Lesson 25: Geometric Interpretation of the Solutions of a Linear System

Student Outcomes

- Students sketch the graphs of two linear equations and find the point of intersection.
- Students identify the point of intersection of the two lines as the solution to the system.
- Students verify by computation that the point of intersection is a solution to each of the equations in the system.

Lesson Notes

In the last lesson, students were introduced to the concept of simultaneous equations. Students compared the graphs of two different equations and found the point of intersection. Students estimated the coordinates of the point of intersection of the lines in order to answer questions about the context related to time and distance. In this lesson, students first graph systems of linear equations and find the precise point of intersection. Then, students verify that the ordered pair that is the intersection of the two lines is a solution to *each* equation in the system. Finally, students informally verify that there can be only one point of intersection of two distinct lines in the system by checking to see if another point that is a solution to one equation is a solution to both equations.

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Students need graph paper to complete the Problem Set.

Classwork

Exploratory Challenge/Exercises 1–5 (25 minutes)

Students work independently on Exercises 1–5.

Exploratory Challenge/Exercises 1–5

1. **Sketch the graphs of the linear system on a coordinate plane:** $\begin{cases} 2y + x = 12 \\ y = \frac{5}{6}x - 2 \end{cases}$

For the equation $2y + x = 12$*:*

$$\begin{aligned} 2y + 0 &= 12 \\ 2y &= 12 \\ y &= 6 \end{aligned}$$

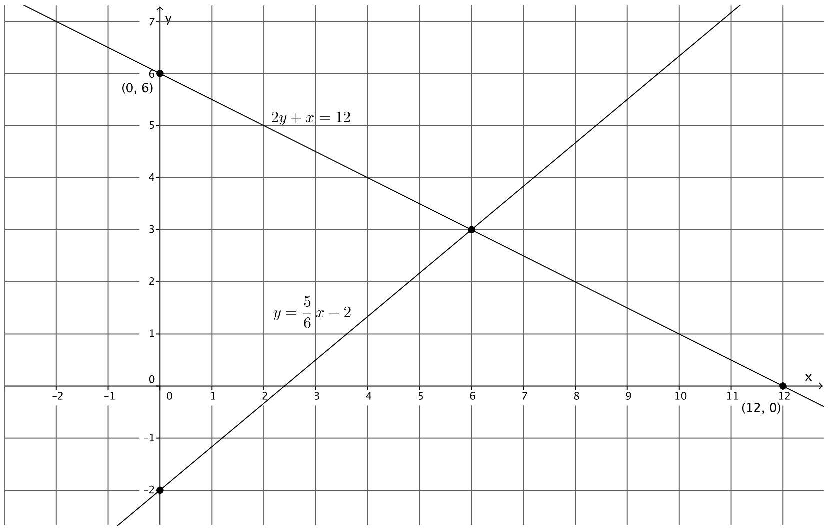
The y*-intercept is* $(0, 6)$*.*

$$\begin{aligned} 2(0) + x &= 12 \\ x &= 12 \\ x &= 12 \end{aligned}$$

The x*-intercept is* $(12, 0)$*.*

For the equation $y = \frac{5}{6}x - 2$*:*

The slope is $\frac{5}{6}$*, and the* y*-intercept is* $(0, -2)$*.*



a. Name the ordered pair where the graphs of the two linear equations intersect.

$(6, 3)$

b. Verify that the ordered pair named in part (a) is a solution to $2y + x = 12$.

$$2(3) + 6 = 12$$
$$6 + 6 = 12$$
$$12 = 12$$

The left and right sides of the equation are equal.

c. Verify that the ordered pair named in part (a) is a solution to $y = \frac{5}{6}x - 2$.

$$3 = \frac{5}{6}(6) - 2$$
$$3 = 5 - 2$$
$$3 = 3$$

The left and right sides of the equation are equal.

d. Could the point $(4, 4)$ be a solution to the system of linear equations? That is, would $(4, 4)$ make both equations true? Why or why not?

No. The graphs of the equations represent all of the possible solutions to the given equations. The point $(4, 4)$ is a solution to the equation $2y + x = 12$ because it is on the graph of that equation. However, the point $(4, 4)$ is not on the graph of the equation $y = \frac{5}{6}x - 2$. Therefore, $(4, 4)$ cannot be a solution to the system of equations.

2. Sketch the graphs of the linear system on a coordinate plane: $\begin{cases} x+y=-2 \\ y=4x+3 \end{cases}$

For the equation $x+y=-2$:

$$0+y=-2$$
$$y=-2$$

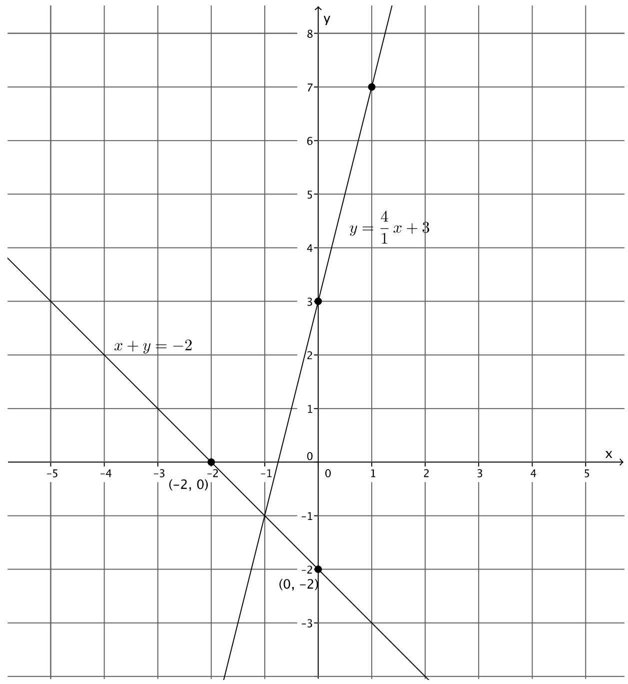
The y-intercept is $(0,-2)$.

$$x+0=-2$$
$$x=-2$$

The x-intercept is $(-2,0)$.

For the equation $y=4x+3$:

The slope is $\frac{4}{1}$, and the y-intercept is $(0,3)$.



a. Name the ordered pair where the graphs of the two linear equations intersect.

$$(-1,-1)$$

b. Verify that the ordered pair named in part (a) is a solution to $x+y=-2$.

$$-1+(-1)=-2$$
$$-2=-2$$

The left and right sides of the equation are equal.

c. Verify that the ordered pair named in part (a) is a solution to $y=4x+3$.

$$-1=4(-1)+3$$
$$-1=-4+3$$
$$-1=-1$$

The left and right sides of the equation are equal.

d. Could the point $(-4,2)$ be a solution to the system of linear equations? That is, would $(-4,2)$ make both equations true? Why or why not?

No. The graphs of the equations represent all of the possible solutions to the given equations. The point $(-4,2)$ is a solution to the equation $x+y=-2$ because it is on the graph of that equation. However, the point $(-4,2)$ is not on the graph of the equation $y=4x+3$. Therefore, $(-4,2)$ cannot be a solution to the system of equations.

3. Sketch the graphs of the linear system on a coordinate plane: $\begin{cases} 3x + y = -3 \\ -2x + y = 2 \end{cases}$

For the equation $3x + y = -3$:

$$3(0) + y = -3$$
$$y = -3$$

The y-intercept is $(0, -3)$.

$$3x + 0 = -3$$
$$3x = -3$$
$$x = -1$$

The x-intercept is $(-1, 0)$.

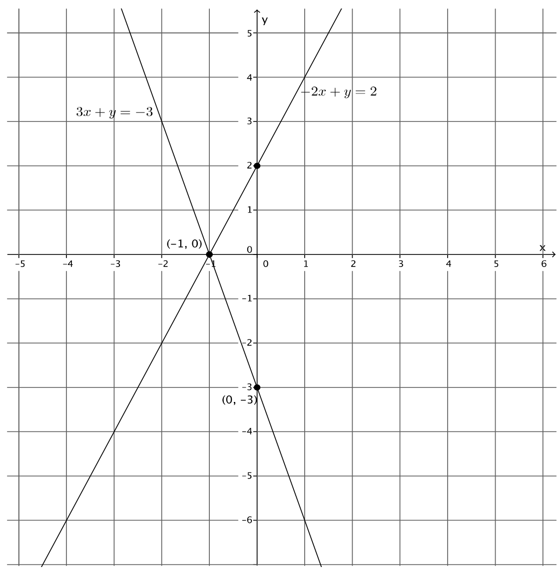
For the equation $-2x + y = 2$:

$$-2(0) + y = 2$$
$$y = 2$$

The y-intercept is $(0, 2)$.

$$-2x + 0 = 2$$
$$-2x = 2$$
$$x = -1$$

The x-intercept is $(-1, 0)$.



a. Name the ordered pair where the graphs of the two linear equations intersect.

$$(-1, 0)$$

b. Verify that the ordered pair named in part (a) is a solution to $3x + y = -3$.

$$3(-1) + 0 = -3$$
$$-3 = -3$$

The left and right sides of the equation are equal.

c. Verify that the ordered pair named in part (a) is a solution to $-2x + y = 2$.

$$-2(-1) + 0 = 2$$
$$2 = 2$$

The left and right sides of the equation are equal.

d. Could the point $(1, 4)$ be a solution to the system of linear equations? That is, would $(1, 4)$ make both equations true? Why or why not?

No. The graphs of the equations represent all of the possible solutions to the given equations. The point $(1, 4)$ is a solution to the equation $-2x + y = 2$ because it is on the graph of that equation. However, the point $(1, 4)$ is not on the graph of the equation $3x + y = -3$. Therefore, $(1, 4)$ cannot be a solution to the system of equations.

4. Sketch the graphs of the linear system on a coordinate plane: $\begin{cases} 2x - 3y = 18 \\ 2x + y = 2 \end{cases}$

For the equation $2x - 3y = 18$:

$$2(0) - 3y = 18$$
$$-3y = 18$$
$$y = -6$$

The y-intercept is $(0, -6)$.

$$2x - 3(0) = 18$$
$$2x = 18$$
$$x = 9$$

The x-intercept is $(9, 0)$.

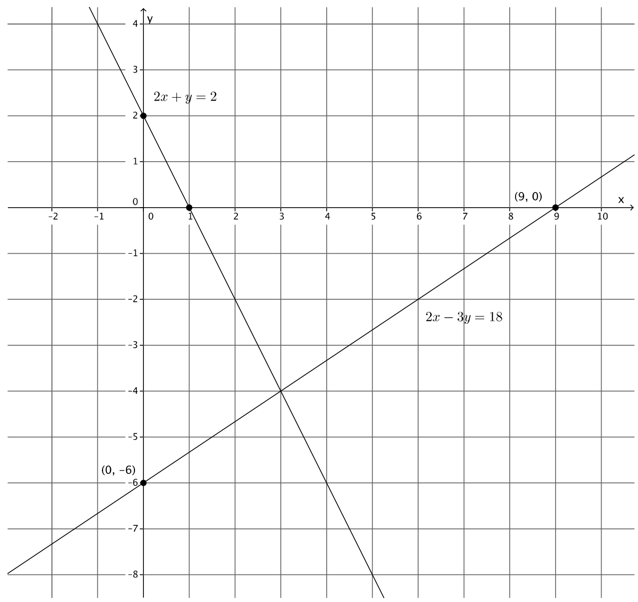
For the equation $2x + y = 2$:

$$2(0) + y = 2$$
$$y = 2$$

The y-intercept is $(0, 2)$.

$$2x + 0 = 2$$
$$2x = 2$$
$$x = 1$$

The x-intercept is $(1, 0)$.



a. Name the ordered pair where the graphs of the two linear equations intersect.

$(3, -4)$

b. Verify that the ordered pair named in part (a) is a solution to $2x - 3y = 18$.

$$2(3) - 3(-4) = 18$$
$$6 + 12 = 18$$
$$18 = 18$$

The left and right sides of the equation are equal.

c. Verify that the ordered pair named in part (a) is a solution to $2x + y = 2$.

$$2(3) + (-4) = 2$$
$$6 - 4 = 2$$
$$2 = 2$$

The left and right sides of the equation are equal.

d. Could the point $(3, -1)$ be a solution to the system of linear equations? That is, would $(3, -1)$ make both equations true? Why or why not?

No. The graphs of the equations represent all of the possible solutions to the given equations. The point $(3, -1)$ is not on the graph of either line; therefore, it is not a solution to the system of linear equations.

5. Sketch the graphs of the linear system on a coordinate plane: $\begin{cases} y - x = 3 \\ y = -4x - 2 \end{cases}$

For the equation $y - x = 3$:

$$y - 0 = 3$$
$$y = 3$$

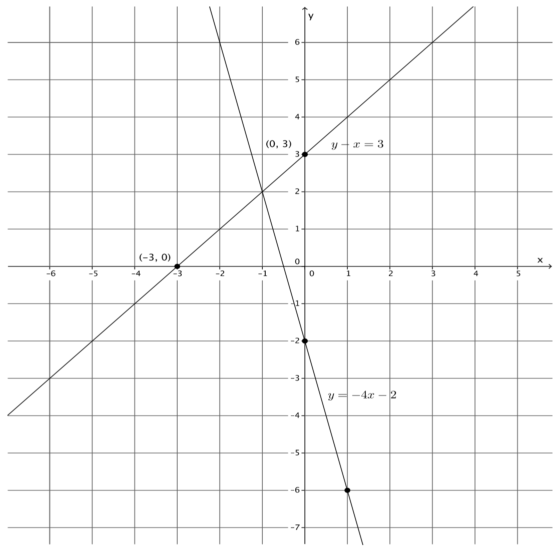
The y-intercept is $(0, 3)$.

$$0 - x = 3$$
$$-x = 3$$
$$x = -3$$

The x-intercept is $(-3, 0)$.

For the equation $y = -4x - 2$:

The slope is $-\frac{4}{1}$, and the y-intercept is $(0, -2)$.



a. Name the ordered pair where the graphs of the two linear equations intersect.

$(-1, 2)$

b. Verify that the ordered pair named in part (a) is a solution to $y - x = 3$.

$$2 - (-1) = 3$$
$$3 = 3$$

The left and right sides of the equation are equal.

c. Verify that the ordered pair named in part (a) is a solution to $y = -4x - 2$.

$$2 = -4(-1) - 2$$
$$2 = 4 - 2$$
$$2 = 2$$

The left and right sides of the equation are equal.

d. Could the point $(-2, 6)$ be a solution to the system of linear equations? That is, would $(-2, 6)$ make both equations true? Why or why not?

No. The graphs of the equations represent all of the possible solutions to the given equations. The point $(-2, 6)$ is a solution to the equation $y = -4x - 2$ because it is on the graph of that equation. However, the point $(-2, 6)$ is not on the graph of the equation $y - x = 3$. Therefore, $(-2, 6)$ cannot be a solution to the system of equations.

Discussion (10 minutes)

The formal proof shown below is optional. The Discussion can be modified by first asking students the questions in the first two bullet points, then having them make conjectures about why there is just one solution to the system of equations.

- How many points of intersection of the two lines were there?
  - *There was only one point of intersection for each pair of lines.*

- Was your answer to part (d) in Exercises 1–5 ever "yes"? Explain.
  - *No, the answer was always "no." Each time, the point given was either a solution to just one of the equations, or it was not a solution to either equation.*
- So far, what you have observed strongly suggests that the solution of a system of linear equations is the point of intersection of two distinct lines. Let's show that this observation is true.

THEOREM: Let

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

be a system of linear equations, where a_1, b_1, c_1, a_2, b_2, and c_2 are constants. At least one of a_1, b_1 is not equal to zero, and at least one of a_2, b_2 is not equal to zero. Let l_1 and l_2 be the lines defined by $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$, respectively. If l_1 and l_2 are not parallel, then the solution of the system is exactly the point of intersection of l_1 and l_2.

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- To prove the theorem, we have to show two things.
  (1) Any point that lies in the intersection of the two lines is a solution of the system.
  (2) Any solution of the system lies in the intersection of the two lines.
- We begin by showing that (1) is true. To show that (1) is true, use the definition of a solution. What does it mean to be a solution to an equation?
  - *A solution is the ordered pair of numbers that makes an equation true. A solution is also a point on the graph of a linear equation.*
- Suppose (x_0, y_0) is the point of intersection of l_1 and l_2. Since the point (x_0, y_0) is on l_1, it means that it is a solution to $a_1x + b_1y = c_1$. Similarly, since the point (x_0, y_0) is on l_2, it means that it is a solution to $a_2x + b_2y = c_2$. Therefore, since the point (x_0, y_0) is a solution to each linear equation, it is a solution to the system of linear equations, and the proof of (1) is complete.
- To prove (2), use the definition of a solution again. Suppose (x_0, y_0) is a solution to the system. Then, the point (x_0, y_0) is a solution to $a_1x + b_1y = c_1$ and, therefore, lies on l_1. Similarly, the point (x_0, y_0) is a solution to $a_2x + b_2y = c_2$ and, therefore, lies on l_2. Since there can be only one point shared by two distinct non-parallel lines, then (x_0, y_0) must be the point of intersection of l_1 and l_2. This completes the proof of (2).
- Therefore, given a system of two distinct non-parallel linear equations, there can be only one point of intersection, which means there is just one solution to the system.

Exercise 6 (3 minutes)

This exercise is optional. It requires students to write two different systems for a given solution. The exercise can be completed independently or in pairs.

Exercise 6

6. Write two different systems of equations with $(1, -2)$ as the solution.

Answers will vary. Two sample solutions are provided:

$$\begin{cases} 3x + y = 1 \\ 2x - 3y = 8 \end{cases} \quad \textit{and} \quad \begin{cases} -x + 3y = -7 \\ 9x - 4y = 17 \end{cases}$$

Closing (5 minutes)

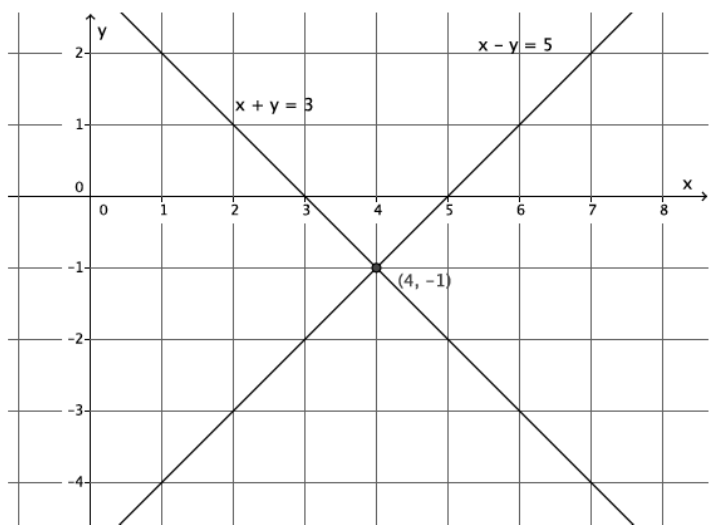
Summarize, or ask students to summarize, the main points from the lesson:

- We know how to sketch the graphs of a system of linear equations and find the point of intersection of the lines.
- We know that the point of intersection represents the solution to the system of linear equations.
- We know that two distinct non-parallel lines will intersect at only one point; therefore, the point of intersection is an ordered pair of numbers that makes both equations in the system true.

Lesson Summary

When the graphs of a system of linear equations are sketched, and if they are not parallel lines, then the point of intersection of the lines of the graph represents the solution to the system. Two distinct lines intersect at most at one point, if they intersect. The coordinates of that point (x, y) represent values that make both equations of the system true.

Example: The system $\begin{cases} x + y = 3 \\ x - y = 5 \end{cases}$ graphs as shown below.



The lines intersect at $(4, -1)$. That means the equations in the system are true when $x = 4$ and $y = -1$.

$$\begin{aligned} x + y &= 3 \\ 4 + (-1) &= 3 \\ 3 &= 3 \end{aligned}$$

$$\begin{aligned} x - y &= 5 \\ 4 - (-1) &= 5 \\ 5 &= 5 \end{aligned}$$

Exit Ticket (5 minutes)

Name ____ Date ____________________

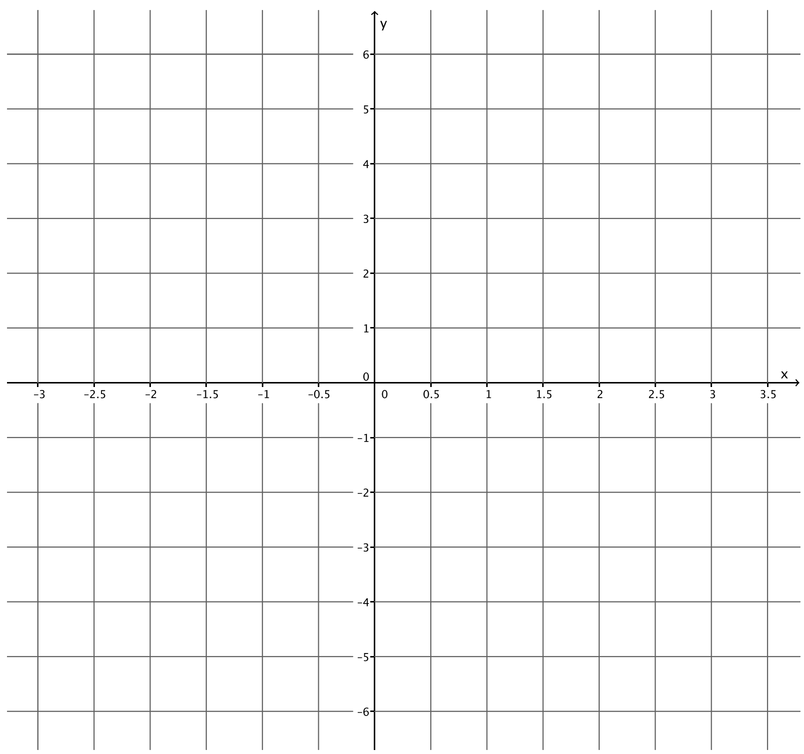
Lesson 25: Geometric Interpretation of the Solutions of a Linear System

Exit Ticket

Sketch the graphs of the linear system on a coordinate plane: $\begin{cases} 2x - y = -1 \\ y = 5x - 5 \end{cases}$

a. Name the ordered pair where the graphs of the two linear equations intersect.

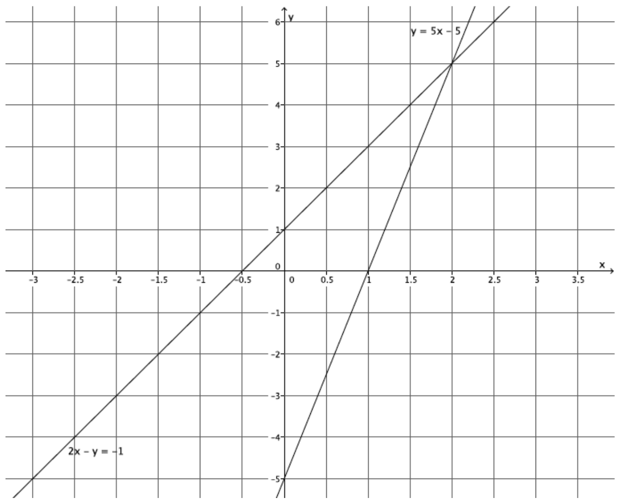
b. Verify that the ordered pair named in part (a) is a solution to $2x - y = -1$.



c. Verify that the ordered pair named in part (a) is a solution to $y = 5x - 5$.

Exit Ticket Sample Solutions

Sketch the graphs of the linear system on a coordinate plane: $\begin{cases} 2x - y = -1 \\ y = 5x - 5 \end{cases}$



a. Name the ordered pair where the graphs of the two linear equations intersect.

$(2, 5)$

b. Verify that the ordered pair named in part (a) is a solution to $2x - y = -1$.

$$2(2) - 5 = -1$$
$$4 - 5 = -1$$
$$-1 = -1$$

The left and right sides of the equation are equal.

c. Verify that the ordered pair named in part (a) is a solution to $y = 5x - 5$.

$$5 = 5(2) - 5$$
$$5 = 10 - 5$$
$$5 = 5$$

The left and right sides of the equation are equal.

Problem Set Sample Solutions

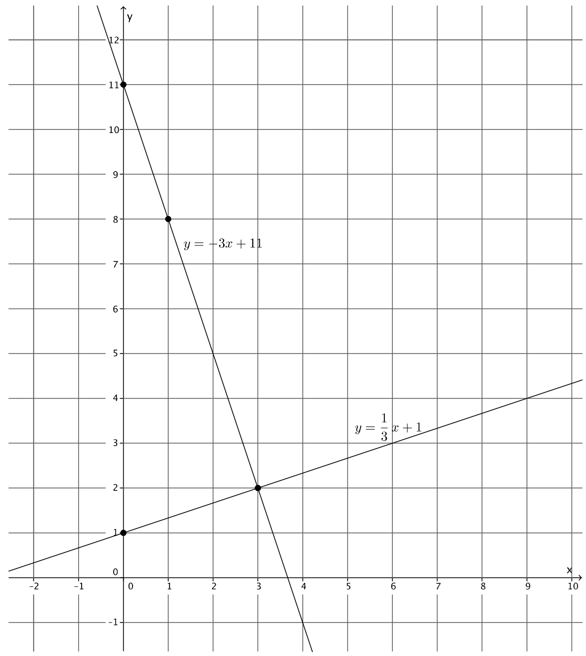
1. Sketch the graphs of the linear system on a coordinate plane: $\begin{cases} y = \frac{1}{3}x + 1 \\ y = -3x + 11 \end{cases}$

For the equation $y = \frac{1}{3}x + 1$:

The slope is $\frac{1}{3}$, and the y-intercept is $(0, 1)$.

For the equation $y = -3x + 11$:

The slope is $-\frac{3}{1}$, and the y-intercept is $(0, 11)$.



a. Name the ordered pair where the graphs of the two linear equations intersect.

$(3, 2)$

b. Verify that the ordered pair named in part (a) is a solution to $y = \frac{1}{3}x + 1$.

$2 = \frac{1}{3}(3) + 1$

$2 = 1 + 1$

$2 = 2$

The left and right sides of the equation are equal.

c. Verify that the ordered pair named in part (a) is a solution to $y = -3x + 11$.

$2 = -3(3) + 11$

$2 = -9 + 11$

$2 = 2$

The left and right sides of the equation are equal.

2. Sketch the graphs of the linear system on a coordinate plane: $\begin{cases} y = \frac{1}{2}x + 4 \\ x + 4y = 4 \end{cases}$

For the equation $y = \frac{1}{2}x + 4$:

The slope is $\frac{1}{2}$, *and the* y*-intercept is* $(0, 4)$.

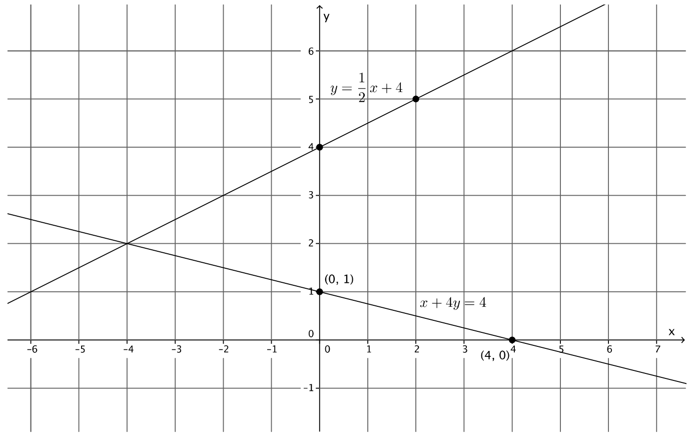
For the equation $x + 4y = 4$:

$$0 + 4y = 4$$
$$4y = 4$$
$$y = 1$$

The y*-intercept is* $(0, 1)$.

$$x + 4(0) = 4$$
$$x = 4$$

The x*-intercept is* $(4, 0)$.



a. Name the ordered pair where the graphs of the two linear equations intersect.

$$(-4, 2)$$

b. Verify that the ordered pair named in part (a) is a solution to $y = \frac{1}{2}x + 4$.

$$2 = \frac{1}{2}(-4) + 4$$
$$2 = -2 + 4$$
$$2 = 2$$

The left and right sides of the equation are equal.

c. Verify that the ordered pair named in part (a) is a solution to $x + 4y = 4$.

$$-4 + 4(2) = 4$$
$$-4 + 8 = 4$$
$$4 = 4$$

The left and right sides of the equation are equal.

3. Sketch the graphs of the linear system on a coordinate plane: $\begin{cases} y = 2 \\ x + 2y = 10 \end{cases}$

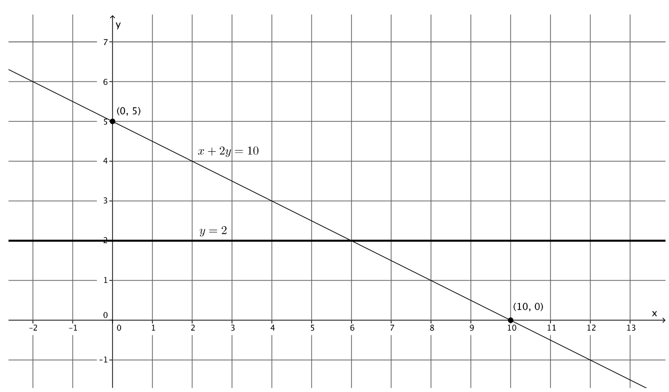
For the equation $x + 2y = 10$:

$$0 + 2y = 10$$
$$2y = 10$$
$$y = 5$$

The y*-intercept is* $(0, 5)$.

$$x + 2(0) = 10$$
$$x = 10$$

The x*-intercept is* $(10, 0)$.



a. Name the ordered pair where the graphs of the two linear equations intersect.

$$(6, 2)$$

b. Verify that the ordered pair named in part (a) is a solution to $y = 2$.

$$2 = 2$$

The left and right sides of the equation are equal.

c. Verify that the ordered pair named in part (a) is a solution to $x + 2y = 10$.

$$6 + 2(2) = 10$$
$$6 + 4 = 10$$
$$10 = 10$$

The left and right sides of the equation are equal.

4. Sketch the graphs of the linear system on a coordinate plane: $\begin{cases} -2x + 3y = 18 \\ 2x + 3y = 6 \end{cases}$

For the equation $-2x + 3y = 18$:

$$-2(0) + 3y = 18$$
$$3y = 18$$
$$y = 6$$

The y-intercept is $(0, 6)$.

$$-2x + 3(0) = 18$$
$$-2x = 18$$
$$x = -9$$

The x-intercept is $(-9, 0)$.

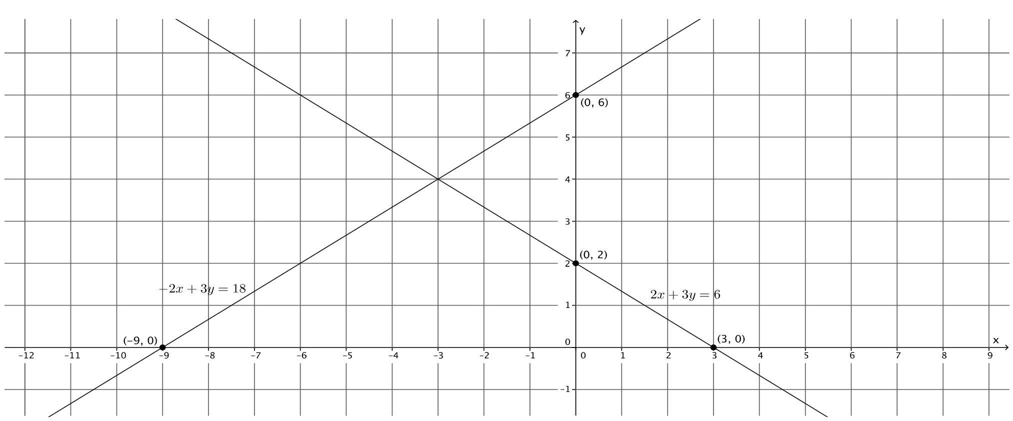
For the equation $2x + 3y = 6$:

$$2(0) + 3y = 6$$
$$3y = 6$$
$$y = 2$$

The y-intercept is $(0, 2)$.

$$2x + 3(0) = 6$$
$$2x = 6$$
$$x = 3$$

The x-intercept is $(3, 0)$.



a. Name the ordered pair where the graphs of the two linear equations intersect.

$(-3, 4)$

b. Verify that the ordered pair named in part (a) is a solution to $-2x + 3y = 18$.

$$-2(-3) + 3(4) = 18$$
$$6 + 12 = 18$$
$$18 = 18$$

The left and right sides of the equation are equal.

c. Verify that the ordered pair named in part (a) is a solution to $2x + 3y = 6$.

$$2(-3) + 3(4) = 6$$
$$-6 + 12 = 6$$
$$6 = 6$$

The left and right sides of the equation are equal.

5. Sketch the graphs of the linear system on a coordinate plane: $\begin{cases} x + 2y = 2 \\ y = \frac{2}{3}x - 6 \end{cases}$

For the equation $x + 2y = 2$:

$$0 + 2y = 2$$
$$2y = 2$$
$$y = 1$$

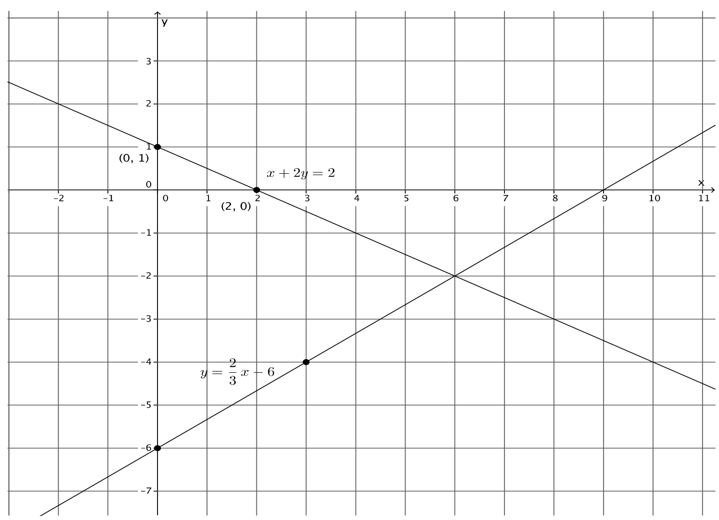
The y-intercept is $(0, 1)$.

$$x + 2(0) = 2$$
$$x = 2$$

The x-intercept is $(2, 0)$.

For the equation $y = \frac{2}{3}x - 6$:

The slope is $\frac{2}{3}$, and the y-intercept is $(0, -6)$.



a. Name the ordered pair where the graphs of the two linear equations intersect.

$(6, -2)$

b. Verify that the ordered pair named in part (a) is a solution to $x + 2y = 2$.

$$6 + 2(-2) = 2$$
$$6 - 4 = 2$$
$$2 = 2$$

The left and right sides of the equation are equal.

c. **Verify that the ordered pair named in part (a) is a solution to $y = \frac{2}{3}x - 6$.**

$$-2 = \frac{2}{3}(6) - 6$$
$$-2 = 4 - 6$$
$$-2 = -2$$

The left and right sides of the equation are equal.

6. **Without sketching the graph, name the ordered pair where the graphs of the two linear equations intersect.**

$$\begin{cases} x = 2 \\ y = -3 \end{cases}$$

$(2, -3)$