Lesson 5: First Consequences of FTS

Student Outcomes

- Students verify the converse of the Fundamental Theorem of Similarity experimentally.
- Students apply the Fundamental Theorem of Similarity to find the location of dilated points on the plane.

Classwork

Concept Development (5 minutes)

Begin by having students restate (in their own words) the Fundamental Theorem of Similarity (FTS) that they learned in the last lesson.

- The Fundamental Theorem of Similarity states:

  Given a dilation with center O and scale factor r, then for any two points P and Q in the plane so that O, P, and Q are not collinear, the lines PQ and $P'Q'$ are parallel, where $P' = Dilation(P)$ and $Q' = Dilation(Q)$, and furthermore, $|P'Q'| = r|PQ|$.

The paraphrased version from the last lesson was: FTS states that given a dilation from center O, and points P and Q (points O, P, and Q are not on the same line), the segments formed when you connect P to Q and P' to Q' are parallel. More surprising is the fact that the segment PQ, even though it was not dilated as points P and Q were, dilates to segment $P'Q'$, and the length of segment $P'Q'$ is the length of segment PQ multiplied by the scale factor.

- The converse of this theorem is also true. That is, *if lines PQ and $P'Q'$ are parallel, and $|P'Q'| = r|PQ|$, then from a center O, $P' = Dilation(P)$, $Q' = Dilation(Q)$, and $|OP'| = r|OP|$ and $|OQ'| = r|OQ|$.*

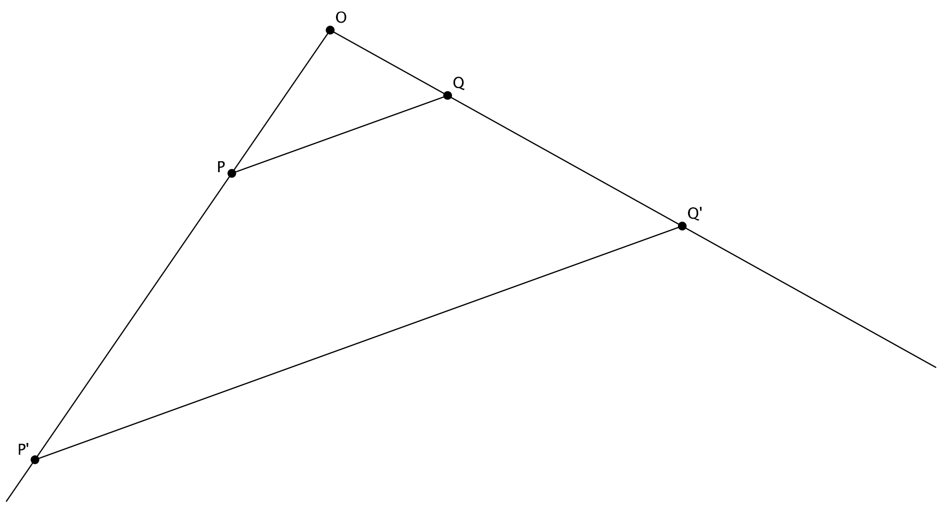
The converse of a theorem begins with the conclusion and produces the hypothesis. FTS concludes that lines are parallel and the length of segment $P'Q'$ is the length of segment PQ multiplied by the scale factor. The converse of FTS begins with the statement that the lines are parallel and the length of segment $P'Q'$ is the length of segment PQ multiplied by the scale factor. It ends with us knowing that the points P' and Q' are dilations of points P and Q by scale factor r, and their respective segments, OP' and OQ', have lengths that are the scale factor multiplied by the original lengths of segment OP and segment OQ. Consider providing students with some simple examples of converses, and discuss whether or not the converses are true. For example, "If it is raining, then I have an umbrella," and its converse, "If I have an umbrella, then it is raining." In this case, the converse is not necessarily true. An example where the converse is true is "If we turn the faucet on, then water comes out," and the converse is "If water comes out, then the faucet is on."

- The converse of the theorem is basically the work we did in the last lesson but backwards. In the last lesson, we knew about dilated points and found out that the segments between the points, P, Q, and their dilations P', Q', were dilated according the same scale factor, i.e., $|P'Q'| = r|PQ|$. We also discovered that the lines containing those segments were parallel, i.e., $PQ \parallel P'Q'$. The converse states that we are given parallel lines, PQ and $P'Q'$, where a segment, PQ, in one line is dilated by scale factor r to a segment, $P'Q'$, in the other line. With that knowledge, we can say something about the center of dilation and the relationship between segments OP' and OP, as well as segments OQ' and OQ.

In Exercise 1 below, students are given the information in the converse of FTS, i.e., $PQ \parallel P'Q'$ and $|P'Q'| = r|PQ|$. Students then verify the conclusion of the converse of FTS by measuring lengths of segments and their dilated images to make sure that they have the properties stated, i.e., $|OP'| = r|OP|$ and $|OQ'| = r|OQ|$.

Consider showing the diagram below and asking students to make conjectures about the relationship between segment OP' and segment OP, as well as segment OQ' and segment OQ. You can record the conjectures and possibly have the class vote on which they believe to be true.

In the diagram below, the lines containing segments PQ and $P'Q'$ are parallel, and the length of segment $P'Q'$ is equal to the length of segment PQ multiplied by the scale factor r, dilated from center O.

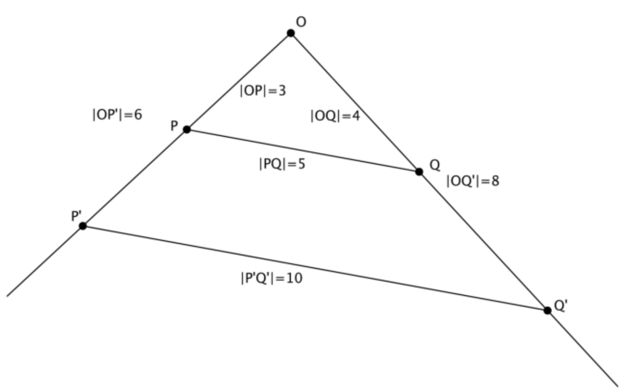


Exercise 1 (5 minutes)

Have students verify experimentally the validity of the converse of the theorem. They can work independently or in pairs. Note that the image in the Student Materials packet is to scale and should be printed at 100% scale to preserve the intended size of the figure for accurate measurements. Adjust your copier or printer settings to actual size and set page scaling to none. The image below is not to scale.

Exercise 1

In the diagram below, points P and Q have been dilated from center O by scale factor r. $PQ \parallel P'Q'$, $|PQ| = 5$ cm, and $|P'Q'| = 10$ cm.



Scaffolding:

If students need help getting started, have them review the activity from Lesson 4. Ask what they should do to find the center O. Students should say draw lines through PP' and QQ', the point of intersection is the center O.

a. **Determine the scale factor r.**

According to FTS, $|P'Q'| = r|PQ|$. Therefore, $10 = r \times 5$, so $r = 2$.

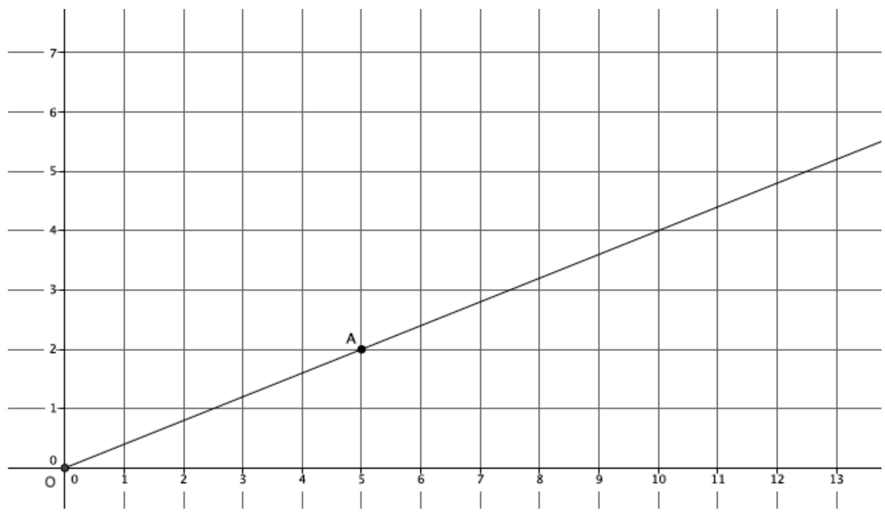
b. **Locate the center O of dilation. Measure the segments to verify that $|OP'| = r|OP|$ and $|OQ'| = r|OQ|$. Show your work below.**

Center O and measurements shown above.

$$|OP'| = r|OP| \qquad |OQ'| = r|OQ|$$
$$6 = 2 \times 3 \qquad 8 = 2 \times 4$$
$$6 = 6 \qquad 8 = 8$$

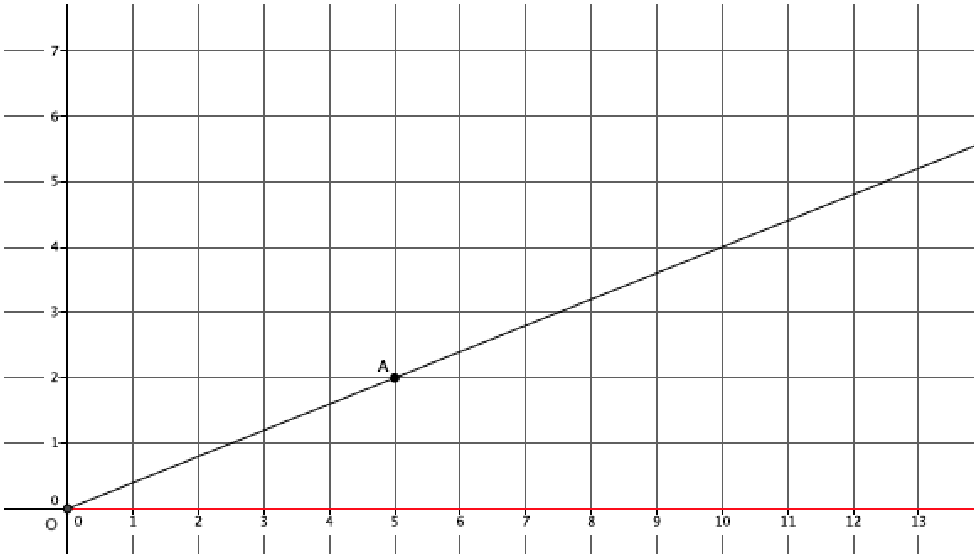
Example 1 (5 minutes)

- Now that we know FTS and the converse of FTS in terms of dilations, we will practice using them to find the coordinates of points and dilated points on a plane. We will begin simply.
- In the diagram, we have center O and ray $\overrightarrow{OA}$. We want to find the coordinates of point A'. We are given that the scale factor of dilation is $r = 2$.



- To find A' we could use a ruler or compass to measure $|OA|$, but now that we know about FTS, we can do this another way. First, we should look for parallel lines that will help us locate point A'. How can we use the coordinate plane to ensure parallel lines?
  - *We could use the vertical or horizontal lines of the coordinate plane to ensure lines are parallel. The coordinate plane is set up so that the lines never intersect. You could also think of those lines as translations of the x-axis and y-axis. Therefore, we are guaranteed to have parallel lines.*

- Let's use the x-axis as one of our rays. (Show picture below). Where should we place a point B, on the ray along the x-axis?



  - *Since we are using the lines on the coordinate plane to verify parallelism, we should place point B directly below point A, on the x-axis. Point B should be at $(5, 0)$.*

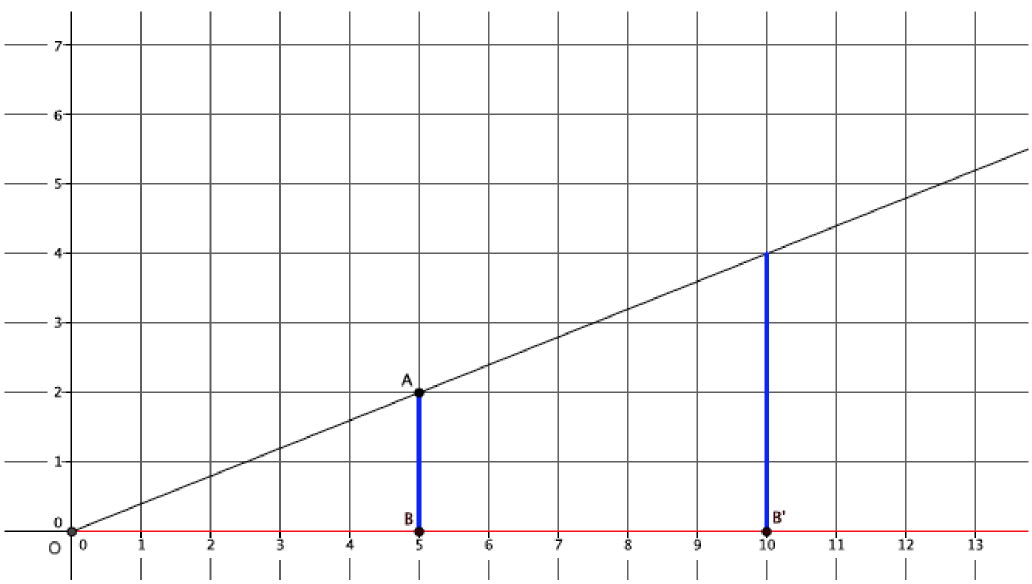
- (Show picture below.) This is beginning to look like the activity we did in Lesson 4. We know that that scale factor $r = 2$. Where should we put point B'?

  - *It is clear that the length of $|OB| = 5$; therefore, $|OB'| = 2 \times 5$, so the point B' should be placed at $(10, 0)$.*

- (Show picture below.) Now that we know the location of B', using FTS, what do we expect to be true about the lines containing segments AB and $A'B'$?

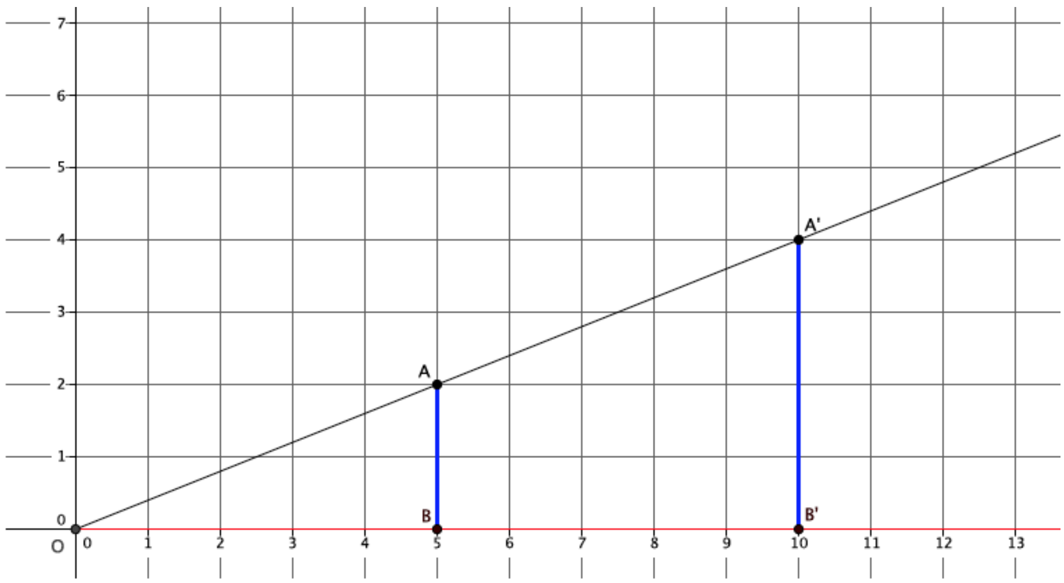
  - *We expect the lines containing segments AB and $A'B'$ to be parallel.*

- (Show picture below.) Then what is the location of point A'?



  - *Point A' will be located at $(10, 4)$.*

- (Show picture below.) Could point A' be located anywhere else? Specifically, could A' have a different x-coordinate? Why or why not?



  - *No. Point A' must be at $(10, 4)$. If it had another x-coordinate, then lines AB and $A'B'$ would have not been dilated by the scale factor $r = 2$.*

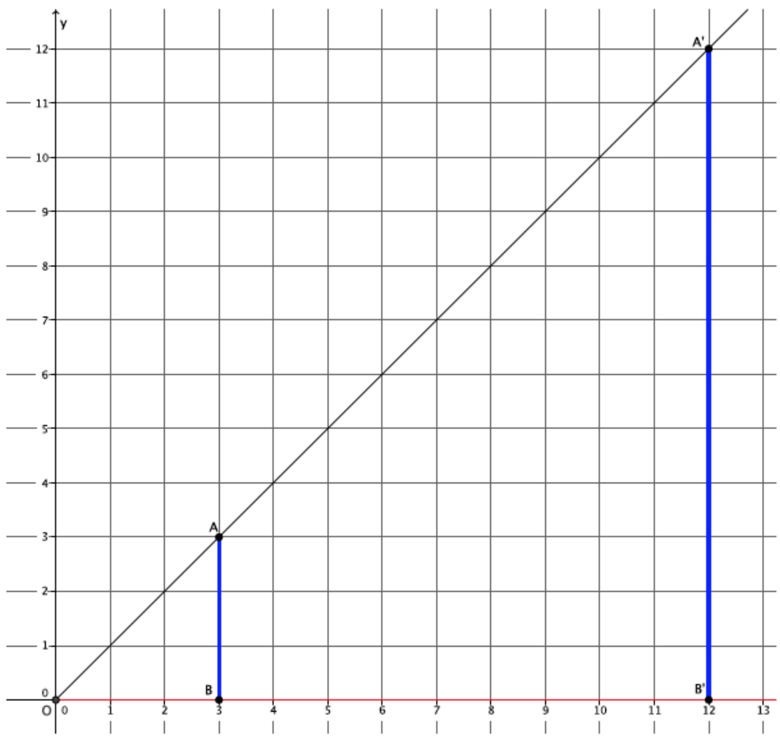
- Could point A' be located at another location that has 10 as its x-coordinate? For example, could A' be at $(10, 5)$? Why or why not?
  - *No. Point A' must be at $(10, 4)$. By definition of dilation, if point A is dilated from center O, then the dilated point must be on the ray $\overrightarrow{OA}$, making points O, A, and A' collinear. If A' were at $(10, 5)$ or at any coordinate other than $(10, 4)$, then the points O, A, and A' would not be collinear.*

Exercise 2 (3 minutes)

Students work independently to find the location of point A' on the coordinate plane.

Exercise 2

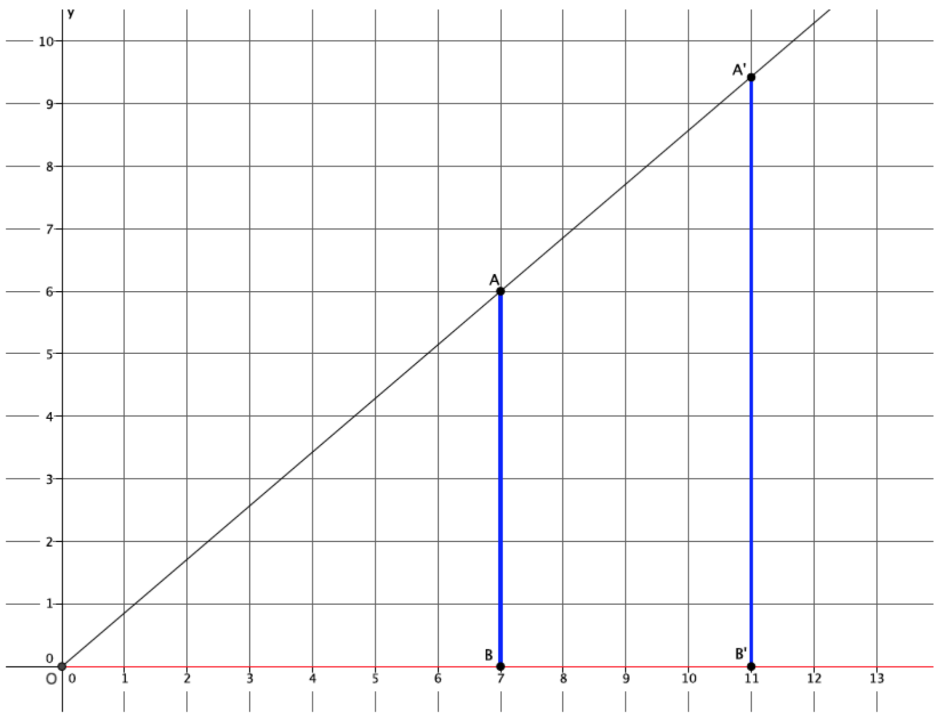
In the diagram below, you are given center O and ray $\overrightarrow{OA}$. Point A is dilated by a scale factor $r = 4$. Use what you know about FTS to find the location of point A'.



Point A' must be located at $(12, 12)$.

Example 2 (6 minutes)

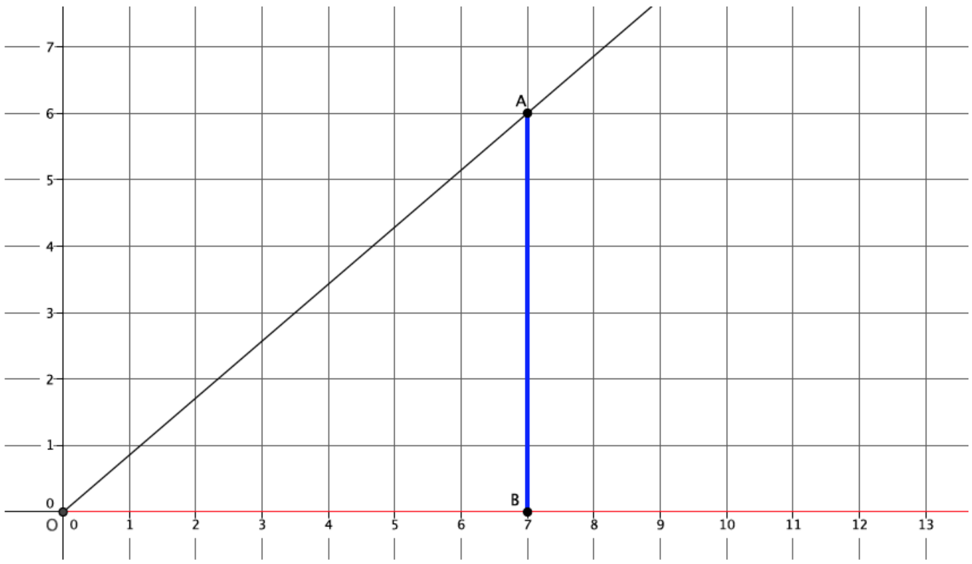
- In the diagram we have center O and ray $\overrightarrow{OA}$. We are given that the scale factor of dilation is $r = \frac{11}{7}$. We want to find the precise coordinates of point A'. Based on our previous work, we know exactly how to begin. Draw a ray $\overrightarrow{OB}$ along the x-axis and mark a point B, directly below point A, on the ray $\overrightarrow{OB}$. (Show picture below.) The question now is how do we locate B'? Think about our work from Lesson 4.



- In Lesson 4, we counted lines to determine the scale factor. Given the scale factor, we know that the point B should be exactly 7 lines from the center O, and that the dilated point, B', should be exactly 11 lines from the center. Therefore, B' is located at $(11, 0)$.

- Now that we know the location of B', where will we find A'?
  - *Point A' will be at the intersection of the ray $\overrightarrow{OA}$, and the line $A'B'$, which must be parallel to line AB.*

- (Show picture below.) Now that we know where A' is we need to find the precise coordinates of it. The x-coordinate is easy, but how can we find the y-coordinate? (Give students time to talk in pairs or small groups.)
  - *The y-coordinate will be the exact length of the segment $A'B'$. To find that length, we can use what we know about the length of segment AB and the scale factor since $|A'B'| = r|AB|$.*



- The length of segment $A'B'$ will give us the y-coordinate. Then $|A'B'| = \frac{11}{7} \times 6 = \frac{66}{7}$. That means that the location of A' is $\left(11, \frac{66}{7}\right)$.

Exercise 3 (4 minutes)

Students work independently or in pairs to find the location of point A' on the coordinate plane.

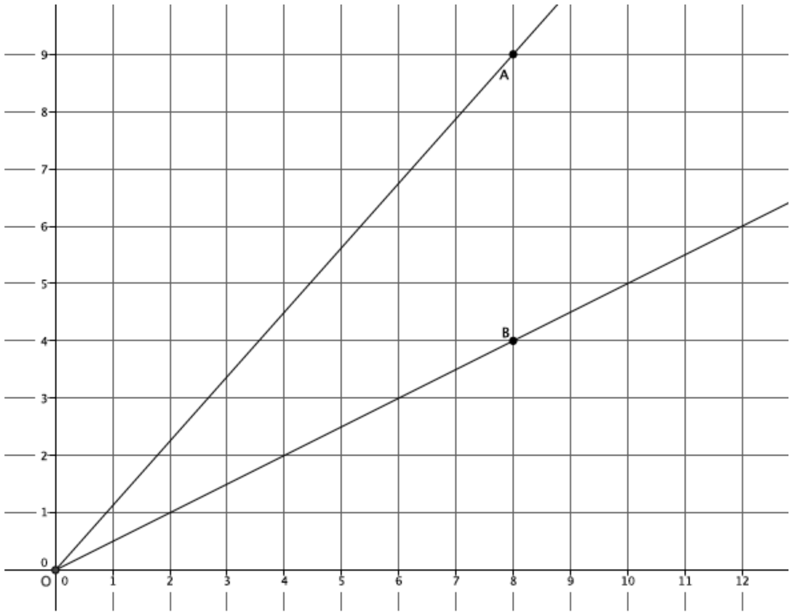
Exercise 3

In the diagram below, you are given center O and ray $\overrightarrow{OA}$. Point A is dilated by a scale factor $r = \frac{5}{12}$. Use what you know about FTS to find the location of point A'.

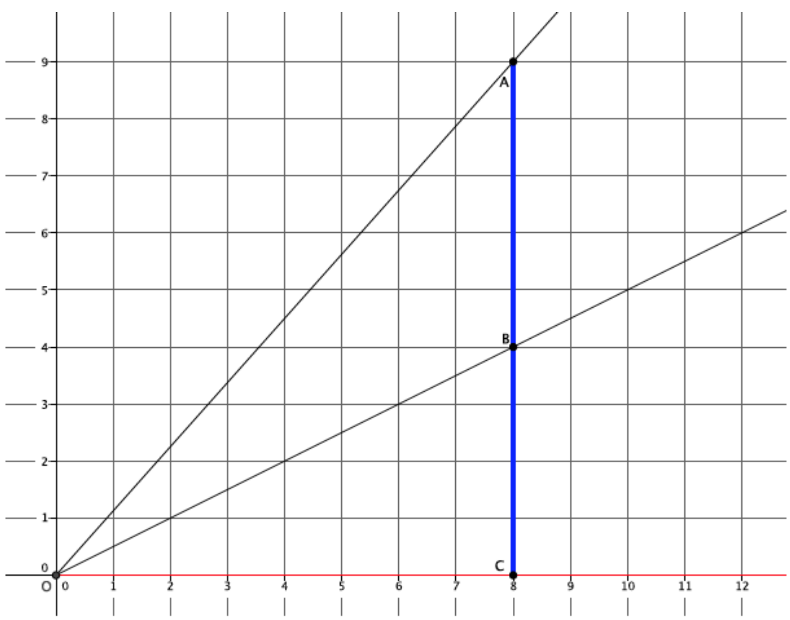
The x-coordinate of A' is 5. The y-coordinate will be equal to the length of segment $A'B'$. Since $|A'B'| = r|AB|$, then $|A'B'| = \frac{5}{12} \times 8 = \frac{40}{12} \approx 3.3$. The location of A' is $(5, 3.3)$.

Example 3 (8 minutes)

- In the diagram below we have center O and rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$. We are given that the scale factor is $r = \frac{5}{8}$. We want to find the precise coordinates of points A' and B'.

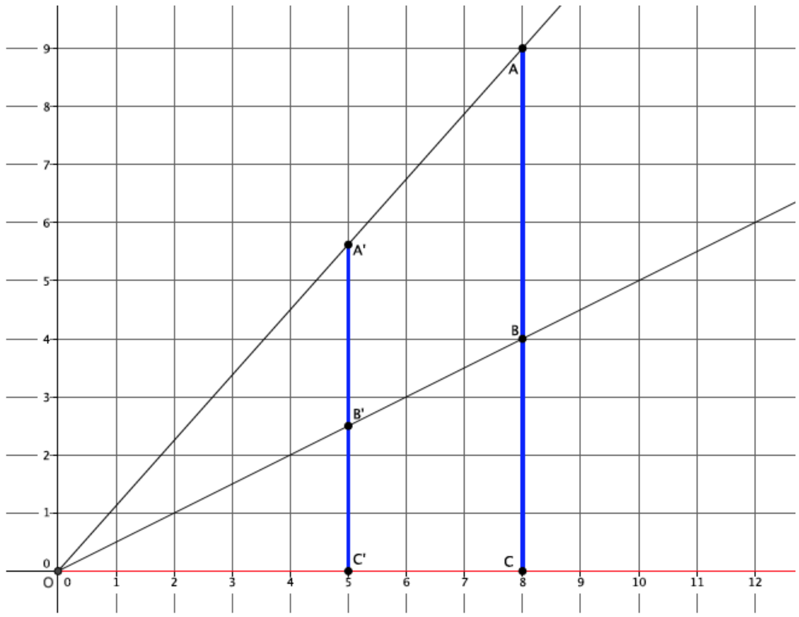


- Based on our previous work, we know exactly how to begin. Describe what we should do.
  - *Draw a ray $\overrightarrow{OC}$ along the x-axis, and mark a point C, directly below point A, on the ray $\overrightarrow{OC}$.*
- (Show picture below.) Based on our work in Lesson 4 and our knowledge of FTS, we know that points B and B' along ray $\overrightarrow{OB}$ enjoy the same properties we have been using in the last few problems. Let's begin by finding the coordinates of A'. What should we do first?



- *First, we need to place the points A', B', and C' on their respective rays by using the scale factor. Since the scale factor $r = \frac{5}{8}$, A', B', and C' will have an x-coordinate of 5. (Also, they are 5 lines from the center.)*

- (Show picture below.) Let's first find the coordinates of A'. What do we need to do to find the y-coordinate of A'?



- *The y-coordinate of A' will be the length of the segment $A'C'$. We can calculate that length by using what we know about segment AC and the scale factor.*

- The y-coordinate of A' is $|A'C'| = r|AC| = \frac{5}{8} \times 9 = \frac{45}{8} \approx 5.6$. Then the location of A' is $(5, 5.6)$.
- Work with a partner to find the y-coordinate of B'.
  - *The y-coordinate of B' is the length of segment $B'C'$. Then $|B'C'| = r|BC| = \frac{5}{8} \times 4 = \frac{20}{8} = 2.5$. The* location of B' is $(5, 2.5)$.

Closing (4 minutes)

Summarize, or ask students to summarize, the main points from the lesson.

- We experimentally verified the converse of FTS. That is, if we are given parallel lines PQ and $P'Q'$ and know $|P'Q'| = r|PQ|$, then we know from a center O, $P' = Dilation(P)$, $Q' = Dilation(Q)$, $|OP'| = r|OP|$, and $|OQ'| = r|OQ|$. In other words, if we are given parallel lines PQ and $P'Q'$, and we know that the length of segment $P'Q'$ is equal to the length of segment PQ multiplied by the scale factor, then we also know that the length of segment OP' is equal to the length of segment OP multiplied by the scale factor and that the length of segment OQ' is equal to the length of segment OQ multiplied by the scale factor.
- We know how to use FTS to find the coordinates of dilated points, even if the dilated point is not on an intersection of graph lines.

Lesson Summary

Converse of the Fundamental Theorem of Similarity:

If lines PQ and $P'Q'$ are parallel, and $|P'Q'| = r|PQ|$, then from a center O, $P' = Dilation(P)$, $Q' = Dilation(Q)$, $|OP'| = r|OP|$, and $|OQ'| = r|OQ|$.

To find the coordinates of a dilated point, we must use what we know about FTS, dilation, and scale factor.

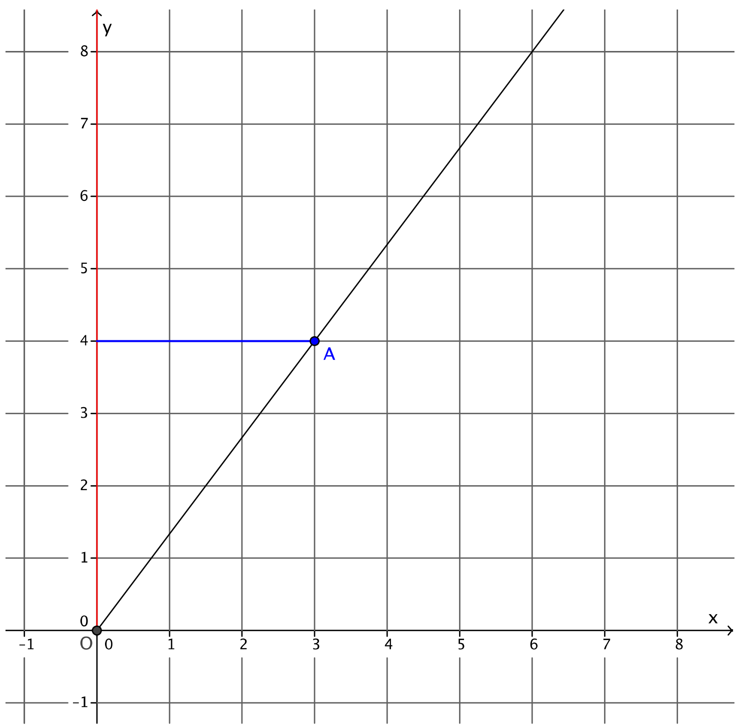
Exit Ticket (5 minutes)

Name ____ Date____________________

Lesson 5: First Consequences of FTS

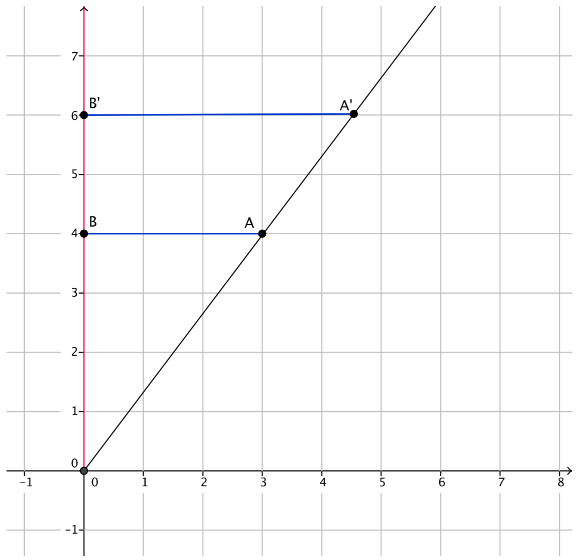
Exit Ticket

In the diagram below, you are given center O and ray $\overrightarrow{OA}$. Point A is dilated by a scale factor $r = \frac{6}{4}$. Use what you know about FTS to find the location of point A'.



Exit Ticket Sample Solutions

In the diagram below, you are given center O and ray $\overrightarrow{OA}$. Point A is dilated by a scale factor $r = \frac{6}{4}$. Use what you know about FTS to find the location of point A'.

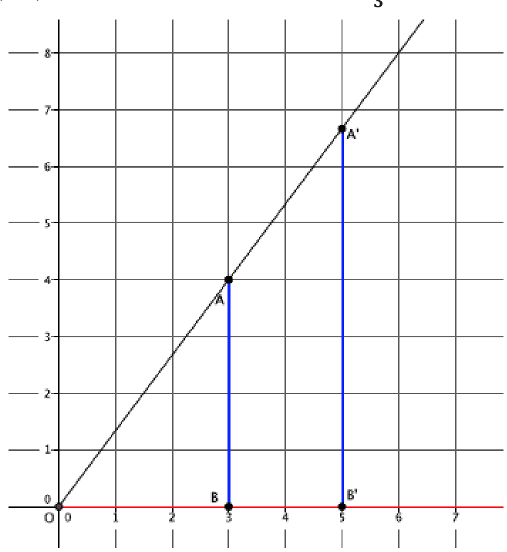


The y-coordinate of A' is 6. The x-coordinate will be equal to the length of segment $A'B'$. Since $|A'B'| = r|AB|$, then $|A'B'| = \frac{6}{4} \times 3 = \frac{18}{4} = 4.5$. The location of A' is $(4.5, 6)$.

Problem Set Sample Solutions

Students practice using the first consequences of FTS in terms of dilated points and their locations on the coordinate plane.

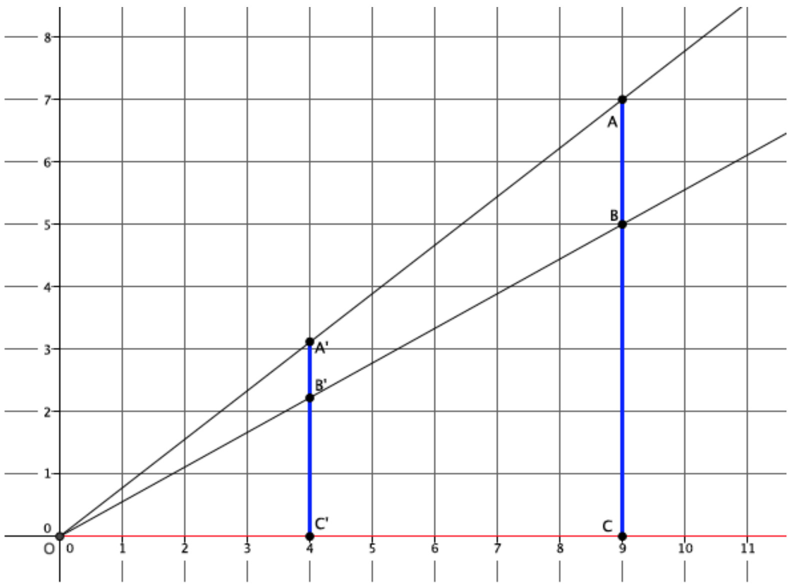
1. **Dilate point A, located at $(3, 4)$ from center O, by a scale factor $r = \frac{5}{3}$.**



What is the precise location of point A'?

The y-coordinate of point A' will be the length of segment $A'B'$. Since $|A'B'| = r|AB|$, then $|A'B'| = \frac{5}{3} \times 4 = \frac{20}{3}$. The location of point A' is $\left(5, \frac{20}{3}\right)$, or approximately $(5, 6.7)$.

2. Dilate point A, located at $(9, 7)$ from center O, by a scale factor $r = \frac{4}{9}$. Then dilate point B, located at $(9, 5)$ from center O, by a scale factor of $r = \frac{4}{9}$. What are the coordinates of A' and B'? Explain.



The y-coordinate of point A' will be the length of $A'C'$. Since $|A'C'| = r|AC|$, then $|A'C'| = \frac{4}{9} \times 7 = \frac{28}{9}$. The location of point A' is $\left(4, \frac{28}{9}\right)$, or approximately $(4, 3.1)$. The y-coordinate of point B' will be the length of $B'C'$. Since $|B'C'| = r|BC|$, then $|B'C'| = \frac{4}{9} \times 5 = \frac{20}{9}$. The location of point B' is $\left(4, \frac{20}{9}\right)$, or approximately $(4, 2.2)$.

3. Explain how you used the Fundamental Theorem of Similarity in Problems 1 and 2.

Using what I knew about scale factor, I was able to determine the placement of points A' and B', but I did not know the actual coordinates. So, one of the ways that FTS was used was actually in terms of the converse of FTS. I had to make sure I had parallel lines. Since the lines of the coordinate plane guarantee parallel lines, I knew that $|A'C'| = r|AC|$. Then, since I knew the length of segment AC and the scale factor, I could find the precise location of A'. The precise location of B' was found in a similar way but using $|B'C'| = r|BC|$.