Lesson 6: Rotations of 180 Degrees

Student Outcomes

- Students learn that a rotation of 180 degrees moves a point on the coordinate plane (a, b) to $(-a, -b)$.
- Students learn that a rotation of 180 degrees around a point, not on the line, produces a line parallel to the given line.

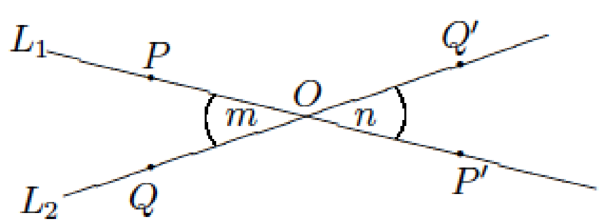
Classwork

Example 1 (5 minutes)

- Rotations of 180 degrees are special. Recall, a rotation of 180 degrees around O is a rigid motion so that if P is any point in the plane, P, O, and $Rotation(P)$ are collinear (i.e., they lie on the same line).
- Rotations of 180 degrees occur in many situations. For example, the frequently cited fact that vertical angles [vert. $\angle s$] at the intersection of two lines are equal, follows immediately from the fact that 180-degree rotations are angle-preserving. More precisely, let two lines L_1 and L_2 intersect at O, as shown:

Example 1

The picture below shows what happens when there is a rotation of $180°$ around center O.



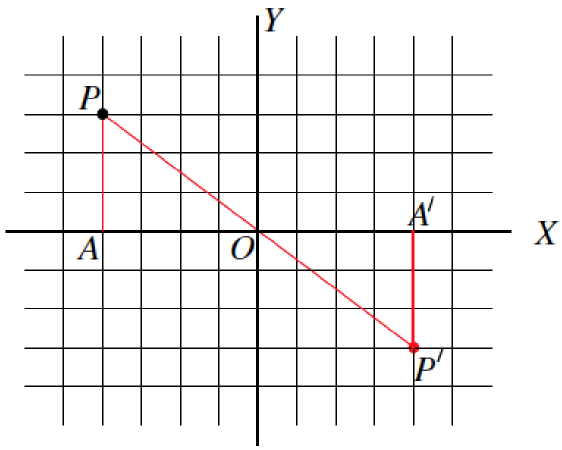
- We want to show that the vertical angles [vert. $\angle s$], $\angle m$ and $\angle n$, are equal (i.e., $\angle m = \angle n$). If we let Rotation_0 be the 180-degree rotation around O, then $Rotation_0$ maps $\angle m$ to $\angle n$. More precisely, if P and Q are points on L_1 and L_2, respectively (as shown above), let $Rotation_0(P) = P'$ and $Rotation_0(Q) = Q'$. Then, $Rotation_0$ maps $\angle POQ$ ($\angle m$) to $\angle Q'OP'$ ($\angle n$), and since $Rotation_0$ is angle-preserving, we have $\angle m = \angle n$.

Example 2 (5 minutes)

- Let's look at a 180-degree rotation, $Rotation_0$ around the origin O of a coordinate system. If a point P has coordinates (a, b), it is generally said that $Rotation_0(P)$ is the point with coordinates $(-a, -b)$.
- Suppose the point P has coordinates $(-4, 3)$; we will show that the coordinates of $Rotation_0(P)$ are $(4, -3)$.

Example 2

The picture below shows what happens when there is a rotation of $180°$ around center O, the origin of the coordinate plane.



- Let $P' = Rotation_0(P)$. Let the *vertical line* (i.e., the line parallel to the y-axis) through P meet the x-axis at a point A. Because the coordinates of P are $(-4, 3)$, the point A has coordinates $(-4, 0)$ by the way coordinates are defined. In particular, A is of distance 4 from O, and since $Rotation_0$ is length-preserving, the point $A' = Rotation_0(A)$ is also of distance 4 from O. However, $Rotation_0$ is a 180-degree rotation around O, so A' also lies on the x-axis but on the opposite side of the x-axis from A. Therefore, the coordinates of A' are $(4,0)$. Now, $\angle PAO$ is a right angle and—since $Rotation_0$ maps it to $\angle P'A'O$, and also preserves degrees—we see that $\angle P'A'O$ is also a right angle. This means that A' is the point of intersection of the vertical line through P' and the x-axis. Since we already know that A' has coordinates of $(4, 0)$, then the x-coordinate of P' is 4, by definition.
- Similarly, the y-coordinate of P being 3 implies that the y-coordinate of P' is – 3. Altogether, we have proved that the 180-degree rotation of a point of coordinates $(-4, 3)$ is a point with coordinates $(4, -3)$.

The reasoning is perfectly general: The same logic shows that the 180-degree rotation around the origin of a point of coordinates (a, b) is the point with coordinates $(-a, -b)$, as desired.

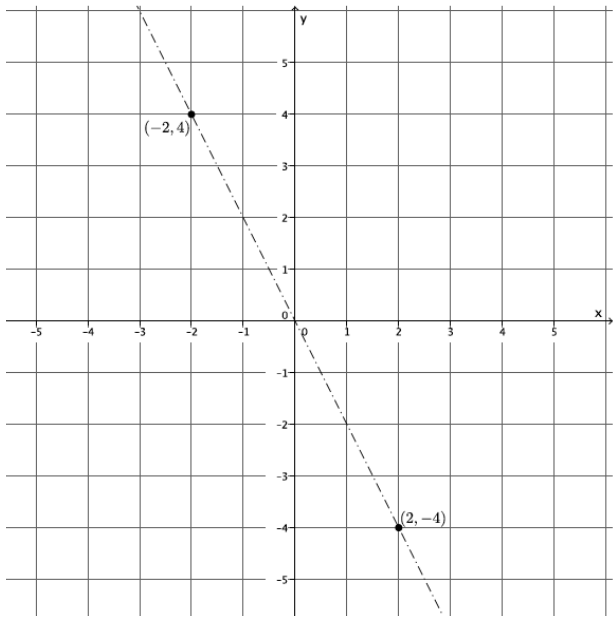
Exercises 1–9 (16 minutes)

Students complete Exercises 1–2 independently. Check solutions. Then, let students work in pairs on Exercises 3–4. Students complete Exercises 5–9 independently in preparation for the example that follows.

Exercises 1–9

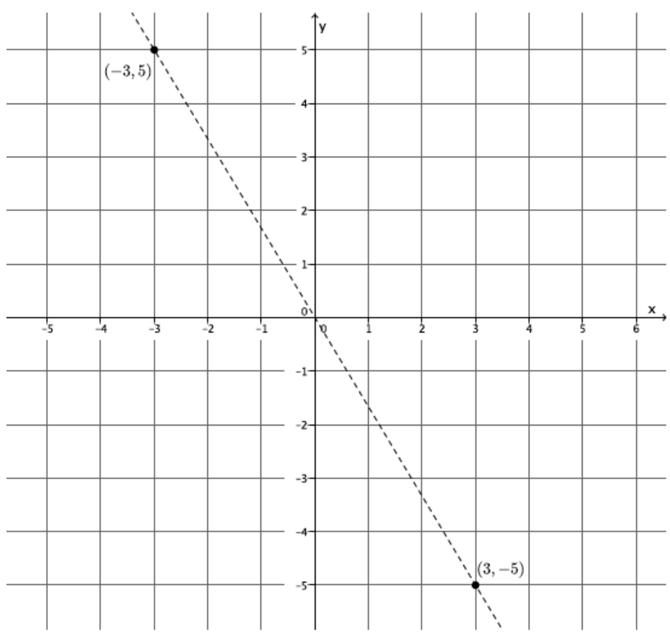
1. **Using your transparency, rotate the plane 180 degrees, about the origin. Let this rotation be $Rotation_0$. What are the coordinates of $Rotation_0(2, -4)$?**

   $Rotation_0(2, -4) = (-2, 4)$.

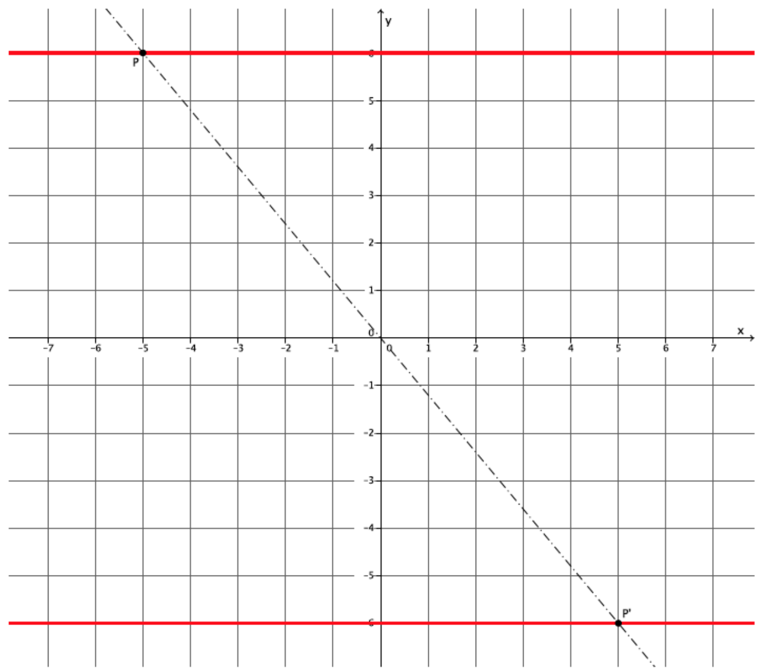


2. **Let $Rotation_0$ be the rotation of the plane by 180 degrees, about the origin. <u>Without</u> using your transparency, find $Rotation_0(-3, 5)$.**

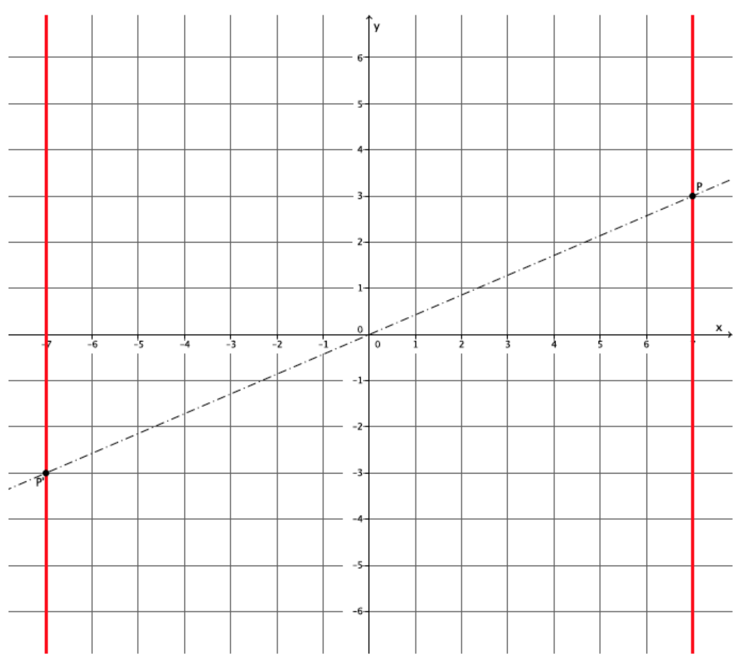
   $Rotation_0(-3, 5) = (3, -5)$.



3. Let $Rotation_0$ be the rotation of 180 degrees around the origin. Let L be the line passing through $(-6, 6)$ parallel to the x-axis. Find $Rotation_0(L)$. Use your transparency if needed.

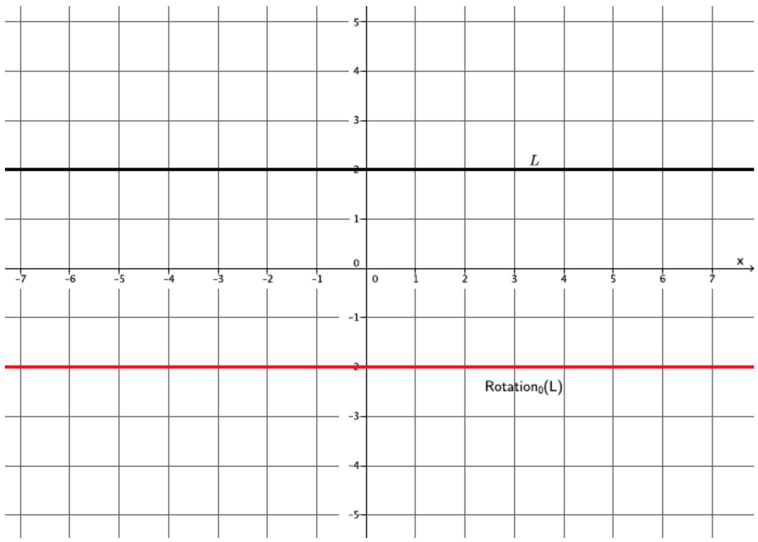


4. Let $Rotation_0$ be the rotation of 180 degrees around the origin. Let L be the line passing through $(7, 0)$ parallel to the y-axis. Find $Rotation_0(L)$. Use your transparency if needed.



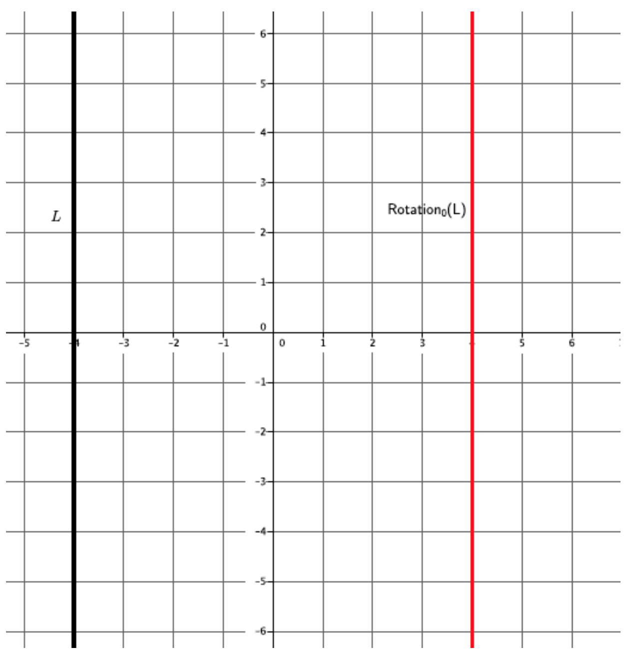
5. Let $Rotation_0$ be the rotation of 180 degrees around the origin. Let L be the line passing through $(0, 2)$ parallel to the x-axis. Is L parallel to $Rotation_0(L)$?

Yes, $L \parallel Rotation_0(L)$.



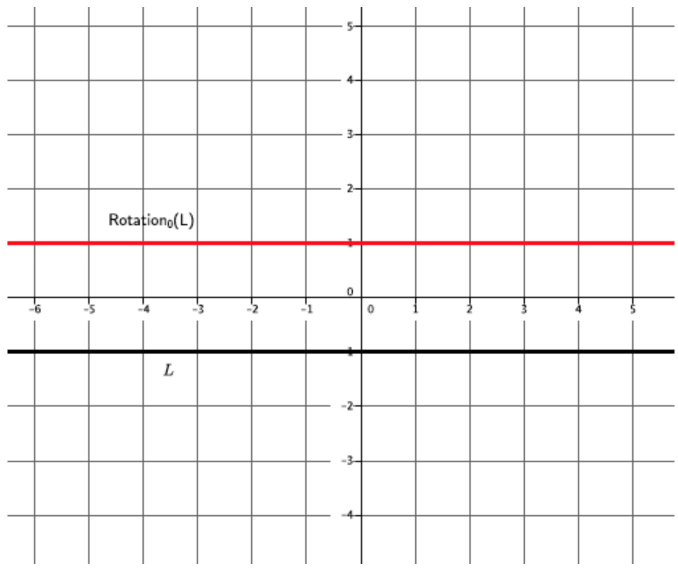
6. Let $Rotation_0$ be the rotation of 180 degrees around the origin. Let L be the line passing through $(4, 0)$ parallel to the y-axis. Is L parallel to $Rotation_0(L)$?

Yes, $L \parallel Rotation_0(L)$.



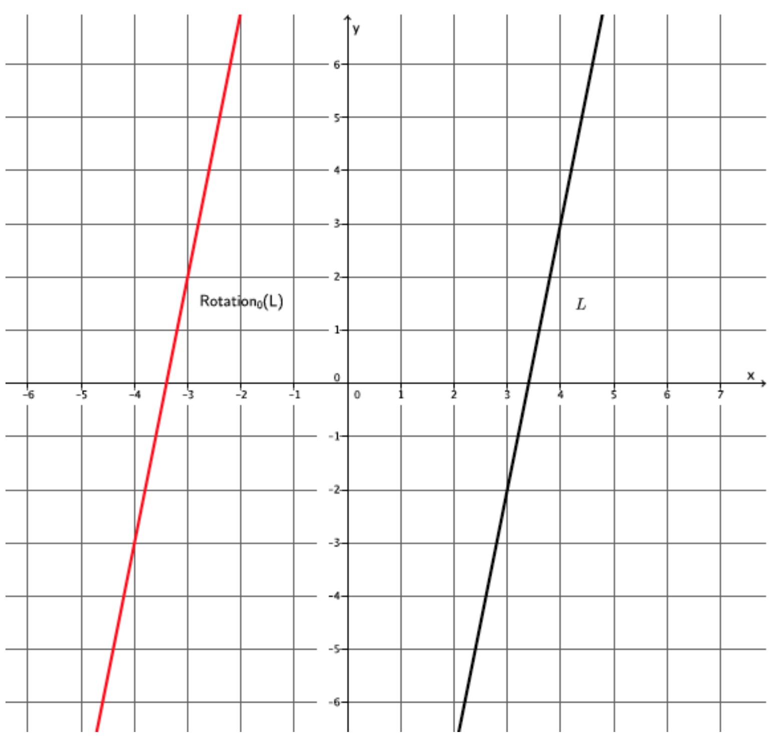
7. Let $Rotation_0$ be the rotation of 180 degrees around the origin. Let L be the line passing through $(0, -1)$ parallel to the x-axis. Is L parallel to $Rotation_0(L)$?

Yes, $L \parallel Rotation_0(L)$.



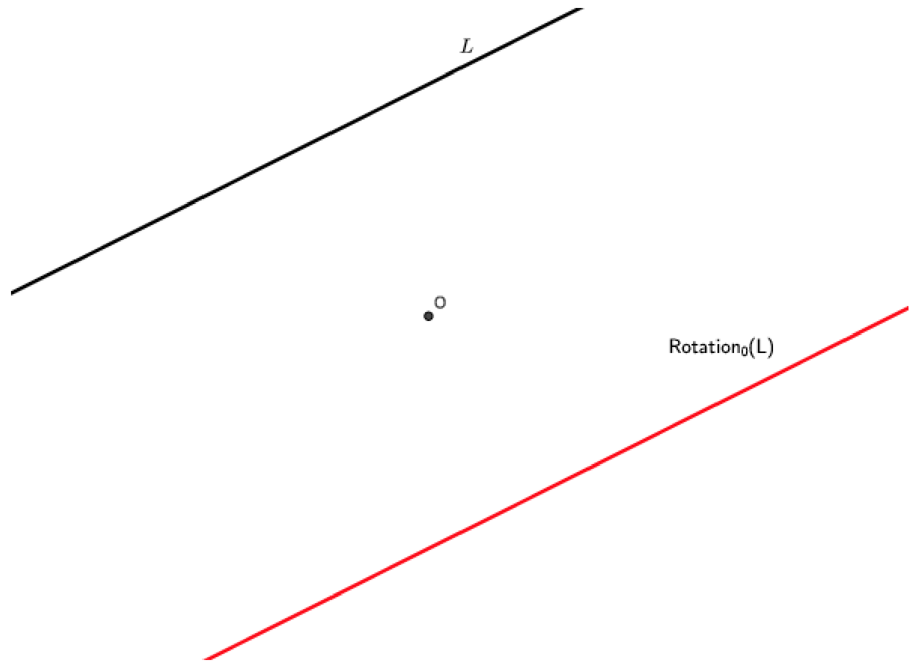
8. Let $Rotation_0$ be the rotation of 180 degrees around the origin. Is L parallel to $Rotation_0(L)$? Use your transparency if needed.

Yes, $L \parallel Rotation_0(L)$.



9. Let $Rotation_0$ be the rotation of 180 degrees around the origin. Is L parallel to $Rotation_0(L)$? Use your transparency if needed.

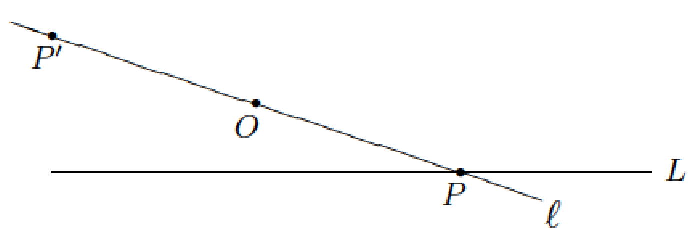
Yes, $L \parallel Rotation_0(L)$.



Example 3 (5 minutes)

MP.2 **Theorem**. Let O be a point not lying on a given line L. Then, the 180-degree rotation around O maps L to a line parallel to L.

Proof: Let $Rotation_0$ be the 180-degree rotation around O, and let P be a point on L. As usual, denote $Rotation_0(P)$ by P'. Since $Rotation_0$ is a 180-degree rotation, P, O, P' lie on the same line (denoted by ℓ).



Scaffolding:

After completing Exercises 5–9, students should be convinced that the theorem is true. Make it clear that their observations can be proven (by contradiction) if we assume something different will happen (e.g., the lines will intersect).

We want to investigate whether P' lies on L or not. Keep in mind that we want to show that the 180-degree rotation maps L to a line parallel to L. If the point P' lies on L, then at some point, the line L and $Rotation_0(L)$ intersect, meaning they are not parallel. If we can eliminate the possibility that P' lies on L, then we have to conclude that P' does not lie on L (rotations of 180 degrees make points that are collinear). If P' lies on L, then ℓ is a line that joins two points, P' and P, on L. However L is already a line that joins P' and P, so ℓ and L must be the same line (i.e., $\ell = L$). This is trouble because we know O lies on ℓ, so $\ell = L$ implies that O lies on L. Look at the hypothesis of the theorem: "Let O be a point not lying on a given line L." We have a contradiction. So, the possibility that P' lies on L is nonexistent. As we said, this means that P' does not lie on L.

What we have proved is that no matter which point P we take from L, we know $Rotation_0(P)$ does not lie on L. But $Rotation_0(L)$ consists of *all* the points of the form $Rotation_0(P)$ where P lies on L, so what we have proved is that no point of $Rotation_0(L)$lies on L. In other words, L and $Rotation_0(L)$ have no point in common (i.e., $L \parallel Rotation_0(L)$). The theorem is proved.

Closing (5 minutes)

Summarize, or have students summarize, the lesson.

- Rotations of 180 degrees are special:
  - A point, P, that is rotated 180 degrees around a center O, produces a point P' so that P, O, P' are collinear.
  - When we rotate around the origin of a coordinate system, we see that the point with coordinates (a, b) is moved to the point $(-a, -b)$.
- We now know that when a line is rotated 180 degrees around a point not on the line, it maps to a line parallel to the given line.

Lesson Summary

- **A rotation of 180 degrees around O is the rigid motion so that if P is any point in the plane, P, O, and $Rotation(P)$ are *collinear* (i.e., lie on the same line).**
- **Given a 180-degree rotation, R_0 around the origin O of a coordinate system, and a point P with coordinates (a, b), it is generally said that $R_0(P)$ is the point with coordinates $(-a, -b)$.**

Theorem: Let O be a point not lying on a given line L. Then, the 180-degree rotation around O maps L to a line parallel to L.

Exit Ticket (5 minutes)

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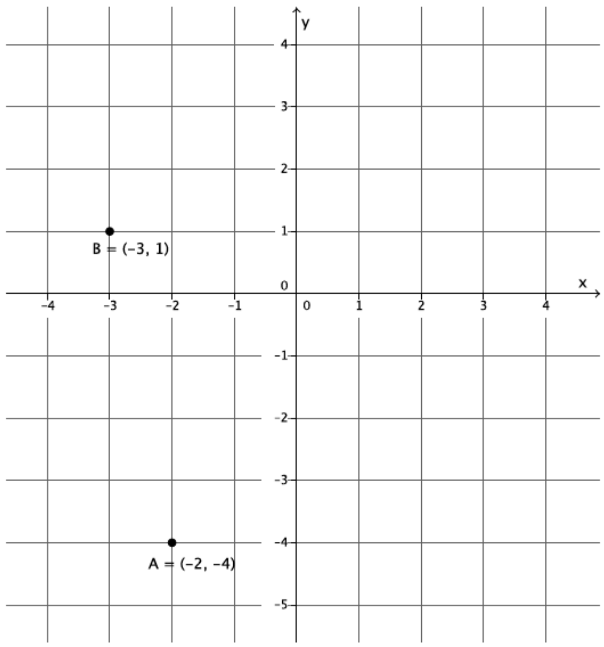
Lesson 6: Rotations of 180 Degrees

Exit Ticket

Let there be a rotation of 180 degrees about the origin. Point A has coordinates $(-2, -4)$, and point B has coordinates $(-3, 1)$, as shown below.



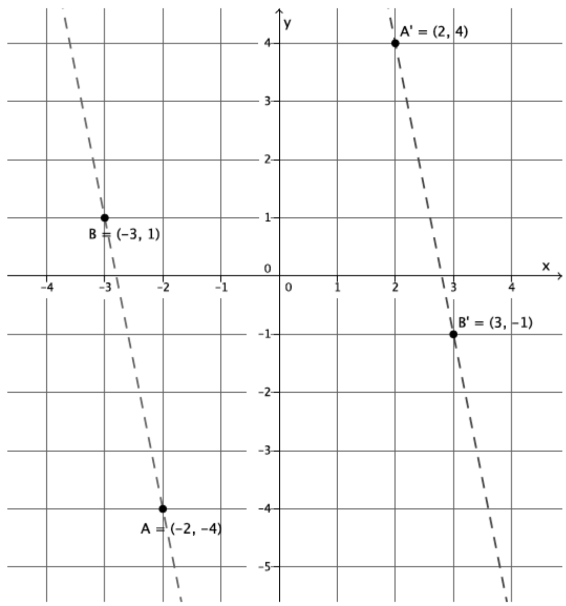
1. What are the coordinates of $Rotation(A)$? Mark that point on the graph so that $Rotation(A) = A'$. What are the coordinates of $Rotation(B)$? Mark that point on the graph so that $Rotation(B) = B'$.

2. What can you say about the points A, A', and O? What can you say about the points B, B', and O?

3. Connect point A to point B to make the line L_{AB}. Connect point A' to point B' to make the line $L_{A'B'}$. What is the relationship between L_{AB} and $L_{A'B'}$?

Exit Ticket Sample Solutions

Let there be a rotation of 180 degrees about the origin. Point A has coordinates $(-2,-4)$, and point B has coordinates $(-3,1)$, as shown below.



1. What are the coordinates of $Rotation(A)$? Mark that point on the graph so that $Rotation(A) = A'$. What are the coordinates of $Rotation(B)$? Mark that point on the graph so that $Rotation(B) = B'$.

   $A' = (2,4)$, $B' = (3,-1)$

2. What can you say about the points A, A', and O? What can you say about the points B, B', and O?

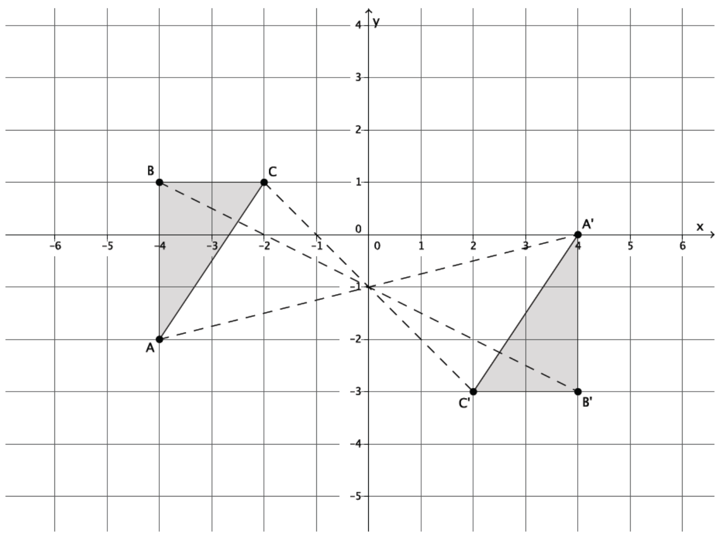
   The points A, A', and O are collinear. The points B, B', and O are collinear.

3. Connect point A to point B to make the line L_{AB}. Connect point A' to point B' to make the line $L_{A'B'}$. What is the relationship between L_{AB} and $L_{A'B'}$?

   $L_{AB} \parallel L_{A'B'}$.

Problem Set Sample Solutions

Use the following diagram for Problems 1–5. Use your transparency as needed.



1. **Looking only at segment BC, is it possible that a $180°$ rotation would map BC onto $B'C'$? Why or why not?**

   It is possible because the segments are parallel.

2. **Looking only at segment AB, is it possible that a $180°$ rotation would map AB onto $A'B'$? Why or why not?**

   It is possible because the segments are parallel.

3. **Looking only at segment AC, is it possible that a $180°$ rotation would map AC onto $A'C'$? Why or why not?**

   It is possible because the segments are parallel.

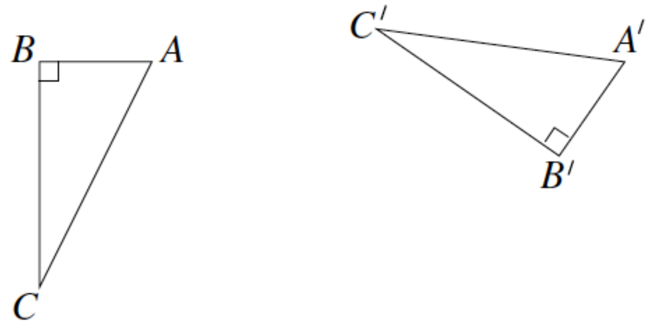
4. **Connect point B to point B', point C to point C', and point A to point A'. What do you notice? What do you think that point is?**

   All of the lines intersect at one point. The point is the center of rotation, I checked by using my transparency.

5. **Would a rotation map triangle ABC onto triangle $A'B'C'$? If so, define the rotation (i.e., degree and center). If not, explain why not.**

   Let there be a rotation $180°$ around point $(0, -1)$. Then, $Rotation(\Delta ABC) = \Delta A'B'C'$.

6. The picture below shows right triangles ABC and $A'B'C'$, where the right angles are at B and B'. Given that $AB = A'B' = 1$, and $BC = B'C' = 2$, and that AB is not parallel to $A'B'$, is there a $180°$ rotation that would map $\triangle ABC$ onto $\triangle A'B'C'$? Explain.



No, because a $180°$ rotation of a segment will map to a segment that is parallel to the given one. It is given that AB is not parallel to $A'B'$; therefore, a rotation of $180°$ will not map $\triangle ABC$ onto $\triangle A'B'C'$.