



Lesson 7: Classification of Solutions

Student Outcomes

- Students know the conditions for which a linear equation will have a unique solution, no solution, or infinitely many solutions.

Lesson Notes

Part of the discussion on the second page in this lesson is optional. The key parts of the discussion are those that point out the characteristics of the constants and coefficients of an equation that allow students to see the structure in equations that lead to a unique solution, no solution, or infinitely many solutions. You can choose to go through the discussion with your students or choose to use the activity at the top of the fourth page of this lesson. The activity requires students to examine groups of equations and make conclusions about the nature of their solutions based on what they observe, leading to the same result as discussion.

Classwork

Exercises 1–3 (6 minutes)

Students complete Exercises 1–3 independently in preparation for the discussion that follows.

Exercises

Solve each of the following equations for x .

1. $7x - 3 = 5x + 5$

$$\begin{aligned} 7x - 3 &= 5x + 5 \\ 7x - 3 + 3 &= 5x + 5 + 3 \\ 7x &= 5x + 8 \\ 7x - 5x &= 5x - 5x + 8 \\ 2x &= 8 \\ x &= 4 \end{aligned}$$

2. $7x - 3 = 7x + 5$

$$\begin{aligned} 7x - 3 &= 7x + 5 \\ 7x - 7x - 3 &= 7x - 7x + 5 \\ -3 &\neq 5 \end{aligned}$$

This equation has no solution.

3. $7x - 3 = -3 + 7x$

$$\begin{aligned} 7x - 3 &= -3 + 7x \\ 7x - 3 + 3 &= -3 + 3 + 7x \\ 7x &= 7x \end{aligned}$$

OR

$$\begin{aligned} 7x - 3 &= -3 + 7x \\ 7x - 7x - 3 &= -3 + 7x - 7x \\ -3 &= -3 \end{aligned}$$

Discussion (15 minutes)

Display the three equations so that students can easily see and compare them throughout the discussion.

$7x - 3 = 5x + 5$	$7x - 3 = 7x + 5$	$7x - 3 = -3 + 7x$
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- Was there anything new or unexpected about Exercise 1?
 - No. We solved the equation for x , and $x = 4$.

You may choose to continue with the discussion of Exercise 1 or complete the activity described at the end of the Discussion on page 78. Be sure to revisit Exercises 2 and 3 and classify the solution to those equations.

- What can you say about the coefficients of terms with x in the equation?
 - They are different.
- Do you think any number other than 4 would make the equation true?
 - I don't think so, but I would have to try different numbers to find out.
- Instead of having to check every single number to see if it makes the equation true, we can look at a general form of an equation to show that there can be only one solution.
- Given a linear equation that has been simplified on both sides, $ax + b = cx + d$, where a , b , c , and d are constants and $a \neq c$, we can use our normal properties of equality to solve for x .

$$\begin{aligned} ax + b &= cx + d \\ ax + b - b &= cx + d - b \\ ax &= cx + d - b \\ ax - cx &= cx - cx + d - b \\ (a - c)x &= d - b \\ \frac{a - c}{a - c}x &= \frac{d - b}{a - c} \\ x &= \frac{d - b}{a - c} \end{aligned}$$

- The only value of x that will make the equation true is the division of the difference of the constants by the difference of the coefficients of x . In other words, if the coefficients of x are different on each side of the equal sign, then the equation will have one solution.

- As an aside, if the coefficients of x are different and the value of the constants are the same, the only solution is $x = 0$. Consider the following equation:

$$\begin{aligned} 2x + 12 &= x + 12 \\ 2x + 12 - 12 &= x + 12 - 12 \\ 2x &= x \\ 2x - x &= x - x \\ x &= 0. \end{aligned}$$

- The only value of x that will make $2x + 12 = x + 12$ true, is $x = 0$.
- Was there anything new or unexpected about Exercise 2?
 - No. We tried to solve the equation and got an untrue statement, $-3 \neq 5$; therefore, the equation had no solutions.*
- What can you say about the coefficients of the terms with x in the second equation?
 - The coefficients were the same.*
- What can you say about the constants in the second equation?
 - The constants were different.*
- It is time to begin formalizing what we have observed as we have solved equations. To begin with, we have learned that some linear equations have no solutions. That is, when we solve an equation using our properties of equality and end up with a statement like $a = b$, where a and b are different numbers, we say that the equation has no solution.
- To produce and/or recognize an equation with no solutions, consider the equation with both sides simplified. The coefficients of the variable x will be the same on both sides of the equal sign, and the value of the constants on each side will be different. This type of equation has no solution because there is no value of x in an equation such as $x + a = x + b$ that will make the equation true.

Give students a moment to compare the equation $x + a = x + b$ with Exercise 2.

- Was there anything new or unexpected about Exercise 3?
 - Yes. When solving the equation, we got $-3 = -3$, which did not give us a value for x .*
 - Yes. When solving the equation, we got to $7x = 7x$, and we weren't sure what to do next or what it meant.*

Provide students with time to discuss in small groups how they should interpret the meaning of such statements. Students have yet to work with equations that yield such results, and they need the time to make sense of the statements. If necessary, ask the questions in the next two bullet points to guide their thinking. Give them time to try different values of x in both equations to verify that it is always true. Next, have students share their thoughts and interpretations and resume the discussion below.

- Is $-3 = -3$ a true statement? Is it always true?
 - Yes.*
- If $7x = 7x$, will we always have an equality, no matter which number we choose for x ?
 - Yes. This is an identity because the same expression is on both sides of the equal sign.*
- What can you say about the coefficients of the terms with x in the equation in Exercise 3?
 - The coefficients are the same.*

- What can you say about the constants in the equation?
 - The constants are the same.*
- We now know that there are equations that produce statements that are always true. That is, -3 will always equal -3 , 0 will always equal 0 , and $7x$ will always equal $7x$, no matter what number x happens to be (because 7 times a number is just a number). When we solve an equation using our properties of equality and end up with a statement like $a = a$ where a is a number, then we say that the equation has infinitely many solutions.
- To produce and/or recognize an equation with infinitely many solutions, the coefficients of the variable x and the value of the constants will be the same in the simplified expressions on both sides of the equal sign. This type of equation has infinitely many solutions because any value of x in an equation like $x + a = x + a$ will make the equation true.

Give students a moment to compare the equation $x + a = x + a$ with Exercise 3.

Activity: Display the table below. Provide students time to work independently to solve the equations; then, allow them to work in small groups to generalize what they notice about each of the equations. That is, they should answer the question: What can we see in an equation that will tell us about the solution to the equation?

$$\begin{aligned} 3x + 4 &= 8x - 9 \\ -4x - 5 &= 6 - 11x \\ 9 + \frac{1}{2}x &= 5x - 1 \end{aligned}$$

$$\begin{aligned} 6x + 5 &= 8 + 6x \\ 12 - 15x &= -2 - 15x \\ \frac{5}{4}x - 1 &= 1 + \frac{5}{4}x \end{aligned}$$

$$\begin{aligned} 10x - 4 &= -4 + 10x \\ -2x + 5 &= -2x + 5 \\ 7 + 9x &= 9x + 7 \end{aligned}$$

The goal of the activity is for students to make the following three observations. (1) In equations where both sides are simplified and the coefficients of x and the constants in the equation are unique, i.e., $ax + b = cx + d$, there will be a unique solution, i.e., $x = \frac{d-b}{a-c}$, as in the first column. (2) Similarly, we want students to observe that when an equation is of the form $x + b = x + c$, that there will be no solution, as in the second column. (3) Lastly, when an equation is of the form $x + a = x + a$, then there are infinitely many solutions, as in the third column.

Exercises 4–10 (14 minutes)

Students complete Exercises 4–10 independently or in pairs.

Give a brief explanation as to what kind of solution(s) you expect the following linear equations to have. Transform the equation into a simpler form if necessary.

4. $11x - 2x + 15 = 8 + 7 + 9x$

If I use the distributive property on the left side, I notice that the coefficients of the x are the same, specifically 9, and when I simplify the constants on the right side, I notice that they are the same. Therefore, this equation has infinitely many solutions.

5. $3(x - 14) + 1 = -4x + 5$

If I use the distributive property on the left side, I notice that the coefficients of x are different. Therefore, the equation has one solution.

6. $-3x + 32 - 7x = -2(5x + 10)$

If I use the distributive property on the each side of the equation, I notice that the coefficients of x are the same, but the constants are different. Therefore, this equation has no solutions.

7. $\frac{1}{2}(8x + 26) = 13 + 4x$

If I use the distributive property on the left side, I notice that the coefficients of x are the same, specifically 4, and the constants are also the same, 13. Therefore, this equation has infinitely many solutions.

8. Write two equations that have no solutions.

Answers will vary. Verify that students have written equations where the coefficients of x on each side of the equal sign are the same and that the constants on each side are unique.

9. Write two equations that have one unique solution each.

Answers will vary. Accept equations where the coefficients of x on each side of the equal sign are unique.

10. Write two equations that have infinitely many solutions.

Answers will vary. Accept equations where the coefficients of x and the constants on each side of the equal sign are the same.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know that equations will either have a unique solution, no solution, or infinitely many solutions.
- We know that equations with no solution will, after being simplified on both sides, have coefficients of x that are the same on both sides of the equal sign and constants that are different.
- We know that equations with infinitely many solutions will, after being simplified on both sides, have coefficients of x and constants that are the same on both sides of the equal sign.

Lesson Summary

There are three classifications of solutions to linear equations: one solution (unique solution), no solution, or infinitely many solutions.

Equations with no solution will, after being simplified, have coefficients of x that are the same on both sides of the equal sign and constants that are different. For example, $x + b = x + c$, where b and c are constants that are not equal. A numeric example is $8x + 5 = 8x - 3$.

Equations with infinitely many solutions will, after being simplified, have coefficients of x and constants that are the same on both sides of the equal sign. For example, $x + a = x + a$, where a is a constant. A numeric example is $6x + 1 = 1 + 6x$.

Exit Ticket (5 minutes)

Name _____

Date _____

Lesson 7: Classification of Solutions

Exit Ticket

Give a brief explanation as to what kind of solution(s) you expect the following linear equations to have. Transform the equation into a simpler form if necessary.

1. $3(6x + 8) = 24 + 18x$

2. $12(x + 8) = 11x - 5$

3. $5x - 8 = 11 - 7x + 12x$

Exit Ticket Sample Solutions

Give a brief explanation as to what kind of solution(s) you expect the following linear equations to have. Transform the equation into a simpler form if necessary.

1. $3(6x + 8) = 24 + 18x$

If I use the distributive property on the left side, I notice that the coefficients of x are the same, and the constants are the same. Therefore, this equation has infinitely many solutions.

2. $12(x + 8) = 11x - 5$

If I use the distributive property on the left side, I notice that the coefficients of x are different, and the constants are different. Therefore, this equation has a unique solution.

3. $5x - 8 = 11 - 7x + 12x$

If I collect the like terms on the right side, I notice that the coefficients of x are the same, but the constants are different. Therefore, this equation has no solution.

Problem Set Sample Solutions

Students apply their knowledge of solutions to linear equations by writing equations with unique solutions, no solutions, and infinitely many solutions.

1. Give a brief explanation as to what kind of solution(s) you expect for the linear equation $18x + \frac{1}{2} = 6(3x + 25)$. Transform the equation into a simpler form if necessary.

If I use the distributive property on the right side of the equation, I notice that the coefficients of x are the same, but the constants are different. Therefore, this equation has no solutions.

2. Give a brief explanation as to what kind of solution(s) you expect for the linear equation $8 - 9x = 15x + 7 + 3x$. Transform the equation into a simpler form if necessary.

If I collect the like terms on the right side of the equation, then I notice that the coefficients of x are different and so are the constants. Therefore, this equation will have a unique solution.

3. Give a brief explanation as to what kind of solution(s) you expect for the linear equation $5(x + 9) = 5x + 45$. Transform the equation into a simpler form if necessary.

This is an identity under the distributive property. Therefore, this equation will have infinitely many solutions.

4. Give three examples of equations where the solution will be unique, that is, only one solution is possible.

Accept equations where the coefficients of x on each side of the equal sign are unique.

5. Solve one of the equations you wrote in Problem 4, and explain why it is the only solution.

Verify that students solved one of the equations. They should have an explanation that includes the statement that there is only one possible number that could make the equation true. They may have referenced one of the simpler forms of their transformed equation to make their case.

6. Give three examples of equations where there will be no solution.

Accept equations where the coefficients of x on each side of the equal sign are the same, and the constants on each side are unique.

7. Attempt to solve one of the equations you wrote in Problem 6, and explain why it has no solution.

Verify that students have solved one of the equations. They should have an explanation that includes the statement about getting a false equation, e.g., $6 \neq 10$.

8. Give three examples of equations where there will be infinitely many solutions.

Accept equations where the coefficients of x and constants on each side of the equal sign are the same.

9. Attempt to solve one of the equations you wrote in Problem 8, and explain why it has infinitely many solutions.

Verify that students have solved one of the equations. They should have an explanation that includes the statement about the linear expressions being exactly the same, an identity; therefore, any rational number x would make the equation true.