Lesson 9: An Application of Linear Equations

Student Outcomes

- Students know how to rewrite an exponential expression that represents a series as a linear equation.

Lesson Notes

The purpose of this lesson is to expose students to applications of linear equations. The discussion below revisits the Facebook problem from Module 1, but this time in the context of a linear equation. This is an opportunity to highlight MP 1: Make sense of problems and persevere in solving them, as the discussion requires students to work with equations in ways they have not before. If you feel that the discussion is too challenging for your students, you can choose to use the series of more accessible applications of linear equations (Exercises 3–11) beginning on page 100.

Classwork

Discussion (30 minutes)

- In Module 1, you saw the following problem:

  You sent a photo of you and your family on vacation to seven Facebook friends. If each of them sends it to five of their friends, and each of those friends sends it to five of their friends, and those friends send it to five more, how many people (not counting yourself) will see your photo? Assume that no friend received the photo twice.

  In Module 1, you were asked to express your answer in exponential notation. The solution is given here:

  (1) The number of friends you sent a photo to $= 7$.

  (2) The number of friends 7 people sent the photo to $= 7 \cdot 5$.

  (3) The number of friends $7 \cdot 5$ people sent the photo to $= (7 \cdot 5) \cdot 5$.

  (4) The number of friends $(7 \cdot 5) \cdot 5$ people sent the photo to $= \big((7 \cdot 5) \cdot 5\big) \cdot 5$.

  Therefore, the total number of people who received the photo is

$$7 + 7 \cdot 5 + 7 \cdot 5^2 + 7 \cdot 5^3.$$

- Let's refer to "you sending the photo" as the first step. Then, "your friends sending the photo to their friends" is the second step, and so on. In the original problem, there were four steps. Assuming the trend continues, how would you find the sum after 10 steps?
  - *We would continue the pattern until we got to the 10^{th} step.*
- What if I asked you how many people received the photo after 100 steps?
  - *It would take a long time to continue the pattern to the 100^{th} step.*
- We want to be able to answer the question for any number of steps. For that reason, we will work towards expressing our answer as a linear equation.

- For convenience, let's introduce some symbols. Since we are talking about steps, we will refer to the sum after step one as S_1, the sum after step two as S_2, the sum after step three as S_3, and so on. Thus,

$$S_1 = 7 \quad (1)$$
$$S_2 = 7 + 7 \cdot 5 \quad (2)$$
$$S_3 = 7 + 7 \cdot 5 + 7 \cdot 5^2 \quad (3)$$
$$S_4 = 7 + 7 \cdot 5 + 7 \cdot 5^2 + 7 \cdot 5^3 \quad (4)$$

- What patterns do you notice within each of the equations (1)–(4)?
  - *They contain some of the same terms. For example, equation (2) is the same as (1), except equation (2) has the term* $7 \cdot 5$. *Similarly, equation (3) is the same as (2), except equation (3) has the term* $7 \cdot 5^2$.
- What you noticed is true. However, we want to generalize in a way that does not require us to know one step before getting to the next step. Let's see what other hidden patterns there are.

> *Scaffolding:*
> Talk students through the manipulation of the equation. For example, "We begin by subtracting 7 from both sides. Next, we will add the number 7 times 5^2 to both sides of the equation. Then, using the distributive property…"

- Let's begin with equation (2):

$$S_2 = 7 + 7 \cdot 5$$
$$S_2 - 7 = 7 \cdot 5$$
$$S_2 - 7 + 7 \cdot 5^2 = 7 \cdot 5 + 7 \cdot 5^2$$
$$S_2 - 7 + 7 \cdot 5^2 = 5(7 + 7 \cdot 5).$$

Notice that the grouping on the right side of the equation is exactly S_2, so we have the following:

$$S_2 - 7 + 7 \cdot 5^2 = 5S_2.$$

This equation is a linear equation in S_2. It is an equation we know how to solve (pretend S_2 is an x if that helps).

- Let's do something similar with equation (3):

$$S_3 = 7 + 7 \cdot 5 + 7 \cdot 5^2$$
$$S_3 - 7 = 7 \cdot 5 + 7 \cdot 5^2$$
$$S_3 - 7 + 7 \cdot 5^3 = 7 \cdot 5 + 7 \cdot 5^2 + 7 \cdot 5^3$$
$$S_3 - 7 + 7 \cdot 5^3 = 5(7 + 7 \cdot 5 + 7 \cdot 5^2).$$

Again, the grouping on the right side of the equation is exactly the equation we began with, S_3, so we have the following:

$$S_3 - 7 + 7 \cdot 5^3 = 5S_3.$$

This is a linear equation in S_3.

- Let's work together to do something similar with equation (4):

$$S_4 = 7 + 7 \cdot 5 + 7 \cdot 5^2 + 7 \cdot 5^3.$$

- What did we do first in each of the equations (2) and (3)?
  - *Subtract 7 from both sides of the equation.*
- Now we have the following:

$$S_4 - 7 = 7 \cdot 5 + 7 \cdot 5^2 + 7 \cdot 5^3.$$

- What did we do next?
  - *We added* $7 \cdot 5$ *raised to a power to both sides of the equation. When it was the second step, the power of* 5 *was* 2*. When it was the third step, the power of* 5 *was* 3*. Now that it is the fourth step, the power of* 5 *should be* 4*.*
- Now we have

$$S_4 - 7 + 7 \cdot 5^4 = 7 \cdot 5 + 7 \cdot 5^2 + 7 \cdot 5^3 + 7 \cdot 5^4.$$

- What did we do after that?
  - *We used the distributive property to rewrite the right side of the equation.*
- Now we have

$$S_4 - 7 + 7 \cdot 5^4 = 5(7 \cdot 5 + 7 \cdot 5^2 + 7 \cdot 5^3).$$

- What do we do now?
  - *We substitute the grouping on the right side of the equation with* S_4*.*
- Finally, we have a linear equation in S_4:

$$S_4 - 7 + 7 \cdot 5^4 = 5S_4.$$

- Let's look at the linear equations altogether.

$$S_2 - 7 + 7 \cdot 5^2 = 5S_2$$
$$S_3 - 7 + 7 \cdot 5^3 = 5S_3$$
$$S_4 - 7 + 7 \cdot 5^4 = 5S_4$$

  What do you think the equation would be for S_{10}?
  - *According to the pattern, it would be* $S_{10} - 7 + 7 \cdot 5^{10} = 5S_{10}$*.*
- Now let's solve the equation. (Note, again, that we do not simplify $(1 - 5)$ for the reason explained above.)

$$\begin{aligned} S_{10} - 7 + 7 \cdot 5^{10} &= 5S_{10} \\ S_{10} - 5S_{10} - 7 + 7 \cdot 5^{10} &= 5S_{10} - 5S_{10} \\ S_{10}(1-5) - 7 + 7 \cdot 5^{10} &= 0 \\ S_{10}(1-5) - 7 + 7 + 7 \cdot 5^{10} - 7 \cdot 5^{10} &= 7 - 7 \cdot 5^{10} \\ S_{10}(1-5) &= 7(1 - 5^{10}) \\ S_{10} &= \frac{7(1-5^{10})}{(1-5)} \\ S_{10} &= 17{,}089{,}842 \end{aligned}$$

  After 10 steps, 17,089,842 will see the photo!

Note to teacher:

This equation is solved differently than the equations we have been solving because we want to show students the normal form of an equation for the summation of a geometric series. In general, if $a \neq 1$, then

$$1 + a + a^2 + \cdots + a^{n-1} + a^n = \frac{1 - a^{n+1}}{1 - a}.$$

Exercises 1–2 (5 minutes)

Students complete Exercises 1–2 independently.

Exercises 1–2

1. **Write the equation for the 15^{th} step.**

$$S_{15} - 7 + 7 \cdot 5^{15} = 5S_{15}$$

2. **How many people would see the photo after 15 steps? Use a calculator if needed.**

$$\begin{aligned} S_{15} - 7 + 7 \cdot 5^{15} &= 5S_{15} \\ S_{15} - 5S_{15} - 7 + +7 \cdot 5^{15} &= 5S_{15} - 5S_{15} \\ S_{15}(1-5) - 7 + 7 \cdot 5^{15} &= 0 \\ S_{15}(1-5) - 7 + 7 + 7 \cdot 5^{15} - 7 \cdot 5^{15} &= 7 - 7 \cdot 5^{15} \\ S_{15}(1-5) &= 7(1-5^{15}) \\ S_{15} &= \frac{7(1-5^{15})}{(1-5)} \\ S_{15} &= 53{,}405{,}761{,}717 \end{aligned}$$

Exercises 3–11 as an Alternative to Discussion (30 minutes)

Students should be able to complete the following problems independently as they are an application of skills learned to this point, namely transcription and solving linear equations in one variable. You may choose to have students work on the problems one at a time and share their work with the whole class, or assign the entire set and allow students to work at their own pace. Provide correct solutions at the end of the lesson.

Exercises 3–11

3. **Marvin paid an entrance fee of $\$5$ plus an additional $\$1.25$ per game at a local arcade. Altogether, he spent $\$26.25$. Write and solve an equation to determine how many games Marvin played.**

Let x represent the number of games he played.

$$\begin{aligned} 5 + 1.25x &= 26.25 \\ 1.25x &= 21.25 \\ x &= \frac{21.25}{1.25} \\ x &= 17 \end{aligned}$$

Marvin played 17 games.

4. **The sum of four consecutive integers is -26. What are the integers?**

Let x be the first integer.

$$\begin{aligned} x + (x+1) + (x+2) + (x+3) &= -26 \\ 4x + 6 &= -26 \\ 4x &= -32 \\ x &= -8 \end{aligned}$$

The integers are -8, -7, -6, and -5.

5. A book has x pages. How many pages are in the book if Maria read 45 pages of a book on Monday, $\frac{1}{2}$ the book on Tuesday, and the remaining 72 pages on Wednesday?

Let x be the number of pages in the book.

$$x = 45 + \frac{1}{2}x + 72$$
$$x = 117 + \frac{1}{2}x$$
$$\frac{1}{2}x = 117$$
$$x = 234$$

The book has 234 pages.

6. A number increased by 5 and divided by 2 is equal to 75. What is the number?

Let x be the number.

$$\frac{x+5}{2} = 75$$
$$x + 5 = 150$$
$$x = 145$$

The number is 145.

7. The sum of thirteen and twice a number is seven less than six times a number. What is the number?

Let x be the number.

$$13 + 2x = 6x - 7$$
$$20 + 2x = 6x$$
$$20 = 4x$$
$$5 = x$$

The number is 5.

8. The width of a rectangle is 7 less than twice the length. If the perimeter of the rectangle is 43.6 inches, what is the area of the rectangle?

Let x represent the length of the rectangle.

$$2x + 2(2x - 7) = 43.6$$
$$2x + 4x - 14 = 43.6$$
$$6x - 14 = 43.6$$
$$6x = 57.6$$
$$x = \frac{57.6}{6}$$
$$x = 9.6$$

The length of the rectangle is 9.6 inches, and the width is 12.2 inches, so the area is 117.12 in^2.

9. **Two hundred and fifty tickets are available for sale for a school dance. On Monday, 35 tickets were sold. An equal number of tickets were sold each day for the next five days. How many tickets were sold on one of those days?**

Let x be the number of tickets sold on one of those days.

$$250 = 35 + 5x$$
$$215 = 5x$$
$$43 = x$$

43 tickets were sold on each of the five days.

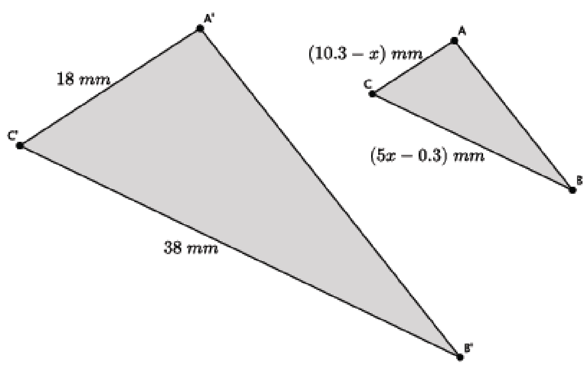
10. **Shonna skateboarded for some number of minutes on Monday. On Tuesday, she skateboarded for twice as many minutes as she did on Monday, and on Wednesday, she skateboarded for half the sum of minutes from Monday and Tuesday. Altogether, she skateboarded for a total of three hours. How many minutes did she skateboard each day?**

Let x be the number of minutes she skateboarded on Monday.

$$x + 2x + \frac{2x + x}{2} = 180$$
$$\frac{2x}{2} + \frac{4x}{2} + \frac{2x + x}{2} = 180$$
$$\frac{9x}{2} = 180$$
$$9x = 360$$
$$x = 40$$

Shonna skateboarded 40 minutes on Monday, 80 minutes on Tuesday, and 60 minutes on Wednesday.

11. **In the diagram below, $\triangle ABC \sim \triangle A'B'C'$. Determine the length of AC and BC.**



$$\frac{18}{10.3 - x} = \frac{38}{5x - 0.3}$$
$$18(5x - 0.3) = 38(10.3 - x)$$
$$90x - 5.4 = 391.4 - 38x$$
$$128x - 5.4 = 391.4$$
$$128x = 396.8$$
$$x = \frac{396.8}{128}$$
$$x = 3.1$$

The length of AC is 7.2 mm, and the length of BC is 15.2 mm.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We can rewrite equations to develop a pattern and make predictions.
- We know that for problems like these we can generalize equations so that we do not have to do each step to get our answer.
- We learned how equations can be used to solve problems.

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 9: An Application of Linear Equations

Exit Ticket

1. Rewrite the equation that would represent the sum in the fifth step of the Facebook problem:

$$S_5 = 7 + 7 \cdot 5 + 7 \cdot 5^2 + 7 \cdot 5^3 + 7 \cdot 5^4.$$

2. The sum of four consecutive integers is 74. Write an equation and solve to find the numbers.

Exit Ticket Sample Solutions

1. **Rewrite the equation that would represent the fifth step of the Facebook problem:**

$$S_5 = 7 + 7\cdot 5 + 7\cdot 5^2 + 7\cdot 5^3 + 7\cdot 5^4.$$

$$\begin{aligned}
S_5 &= 7 + 7\cdot 5 + 7\cdot 5^2 + 7\cdot 5^3 + 7\cdot 5^4\\
S_5 - 7 &= 7\cdot 5 + 7\cdot 5^2 + 7\cdot 5^3 + 7\cdot 5^4\\
S_5 - 7 + 7\cdot 5^5 &= 7\cdot 5 + 7\cdot 5^2 + 7\cdot 5^3 + 7\cdot 5^4 + 7\cdot 5^5\\
S_5 - 7 + 7\cdot 5^5 &= 5(7 + 7\cdot 5 + 7\cdot 5^2 + 7\cdot 5^3 + 7\cdot 5^4)\\
S_5 - 7 + 7\cdot 5^5 &= 5(S_5)\\
S_5 - 5S_5 - 7 + 7\cdot 5^5 &= 0\\
S_5 - 5S_5 &= 7 - (7\cdot 5^5)\\
(1-5)S_5 &= 7 - (7\cdot 5^5)\\
(1-5)S_5 &= 7(1-5^5)\\
S_5 &= \frac{7(1-5^5)}{1-5}
\end{aligned}$$

2. **The sum of four consecutive integers is 74. Write an equation and solve to find the numbers.**

Let x be the first number.

$$\begin{aligned}
x + (x+1) + (x+2) + (x+3) &= 74\\
4x + 6 &= 74\\
4x &= 68\\
x &= 17
\end{aligned}$$

The numbers are 17, 18, 19, and 20.

Problem Set Sample Solutions

Assign the problems that relate to the elements of the lesson you chose to use with students.

1. **You forward an e-card that you found online to three of your friends. They liked it so much that they forwarded it on to four of their friends, who then forwarded it on to four of their friends, and so on. The number of people who saw the e-card is shown below. Let S_1 represent the number of people who saw the e-card after one step, let S_2 represent the number of people who saw the e-card after two steps, and so on.**

$$S_1 = 3$$
$$S_2 = 3 + 3\cdot 4$$
$$S_3 = 3 + 3\cdot 4 + 3\cdot 4^2$$
$$S_4 = 3 + 3\cdot 4 + 3\cdot 4^2 + 3\cdot 4^3$$

a. **Find the pattern in the equations.**

$$S_2 = 3 + 3 \cdot 4$$
$$S_2 - 3 = 3 \cdot 4$$
$$S_2 - 3 + 3 \cdot 4^2 = 3 \cdot 4 + 3 \cdot 4^2$$
$$S_2 - 3 + 3 \cdot 4^2 = 4(3 + 3 \cdot 4)$$
$$S_2 - 3 + 3 \cdot 4^2 = 4S_2$$

$$S_3 = 3 + 3 \cdot 4 + 3 \cdot 4^2$$
$$S_3 - 3 = 3 \cdot 4 + 3 \cdot 4^2$$
$$S_3 - 3 + 3 \cdot 4^3 = 3 \cdot 4 + 3 \cdot 4^2 + 3 \cdot 4^3$$
$$S_3 - 3 + 3 \cdot 4^3 = 4(3 + 3 \cdot 4 + 3 \cdot 4^2)$$
$$S_3 - 3 + 3 \cdot 4^3 = 4S_3$$

$$S_4 = 3 + 3 \cdot 4 + 3 \cdot 4^2 + 3 \cdot 4^3$$
$$S_4 - 3 = 3 \cdot 4 + 3 \cdot 4^2 + 3 \cdot 4^3$$
$$S_4 - 3 + 3 \cdot 4^4 = 3 \cdot 4 + 3 \cdot 4^2 + 3 \cdot 4^3 + 3 \cdot 4^4$$
$$S_4 - 3 + 3 \cdot 4^4 = 4(3 + 3 \cdot 4 + 3 \cdot 4^2 + 3 \cdot 4^3)$$
$$S_4 - 3 + 3 \cdot 4^4 = 4S_4$$

b. **Assuming the trend continues, how many people will have seen the e-card after 10 steps?**

$$S_{10} - 3 + 3 \cdot 4^{10} = 4S_{10}$$
$$S_{10} - 4S_{10} - 3 + 3 \cdot 4^{10} = 0$$
$$S_{10}(1-4) = 3 - 3 \cdot 4^{10}$$
$$S_{10}(1-4) = 3(1 - 4^{10})$$
$$S_{10} = \frac{3(1-4^{10})}{(1-4)}$$
$$S_{10} = 1{,}048{,}575$$

After 10 steps, $1{,}048{,}575$ people will have seen the e-card.

c. **How many people will have seen the e-card after n steps?**

$$S_n = \frac{3(1-4^n)}{(1-4)}$$

For each of the following questions, write an equation and solve to find each answer.

2. **Lisa has a certain amount of money. She spent $\$39$ and has $\frac{3}{4}$ of the original amount left. How much money did she have originally?**

Let x be the amount of money Lisa had originally.

$$x - 39 = \frac{3}{4}x$$
$$-39 = -\frac{1}{4}x$$
$$156 = x$$

Lisa had $\$156$ originally.

3. The length of a rectangle is 4 more than 3 times the width. If the perimeter of the rectangle is 18.4 cm, what is the area of the rectangle?

Let x represent the width of the rectangle.

$$\begin{aligned} 2(4+3x)+2x &= 18.4 \\ 8+6x+2x &= 18.4 \\ 8+8x &= 18.4 \\ 8x &= 10.4 \\ x &= \frac{10.4}{8} \\ x &= 1.3 \end{aligned}$$

The width of the rectangle is 1.3 cm, and the length is 7.9 cm, so the area is 10.27 cm^2.

4. Eight times the result of subtracting 3 from a number is equal to the number increased by 25. What is the number?

Let x be the number.

$$\begin{aligned} 8(x-3) &= x+25 \\ 8x-24 &= x+25 \\ 7x-24 &= 25 \\ 7x &= 49 \\ x &= 7 \end{aligned}$$

The number is 7.

5. Three consecutive odd integers have a sum of 3. What are the numbers?

Let x be the first odd number.

$$\begin{aligned} x+(x+2)+(x+4) &= 3 \\ 3x+6 &= 3 \\ 3x &= -3 \\ x &= -1 \end{aligned}$$

The three consecutive odd integers are -1, 1, and 3.

6. Each month, Liz pays $\$35$ to her phone company just to use the phone. Each text she sends costs her an additional $\$0.05$. In March, her phone bill was $\$72.60$. In April, her phone bill was $\$65.85$. How many texts did she send each month?

Let x be the number of texts she sent in March.

$$\begin{aligned} 35+0.05x &= 72.60 \\ 0.05x &= 37.6 \\ x &= \frac{37.6}{0.05} \\ x &= 752 \end{aligned}$$

She sent 752 texts in March.

Let y be the number of texts she sent in April.

$$\begin{aligned} 35+0.05y &= 65.85 \\ 0.05y &= 30.85 \\ y &= \frac{30.85}{0.05} \\ y &= 617 \end{aligned}$$

She sent 617 texts in April.

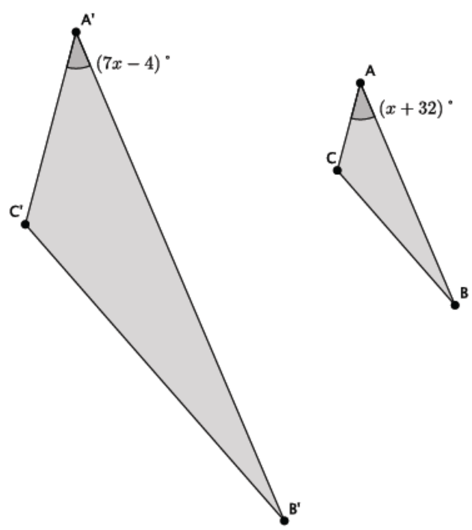
7. Claudia is reading a book that has 360 pages. She read some of the book last week. She plans to read 46 pages today. When she does, she will be $\frac{4}{5}$ of the way through the book. How many pages did she read last week?

Let x be the number of pages she read last week.

$$x + 46 = \frac{4}{5}(360)$$
$$x + 46 = 288$$
$$x = 242$$

Claudia read 242 pages last week.

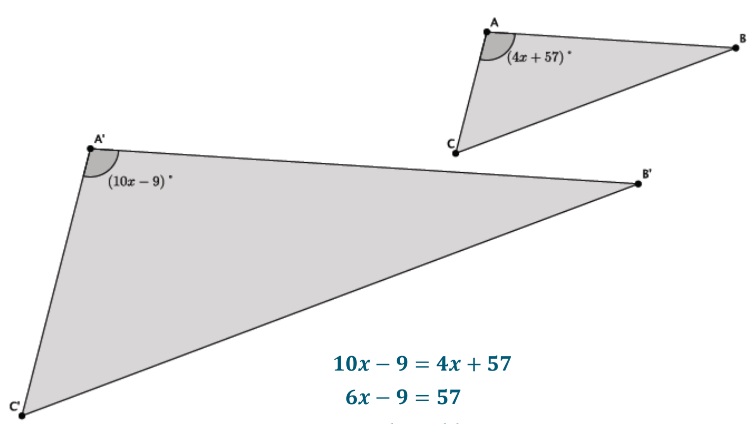
8. In the diagram below, $\triangle ABC \sim \triangle A'B'C'$. Determine the measure of $\angle A$.



$$7x - 4 = x + 32$$
$$6x - 4 = 32$$
$$6x = 36$$
$$x = 6$$

The measure of $\angle A$ is $38°$.

9. In the diagram below, $\triangle ABC \sim \triangle A'B'C'$. Determine the measure of $\angle A$.



$$10x - 9 = 4x + 57$$
$$6x - 9 = 57$$
$$6x = 66$$
$$x = 11$$

The measure of $\angle A$ is $101°$.