can be assumed. Determine the rule that describes the function.**

Input (x)	-1	0	1	2	3	4	5	6
Output (y)	3	8	13	18	23	28	33	38

The rule that describes this function is $y = 5x + 8$.

7. Examine the data in the table below. The inputs represent the number of bags of candy purchased, and the outputs represent the cost. Determine the cost of one bag of candy, assuming the price per bag is the same no matter how much candy is purchased. Then, complete the table.

Bags of Candy (x)	1	2	3	4	5	6	7	8
Cost (y)	*$\$1.25$*	*$\$2.50$*	*$\$3.75$*	$\$5.00$	$\$6.25$	*$\$7.50$*	*$\$8.75$*	$\$10.00$

a. Write the rule that describes the function.

$y = 1.25x$

b. Can you determine the value of the output for an input of $x = -4$? If so, what is it?

When $x = -4$, the output is -5.

c. Does an input of -4 make sense in this situation? Explain.

No, an input of -4 does not make sense for the situation. It would mean -4 bags of candy. You cannot purchase -4 bags of candy.

8. A local grocery store sells 2 pounds of bananas for $\$1.00$. Can this situation be represented by a function? Explain.

Yes, this situation can be represented by a function if the cost of 2 pounds of bananas is $\$1.00$. That is, at all times the cost of 2 pounds will be $\$1.00$, not any more or any less. The function assigns the cost of $\$1.00$ to 2 pounds of bananas.

9. Write a brief explanation to a classmate who was absent today about why the table in part (a) is a function and the table in part (b) is not.

a.

Input (x)	-1	-2	-3	-4	4	3	2	1
Output (y)	81	100	320	400	400	320	100	81

b.

Input (x)	1	6	-9	-2	1	-10	8	14
Output (y)	2	6	-47	-8	19	-2	15	31

The table in part (a) is a function because each input has exactly one output. This is different from the information in the table in part (b). Notice that the input of 1 has been assigned two different values. The input of 1 is assigned 2 and 19. Because the input of 1 has more than one output, this table cannot represent a function.