Lesson 12: From Unit Cubes to the Formulas for Volume

Student Outcomes

- Students extend the volume formula for a right rectangular prism to the formula $V = \text{Area of base} \cdot \text{height}$. They understand that any face can be the base.

Lesson Notes

This lesson is a continuation of the ideas in Lesson 11 and the lessons in Grade 5, Module 5, Topics A and B.

The word *face*, though referenced in the last lesson, should be taught to students who may not know this meaning of it. A student-friendly definition and illustration can be posted on the wall (along with definitions of *edge(s)* and *vertex/vertices*). Here is a link to a useful illustration: http://www.11plusforparents.co.uk/Maths/shape8.html.

Classwork

Example 1 (10 minutes)

> *Scaffolding:*
> You may want to use unit cubes to help students visualize the layers in this problem.

- Look at the rectangular prisms in the first example. Write a numerical expression for the volume of each rectangular prism.
  - *Answers provided below.*
- What do these expressions have in common?
  - *They have the same dimensions for the lengths and widths.*
- What do these dimensions represent?
  - *They represent the area of the bases of the rectangular prisms.*

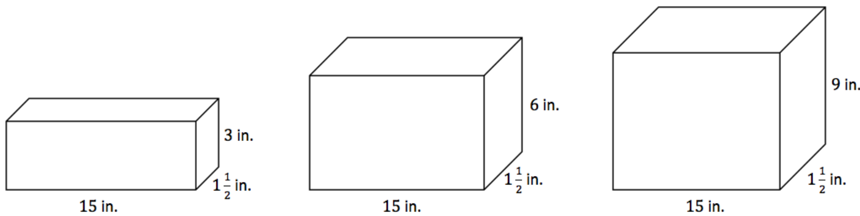
MP.7

- Rewrite each of the numerical expressions to show what they have in common.
  - *Answers provided below.*

MP.8

- If we know volume for a rectangular prism as length times width times height, what is another formula for volume that we could use based on these examples?
  - *We could use area of the base times the height.*
- What is the area of the base of each of the rectangular prisms?
  - $A = l\,w$*; $A = (15 \text{ in.})\left(1\frac{1}{2} \text{ in.}\right)$; and $A = 22\frac{1}{2} \text{ in}^2$*
- How would we use the area of the base to determine the volumes? (Think about the unit cubes we have been using. The area of the base would be the first layer of unit cubes. What would the height represent?)
  - *We would multiply the area of the base times the height. The height would represent how many layers of cubes it would take to fill up the rectangular prism. Sample answers are below.*
- How do the volumes of the first and second rectangular prisms compare? The first and third?
  - *The volume of the second prism is twice that of the first because the height is doubled. The volume of the third prism is three times that of the first because the height is tripled.*

Example 1



a. **Write a numerical expression for the volume of each of the rectangular prisms above.**

$(15 \text{ in.})\left(1\frac{1}{2} \text{ in.}\right)(3 \text{ in.})$ $(15 \text{ in.})\left(1\frac{1}{2} \text{ in.}\right)(6 \text{ in.})$ $(15 \text{ in.})\left(1\frac{1}{2} \text{ in.}\right)(9 \text{ in.})$

b. **What do all of these expressions have in common? What do they represent?**

All of the expressions have $(15 \text{ in.})\left(1\frac{1}{2} \text{in.}\right)$. *This is the area of the base.*

c. **Rewrite the numerical expressions to show what they have in common.**

$\left(22\frac{1}{2} \text{ in}^2\right)(3 \text{ in.})$ $\left(22\frac{1}{2} \text{ in}^2\right)(6 \text{ in.})$ $\left(22\frac{1}{2} \text{ in}^2\right)(9 \text{ in.})$

d. **If we know volume for a rectangular prism as length times width times height, what is another formula for volume that we could use based on these examples?**

We could use $(\text{area of the base})(\text{height})$*, or area of the base times height.*

e. **What is the area of the base for all of the rectangular prisms?**

$(15 \text{ in.})\left(1\frac{1}{2} \text{ in.}\right) = 22\frac{1}{2} \text{ in}^2$

f. **Determine the volume of each rectangular prism using either method.**

$(15 \text{ in.})\left(1\frac{1}{2} \text{ in.}\right)(3 \text{ in.}) = 67\frac{1}{2} \text{ in}^3$ *or* $\left(22\frac{1}{2} \text{ in}^2\right)(3 \text{ in.}) = 67\frac{1}{2} \text{ in}^3$

$(15 \text{ in.})\left(1\frac{1}{2} \text{ in.}\right)(6 \text{ in.}) = 135 \text{ in}^3$ *or* $\left(22\frac{1}{2} \text{ in}^2\right)(6 \text{ in.}) = 135 \text{ in}^3$

$(15 \text{ in.})\left(1\frac{1}{2} \text{ in.}\right)(9 \text{ in.}) = 202\frac{1}{2} \text{ in}^3$ *or* $\left(22\frac{1}{2} \text{ in}^2\right)(9 \text{ in.}) = 202\frac{1}{2} \text{ in}^3$

g. **How do the volumes of the first and second rectangular prisms compare? The volumes of the first and third?**

$135 \text{ in}^3 = 67\frac{1}{2} \text{ in}^3 \times 2$

$202\frac{1}{2} \text{ in}^3 = 67\frac{1}{2} \text{ in}^3 \times 3$

The volume of the second prism is twice that of the first because the height is doubled. The volume of the third prism is three times as much as the first because the height is triple the first prism's height.

- What do you think would happen to the volume if we turn this prism on its side so that a different face is the base? (Have students calculate the area of the base times the height for this new prism. To help students visualize what is happening with this rotation, you could use a textbook or a stack of index cards and discuss how this prism is similar and/or different to the rectangular prisms in part (a).)
  - *Answers will vary. Some students may see that the volume will be the same no matter which face is the base.*

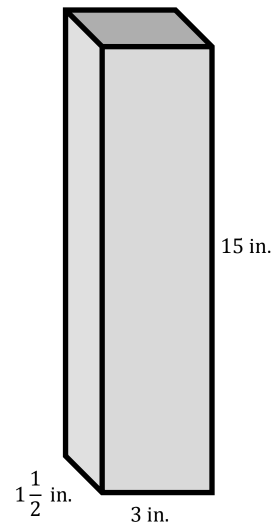
    $\text{Area of the base} = (3 \text{ in.})\left(1\frac{1}{2}\text{in.}\right)$

    $\text{Area of the base} = 4.5 \text{ in}^2$

    $\text{Volume} = \text{Area of the base} \times \text{height}$

    $\text{Volume} = \left(4\frac{1}{2} \text{ in}^2\right)(15 \text{ in.})$

    $\text{Volume} = 67\frac{1}{2} \text{ in}^3$



- How does this volume compare with the volume you calculated using the other face as the base?
  - *The volumes in both solutions are the same.*

MP.7

- What other expressions could we use to determine the volume of the prism?
  - *Answers will vary. Some possible variations are included below.*

$$15 \text{ in.} \times 1\frac{1}{2} \text{ in.} \times 3 \text{ in.}$$

$$15 \text{ in.} \times 3 \text{ in.} \times 1\frac{1}{2} \text{ in.}$$

$$3 \text{ in.} \times 15 \text{ in.} \times 1\frac{1}{2} \text{ in.}$$

$$45 \text{ in}^2 \times 1\frac{1}{2} \text{ in.}$$

- We notice that $3 \text{ in.} \times 15 \text{ in.} \times 1\frac{1}{2}$ in. and $45 \text{ in}^2 \times 1\frac{1}{2}$ in. are equivalent, and both represent the volume. How do they communicate different information?
  - *The first expression* ($3 \text{ in.} \times 15 \text{ in.} \times 1\frac{1}{2}$ in.) *shows that the volume is the product of three edge lengths. The second* ($45 \text{ in}^2 \times 1\frac{1}{2}$ in.) *shows that the volume is the product of the area of the base and the height.*

Example 2 (5 minutes)

Example 2

The base of a rectangular prism has an area of $3\frac{1}{4}$ in^2. The height of the prism is $2\frac{1}{2}$ in. Determine the volume of the rectangular prism.

$\mathbf{V = \text{Area of base} \times \text{height}}$

$\mathbf{V = \left(3\frac{1}{4}\ \text{in}^2\right)\left(2\frac{1}{2}\ \text{in.}\right)}$

$\mathbf{V = \left(\frac{13}{4}\ \text{in}^2\right)\left(\frac{5}{2}\ \text{in.}\right)}$

$\mathbf{V = \frac{65}{8}\ \text{in}^3}$

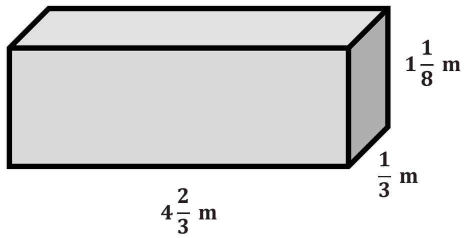
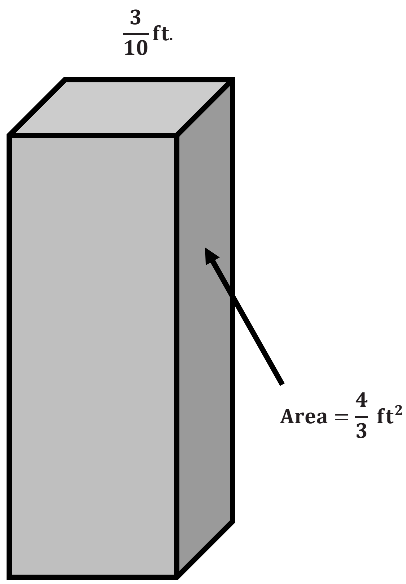
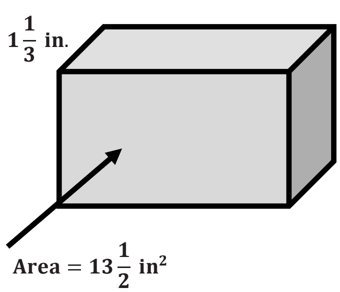
- Do we need to know the length and the width to find the volume of the rectangular prism?
  - *The length and width are needed to calculate the area of the base, and we already know the area of the base. Therefore, we do not need the length and width. The length and width are used to calculate the area, and we are already given the area.*

Exercises (20 minutes)

The cards are printed out and used as stations or hung on the classroom walls so that students can move from question to question. Copies of the questions can be found at the end of the lesson. Multiple copies of each question can be printed so that a small number of students visit each question at a time. Students should spend about three minutes at each station, where they will show their work by first writing a numerical expression, and then using the expression to calculate the volume of the rectangular prism described. They will use the rest of the time to discuss the answers, and the teacher can answer any questions students have about the lesson.

Card	Sample Response
a. Draw a sketch of the figure. Then, calculate the volume. Rectangular Prism Area of the base $= 4\frac{3}{8}$ ft^2 Height $= 2\frac{1}{2}$ ft.	$V = \text{Area of base} \times \text{height}$ $V = \left(4\frac{3}{8}\ \text{ft}^2\right)\left(2\frac{1}{2}\ \text{ft.}\right)$ $V = \left(\frac{35}{8}\ \text{ft}^2\right)\left(\frac{5}{2}\ \text{ft.}\right)$ $V = \frac{175}{16}\ \text{ft}^3$
b. Draw a sketch of the figure. Write the length, width, and height in feet. Then, calculate the volume. Rectangular Prism Length is $2\frac{1}{2}$ times as long as the height. Width is $\frac{3}{4}$ as long as the height. Height $= 3$ ft.	$\text{Length} = 3\ \text{ft.} \times 2\frac{1}{2} = \frac{15}{2}\ \text{ft.}$ $\text{Width} = 3\ \text{ft.} \times \frac{3}{4} = \frac{9}{4}\ \text{ft.}$ $V = l\,w\,h$ $V = \left(\frac{15}{2}\ \text{ft.}\right)\left(\frac{9}{4}\ \text{ft.}\right)(3\ \text{ft.})$ $V = \frac{405}{8}\ \text{ft}^3$

MP.7

c. Write two different expressions to represent the volume, and explain what each one represents.  	Answers will vary. Some possible solutions include $\left(4\frac{2}{3}\text{ m}\right)\left(\frac{1}{3}\text{ m}\right)\left(1\frac{1}{8}\text{ m}\right)$ and $\left(\frac{14}{9}\text{ m}^2\right)\left(1\frac{1}{8}\text{ m}\right)$. The first expression shows the volume as a product of the three edge lengths. The second expression, $\left(4\frac{2}{3}\text{ m}\right)\left(\frac{1}{3}\text{ m}\right)$, shows the volume as a product of the base area times the height.
d. Calculate the volume.  	$V =$ Area of base $\times$ height $V = \left(\frac{4}{3}\text{ ft}^2\right)\left(\frac{3}{10}\text{ ft.}\right)$ $V = \frac{12}{30}\text{ ft}^3$ $V = \frac{2}{5}\text{ ft}^3$
e. Calculate the volume.  	$V =$ Area of base $\times$ height $V = \left(13\frac{1}{2}\text{ in}^2\right)\left(1\frac{1}{3}\text{ in.}\right)$ $V = \frac{108}{6}\text{ in}^3$ $V = 18\text{ in}^3$

f. Challenge: Determine the volume of a rectangular prism whose length and width are in a ratio of $3:1$. The width and height are in a ratio of $2:3$. The length of the rectangular prism is 5 ft.	Length $= 5$ ft. Width $= 5\text{ ft.} \div 3 = \frac{5}{3}$ ft. Height $= \frac{5}{3}\text{ ft.} \times \frac{3}{2} = \frac{5}{2}$ ft. $V = l\,w\,h$ $V = (5\text{ ft.})\left(\frac{5}{3}\text{ ft.}\right)\left(\frac{5}{2}\text{ ft.}\right)$ $V = \frac{125}{6}\text{ ft}^3$

Extension (3 minutes)

Extension

A company is creating a rectangular prism that must have a volume of 6 ft^3. The company also knows that the area of the base must be $2\frac{1}{2}\text{ ft}^2$. How can you use what you learned today about volume to determine the height of the rectangular prism?

I know that the volume can be calculated by multiplying the area of the base times the height. So, if I needed the height instead, I would do the opposite. I would divide the volume by the area of the base to determine the height.

$$V = \text{Area of base} \times \text{height}$$
$$6\text{ ft}^3 = \left(2\frac{1}{2}\text{ ft}^2\right)h$$
$$6\text{ ft}^3 \div 2\frac{1}{2}\text{ ft}^2 = h$$
$$2\frac{2}{5}\text{ ft.} = h$$

Closing (2 minutes)

- How is the formula $V = l \cdot w \cdot \text{h}$ related to the formula $V = \text{Area of the base} \cdot \text{height}$?
  - *When we multiply the length and width of the rectangular prism, we are actually finding the area of the base. Therefore, the two formulas both determine the volume of the rectangular prism.*

Exit Ticket (5 minutes)

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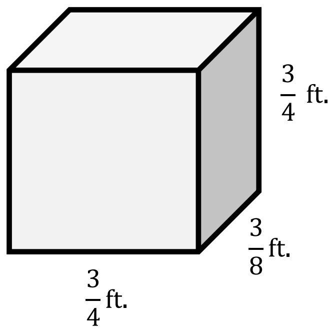
Lesson 12: From Unit Cubes to the Formulas for Volume

Exit Ticket

1. Determine the volume of the rectangular prism in two different ways.



2. The area of the base of a rectangular prism is 12 cm^2, and the height is $3\frac{1}{3}$ cm. Determine the volume of the rectangular prism.

Exit Ticket Sample Solutions

1. Determine the volume of the rectangular prism in two different ways.

$V = l \cdot w \cdot h$

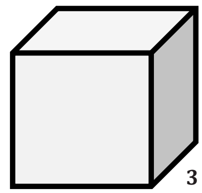
$V = \left(\frac{3}{4}\text{ ft.}\right)\left(\frac{3}{8}\text{ ft.}\right)\left(\frac{3}{4}\text{ ft.}\right)$

$V = \frac{27}{128}\text{ ft}^3$

$V = \text{Area of base} \cdot \text{height}$

$V = \left(\frac{9}{32}\text{ ft}^2\right)\left(\frac{3}{4}\text{ ft.}\right)$

$V = \frac{27}{128}\text{ ft}^3$



2. The area of the base of a rectangular prism is 12 cm^2, and the height is $3\frac{1}{3}\text{ cm}$. Determine the volume of the rectangular prism.

$V = \text{Area of base} \cdot \text{height}$

$V = (12\text{ cm}^2)\left(3\frac{1}{3}\text{ cm}\right)$

$V = \frac{120}{3}\text{ cm}^3$

$V = 40\text{ cm}^3$

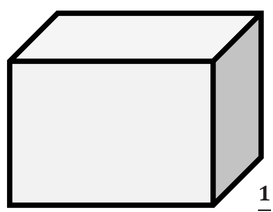
Problem Set Sample Solutions

1. Determine the volume of the rectangular prism.

$V = l\,w\,h$

$V = \left(1\frac{1}{2}\text{ m}\right)\left(\frac{1}{2}\text{ m}\right)\left(\frac{7}{8}\text{ m}\right)$

$V = \frac{21}{32}\text{ m}^3$



2. The area of the base of a rectangular prism is $4\frac{3}{4}\text{ ft}^2$, and the height is $2\frac{1}{3}$ ft. Determine the volume of the rectangular prism.

$V = \text{Area of base} \times \text{height}$

$V = \left(4\frac{3}{4}\text{ ft}^2\right)\left(2\frac{1}{3}\text{ ft.}\right)$

$V = \left(\frac{19}{4}\text{ ft}^2\right)\left(\frac{7}{3}\text{ ft.}\right)$

$V = \frac{133}{12}\text{ ft}^3$

3. The length of a rectangular prism is $3\frac{1}{2}$ times as long as the width. The height is $\frac{1}{4}$ of the width. The width is 3 cm. Determine the volume.

Width $= 3$ cm

Length $= 3 \text{ cm} \times 3\frac{1}{2} = \frac{21}{2}$ cm

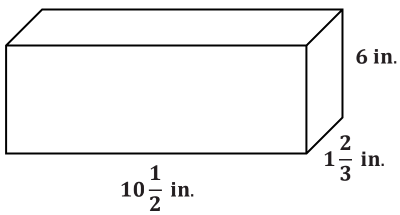
Height $= 3 \text{ cm} \times \frac{1}{4} = \frac{3}{4}$ cm

$V = l\,w\,h$

$V = \left(\frac{21}{2} \text{ cm}\right)(3 \text{ cm})\left(\frac{3}{4} \text{ cm}\right)$

$V = \frac{189}{8} \text{ cm}^3$

4.



a. Write numerical expressions to represent the volume in two different ways, and explain what each reveals.

$\left(10\frac{1}{2} \text{ in.}\right)\left(1\frac{2}{3} \text{ in.}\right)(6 \text{ in.})$ *represents the product of three edge lengths.* $\left(\frac{35}{2} \text{ in}^2\right)(6 \text{ in})$ *represents the product of the base area times the height. Answers will vary.*

b. Determine the volume of the rectangular prism.

$\left(10\frac{1}{2} \text{ in.}\right)\left(1\frac{2}{3} \text{ in.}\right)(6 \text{ in.}) = 105 \text{ in}^3$ or $\left(\frac{35}{2} \text{ in}^2\right)(6 \text{ in.}) = 105 \text{ in}^3$

5. An aquarium in the shape of a rectangular prism has the following dimensions: length $= 50$ cm, width $= 5\frac{1}{2}$ cm, and height $= 30\frac{1}{2}$ cm.

a. Write numerical expressions to represent the volume in two different ways, and explain what each reveals.

$(50 \text{ cm})\left(25\frac{1}{2} \text{ cm}\right)\left(30\frac{1}{2} \text{ cm}\right)$ *represents the product of the three edge lengths.*

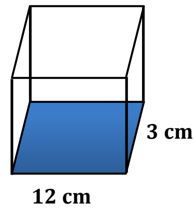
$(1275 \text{ cm}^2)\left(30\frac{1}{2} \text{ cm}\right)$ *represents the base area times the height.*

b. Determine the volume of the rectangular prism.

$(1275 \text{ cm}^2)\left(30\frac{1}{2} \text{ cm}\right) = 38887\frac{1}{2} \text{ cm}^3$

6. The area of the base in this rectangular prism is fixed at 36 cm^2. This means that for the varying heights, there will be various volumes.

   a. Complete the table of values to determine the various heights and volumes.



Height in Centimeters	Volume in Cubic Centimeters
1	*36*
2	72
3	108
4	144
5	180
6	*216*
7	*252*
8	288

   b. Write an equation to represent the relationship in the table. Be sure to define the variables used in the equation.

   Let x be the height of the rectangular prism in centimeters.

   Let y be the volume of the rectangular prism in cubic centimeters.

   $36x = y$

   c. What is the unit rate for this proportional relationship? What does it mean in this situation?

   The unit rate is 36.

   For every centimeter of height, the volume increases by 36 cm^3 because the area of the base is 36 cm^2. In order to determine the volume, multiply the height by 36.

7. The volume of a rectangular prism is 16.328 cm^3. The height is 3.14 cm.

   a. Let B represent the area of the base of the rectangular prism. Write an equation that relates the volume, the area of the base, and the height.

   $16.328 = 3.14B$

   b. Solve the equation for B.

$$\frac{16.328}{3.14} = \frac{3.14B}{3.14}$$
$$B = 5.2$$

   The area of the base is 5.2 cm^2.

Station A

Make a sketch of the figure. Then, calculate the volume.

Rectangular prism:

Area of the base = $4\frac{3}{8}\text{ ft}^2$

Height = $2\frac{1}{2}\text{ ft}$.

Station B

Make a sketch of the figure. Write the length, the width, and height in feet. Then, calculate the volume.

Rectangular prism:

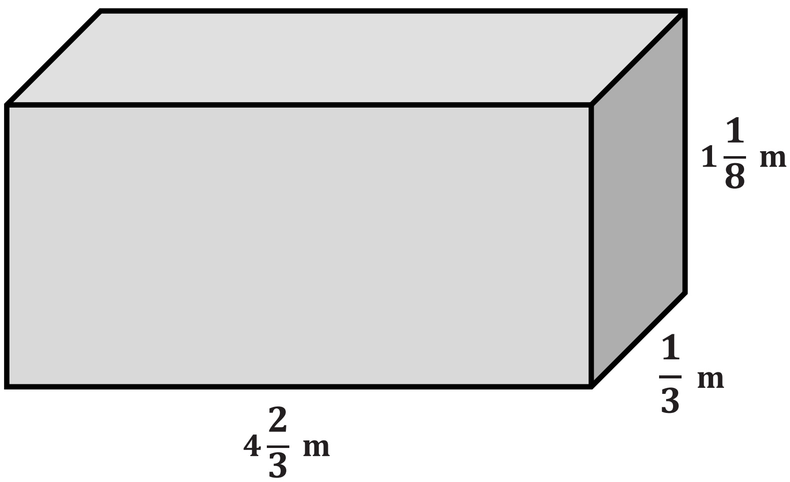
Length is $2\frac{1}{2}$ times the height.

Width is $\frac{3}{4}$ as long as the height.

Height = 3 ft.

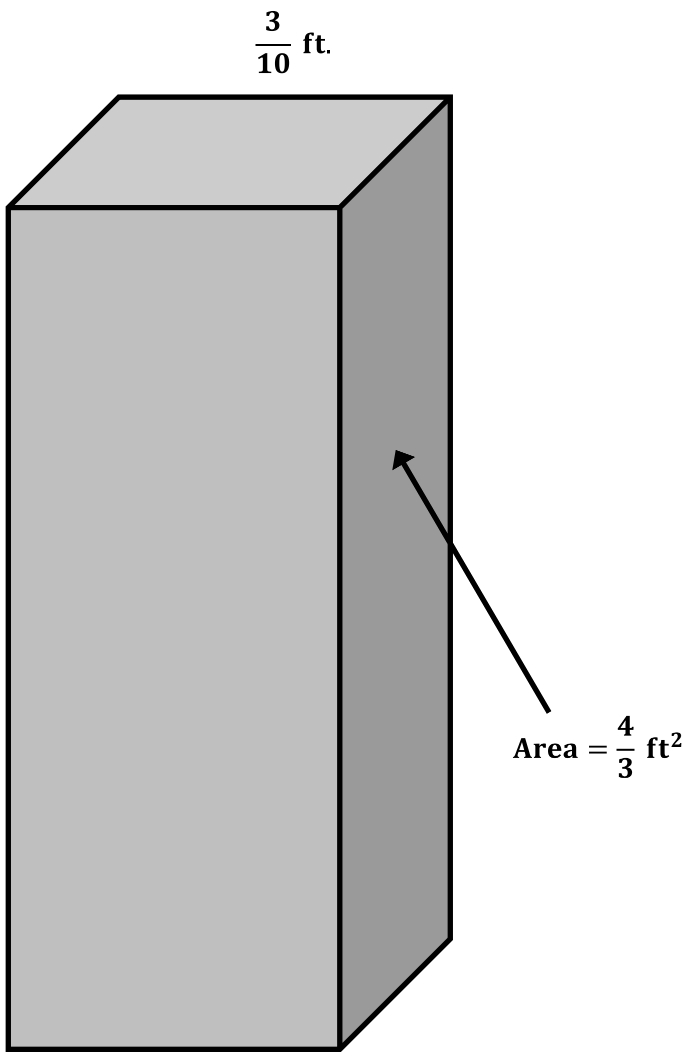
Station C

Write two different expressions to represent the volume, and explain what each expression represents.



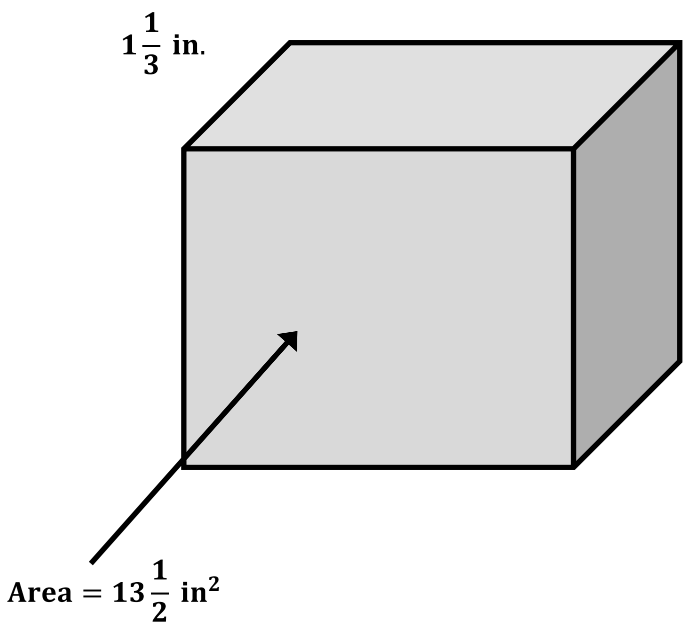
Station D

Calculate the volume.



Station E

Calculate the volume.



Station F

Challenge:

Determine the volume of a rectangular prism whose length and width are in a ratio of $3:1$. The width and height are in a ratio of $2:3$. The length of the rectangular prism is 5 ft.