Table of Contents[1]

The Concept of Congruence

Module Overview ... 2

Topic A: Definitions and Properties of the Basic Rigid Motions (**8.G.A.1**) ... 7

Lesson 1: Why Move Things Around? ... 9

Lesson 2: Definition of Translation and Three Basic Properties ... 17

Lesson 3: Translating Lines ... 30

Lesson 4: Definition of Reflection and Basic Properties ... 38

Lesson 5: Definition of Rotation and Basic Properties ... 51

Lesson 6: Rotations of 180 Degrees ... 64

Topic B: Sequencing the Basic Rigid Motions (**8.G.A.2**) ... 76

Lesson 7: Sequencing Translations ... 77

Lesson 8: Sequencing Reflections and Translations ... 86

Lesson 9: Sequencing Rotations ... 96

Lesson 10: Sequences of Rigid Motions ... 105

Mid-Module Assessment and Rubric ... 116
Topics A and B (assessment 1 day, return 1 day, remediation or further applications 2 days)

Topic C: Congruence and Angle Relationships (**8.G.A.2**, **8.G.A.5**) ... 128

Lesson 11: Definition of Congruence and Some Basic Properties ... 129

Lesson 12: Angles Associated with Parallel Lines ... 138

Lesson 13: Angle Sum of a Triangle ... 146

Lesson 14: More on the Angles of a Triangle ... 156

End-of-Module Assessment and Rubric ... 169
Topics A through C (assessment 1 day, return 1 day, remediation or further applications 3 days)

(Optional) Topic D: The Pythagorean Theorem (**8.G.B.6**, **8.G.B.7**) ... 177

Lesson 15: Informal Proof of the Pythagorean Theorem ... 178

Lesson 16: Applications of the Pythagorean Theorem ... 190

[1] Each lesson is ONE day, and ONE day is considered a 45-minute period.

Grade 8 • Module 2

The Concept of Congruence

OVERVIEW

In this module, students learn about translations, reflections, and rotations in the plane and, more importantly, how to use them to precisely define the concept of *congruence*. Up to this point, "congruence" has been taken to mean, intuitively, "same size and same shape." Because this module begins with a serious study of geometry, this intuitive definition must be replaced by a precise definition. This module is a first step; its goal is to provide the needed intuitive background for the precise definitions that are introduced in this module for the first time.

Translations, reflections, and rotations are examples of *rigid motions*, which are, intuitively, rules of moving points in the plane in such a way that preserves distance. For the sake of brevity, these three rigid motions will be referred to exclusively as the *basic rigid motions*. Initially, the exploration of these basic rigid motions is done via hands-on activities using an overhead projector transparency, but with the availability of geometry software, the use of technology in this learning environment is inevitable, and some general guidelines for this usage will be laid out at the end of Lesson 2. What needs to be emphasized is that the importance of these basic rigid motions lies not in the fun activities they bring but in the *mathematical* purpose they serve in clarifying the meaning of congruence.

Throughout Topic A, on the definitions and properties of the basic rigid motions, students verify experimentally their basic properties and, when feasible, deepen their understanding of these properties using reasoning. In particular, what students learned in Grade 4 about angles and angle measurement (**4.MD.C.5**) will be put to good use here. They learn that the basic rigid motions preserve angle measurements, as well as segment lengths.

Topic B is a critical foundation to the understanding of congruence. All the lessons of Topic B demonstrate to students the ability to sequence various combinations of rigid motions while maintaining the basic properties of individual rigid motions. Lesson 7 begins this work with a sequence of translations. Students verify experimentally that a sequence of translations have the same properties as a single translation. Lessons 8 and 9 demonstrate sequences of reflections and translations and sequences of rotations. The concept of sequencing a combination of all three rigid motions is introduced in Lesson 10; this paves the way for the study of congruence in the next topic.

In Topic C, which introduces the definition and properties of congruence, students learn that congruence is just a sequence of basic rigid motions. The fundamental properties shared by all the basic rigid motions are then inherited by congruence: Congruence moves lines to lines and angles to angles, and it is both distance- and angle-preserving (Lesson 11). In Grade 7, students used facts about supplementary, complementary, vertical, and adjacent angles to find the measures of unknown angles (**7.G.B.5**). This module extends that knowledge to angle relationships that are formed when two parallel lines are cut by a transversal. In Topic C, on angle relationships related to parallel lines, students learn that pairs of angles are congruent because they are angles that have been translated along a transversal, rotated around a point, or reflected across a line.

Students use this knowledge of angle relationships in Lessons 13 and 14 to show why a triangle has a sum of interior angles equal to $180°$ and why the measure of each exterior angle of a triangle is the sum of the measures of the two remote interior angles of the triangle.

Optional Topic D introduces the Pythagorean theorem. Students are shown the "square within a square" proof of the Pythagorean theorem. The proof uses concepts learned in previous topics of the module, i.e., the concept of congruence and concepts related to degrees of angles. Students begin the work of finding the length of a leg or hypotenuse of a right triangle using $a^2 + b^2 = c^2$. Note that this topic will not be assessed until Module 7.

Focus Standards

Understand congruence and similarity using physical models, transparencies, or geometry software.

8.G.A.1 Verify experimentally the properties of rotations, reflections, and translations:

a. Lines are taken to lines, and line segments to line segments of the same length.

b. Angles are taken to angles of the same measure.

c. Parallel lines are taken to parallel lines.

8.G.A.2 Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.

8.G.A.5 Use informal arguments to establish facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles. *For example, arrange three copies of the same triangle so that the sum of the three angles appears to form a line, and give an argument in terms of transversals why this is so.*

Understand and apply the Pythagorean Theorem.

8.G.B.6 Explain a proof of the Pythagorean Theorem and its converse.

8.G.B.7 Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.

Foundational Standards

Geometric measurement: understand concepts of angle and measure angles.

4.MD.C.5 Recognize angles as geometric shapes that are formed wherever two rays share a common endpoint, and understand concepts of angle measurement:

a. An angle is measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays intersect the circle. An angle that turns through 1/360 of a circle is called a "one-degree angle," and can be used to measure angles.

b. An angle that turns through n one-degree angles is said to have an angle measure of n degrees.

Draw and identify lines and angles, and classify shapes by properties of their lines and angles.

4.G.A.1 Draw points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Identify these in two-dimensional figures.

4.G.A.2 Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles.

4.G.A.3 Recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. Identify line-symmetric figures and draw lines of symmetry.

Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.

7.G.B.5 Use facts about supplementary, complementary, vertical, and adjacent angles in a multi-step problem to write and solve simple equations for an unknown angle in a figure.

Focus Standards for Mathematical Practice

MP.2 **Reason abstractly and quantitatively.** This module is rich with notation that requires students to decontextualize and contextualize throughout. Students work with figures and their transformed images using symbolic representations and need to attend to the meaning of the symbolic notation to contextualize problems. Students use facts learned about rigid motions in order to make sense of problems involving congruence.

MP.3 **Construct viable arguments and critique the reasoning of others.** Throughout this module, students construct arguments around the properties of rigid motions. Students make assumptions about parallel and perpendicular lines and use properties of rigid motions to directly or indirectly prove their assumptions. Students use definitions to describe a sequence of rigid motions to prove or disprove congruence. Students build a logical progression of statements to show relationships between angles of parallel lines cut by a transversal, the angle sum of triangles, and properties of polygons like rectangles and parallelograms.

MP.5 **Use appropriate tools strategically**. This module relies on students' fundamental understanding of rigid motions. As a means to this end, students use a variety of tools but none as important as an overhead transparency. Students verify experimentally the properties of rigid motions using physical models and transparencies. Students use transparencies when learning about translation, rotation, reflection, and congruence in general. Students determine when they need to use the transparency as a tool to justify conjectures or when critiquing the reasoning of others.

MP.6 **Attend to precision**. This module begins with precise definitions related to transformations and statements about transformations being distance- and angle-preserving. Students are expected to attend to the precision of these definitions and statements consistently and appropriately as they communicate with others. Students describe sequences of motions precisely and carefully label diagrams so that there is clarity about figures and their transformed images. Students attend to precision in their verbal and written descriptions of rays, segments, points, angles, and transformations in general.

Terminology

New or Recently Introduced Terms

- **Transformation** (A *transformation* is a rule, to be denoted by F, that assigns each point P of the plane a unique point which is denoted by $F(P)$.)
- **Basic Rigid Motion** (A *basic rigid motion* is a rotation, reflection, or translation of the plane.
  - Basic rigid motions are examples of transformations. Given a transformation, the image of a point A is the point the transformation maps the point A to in the plane.)
- **Translation** (A *translation* is a basic rigid motion that moves a figure along a given vector.)
- **Rotation** (A *rotation* is a basic rigid motion that moves a figure around a point, d degrees.)
- **Reflection** (A *reflection* is a basic rigid motion that moves a figure across a line.)
- **Image of a point, image of a figure** (*Image* refers to the location of a point or figure after it has been transformed.)
- **Sequence (Composition) of Transformations** (A *sequence of transformations* is more than one transformation. Given transformations G and F, $G \circ F$ is called the composition of F and G.)
- **Vector** (A Euclidean *vector* (or directed segment) $\overrightarrow{AB}$ is the line segment AB together with a direction given by connecting an initial point A to a terminal point B.)
- **Congruence** (A *congruence* is a sequence of basic rigid motions (rotations, reflections, translations) of the plane.)
- **Transversal** (Given a pair of lines L and M in a plane, a third line T is a *transversal* if it intersects L at a single point and intersects M at a single but different point.)

Familiar Terms and Symbols[2]

- Ray, line, line segment, angle
- Parallel and perpendicular lines

[2] These are terms and symbols students have seen previously.

- Supplementary, complementary, vertical, and adjacent angles
- Triangle, quadrilateral
- Area and perimeter

Suggested Tools and Representations

- Transparency or patty paper
- Wet or dry erase markers for use with transparency
- Optional: geometry software
- Composition of Rigid Motions: http://youtu.be/O2XPy3ZLU7Y
- ASA: http://www.youtube.com/watch?v=-yIZdenw5U4

Assessment Summary

Assessment Type	Administered	Format	Standards Addressed
Mid-Module Assessment Task	After Topic B	Constructed response with rubric	8.G.A.1
End-of-Module Assessment Task	After Topic C	Constructed response with rubric	8.G.A.2, 8.G.A.5

Topic A:

Definitions and Properties of the Basic Rigid Motions

8.G.A.1

Focus Standard:	8.G.A.1	Verify experimentally the properties of rotations, reflections, and translations: a. Lines are taken to lines, and line segments to line segments of the same length. b. Angles are taken to angles of the same measure. c. Parallel lines are taken to parallel lines.
Instructional Days:	6	
Lesson 1:	Why Move Things Around? (E)[1]	
Lesson 2:	Definition of Translation and Three Basic Properties (P)	
Lesson 3:	Translating Lines (S)	
Lesson 4:	Definition of Reflection and Basic Properties (P)	
Lesson 5:	Definition of Rotation and Basic Properties (S)	
Lesson 6:	Rotations of 180 Degrees (P)	

In Topic A, students learn about the *mathematical* needs for rigid motions and begin by exploring the *possible* effects of rigid motions in Lesson 1. In particular, the study of rigid motions in this module will not just be about moving geometric figures around by the use of reflections, translations, and rotations. Rather, students explore the geometric implications of having an abundance of these basic rigid motions in the plane. Lessons on translation, reflection, and rotation show students that lines are taken to lines, line segments are taken to line segments, and parallel lines are taken to parallel lines. In addition to the intuitive notion of figures "retaining the same shape" under such motions, students learn to express precisely the fact that lengths of segments and size of angles are preserved.

Lessons 2 and 3 focus on translation but also set up precise definitions and statements related to transformations that are used throughout the remainder of the module. In Lesson 2, students learn the basics of translation by translating points, lines, and figures along a vector, and students verify experimentally that translations map lines to lines, segments to segments, rays to rays, and angles to angles. Students also

[1] Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson

verify experimentally that translations preserve length and angle measure. Lesson 3 focuses on the translation of lines, specifically the idea that a translation maps a line either to itself or to a parallel line.

In Lesson 4, students verify experimentally that reflections are distance- and angle-preserving. In Lesson 5, rotation around a point is investigated in a similar manner as the other rigid motions. Students verify experimentally that rotations take lines to lines, etc. and are distance- and angle-preserving. In Lesson 6, students are provided proof that 180-degree rotations map a line to a parallel line and use that knowledge to prove that vertical angles are equal.

Lesson 1: Why Move Things Around?

Student Outcomes

- Students are introduced to vocabulary and notation related to rigid motions (e.g., transformation, image, and map).
- Students are introduced to transformations of the plane and learn that a rigid motion is a transformation that is distance-preserving.
- Students use transparencies to imitate a rigid motion that moves or maps one figure to another figure in the plane.

Materials (needed for this and subsequent lessons)

- Overhead projector transparencies (one per student)
- Fine point dry-erase markers (one per student)
- Felt cloth or other eraser (one per student, or per pair)

Lesson Notes

The goal of this module is to arrive at a clear understanding of the concept of *congruence.* (i.e., What does it mean for two geometric figures to have the same size and shape?) We introduce the basic rigid motions (i.e., translations, reflections, and rotations) using overhead projector transparencies. We then explain *congruence* in terms of a sequence of these basic rigid motions. It may be worth pointing out that we are studying the basic rigid motions for a definite *mathematical* purpose, not for artistic reasons or for the purpose of studying "transformational geometry," per se.

The traditional way of dealing with congruence in Euclidian geometry is to write a set of axioms that abstractly guarantees that two figures are "the same" (i.e., congruent). This method can be confusing to students. Today, we take a more direct approach so that the concept of congruence ceases to be abstract and intangible, becoming instead, susceptible to concrete realizations through hands-on activities using an overhead projector transparency. It will be important not only that teachers use transparencies for demonstration purposes, but also that students have access to them for hands-on experience.

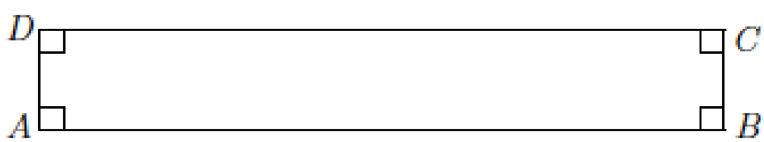
The two basic references for this module are "Teaching Geometry According to the Common Core Standards," and "Pre-Algebra," both by Hung-Hsi Wu. Incidentally, the latter is identical to the document cited on page 92 of the *Common Core State Standards for Mathematics*, which is "Lecture Notes for the 2009 Pre-Algebra Institute" by Hung-Hsi Wu, September 15, 2009.

Classwork

Concept Development (20 minutes)

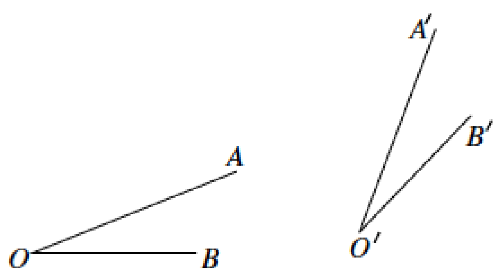
- Given two segments AB and CD, which could be very far apart, how can we find out if they have the same length without measuring them individually? Do you think they have the same length? How do you check? (We will revisit this question later.)

- For example, given a quadrilateral $ABCD$ where all four angles at A, B, C, D are right angles, are the opposite sides AD, BC of equal length?

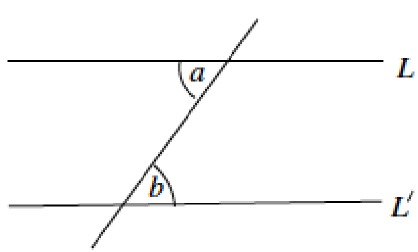


Later, we will *prove* that they have the same length.

- Similarly, given angles $\angle AOB$ and $\angle A'O'B'$, how can we tell whether they have the same degree without having to measure each angle individually?



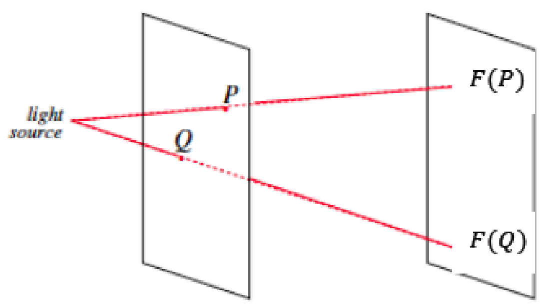
- For example, if two lines L and L' are parallel and they are intersected by another line, how can we tell if the angles $\angle a$ and $\angle b$ (as shown) have the same degree when measured?



- We are, therefore, confronted with having two geometric figures (two segments, two angles, two triangles, etc.) in different parts of the plane, and we have to find out if they are, in some sense, *the same* (e.g., same length, same degree, same shape).
- To this end, there are three standard *moves* we can use to bring one figure on top of another to see if they coincide.
- So, the key question is how do we *move* things around in a plane, keeping in mind that lines are still lines after being moved, and that the lengths of segments and degrees of the measures of angles remain unchanged in the process.
- "Moving things around in a plane" is exactly where the concept of *transformation* comes in.

MP.2 & MP.4

  - A **transformation** of the plane, to be denoted by F, is a rule that associates (or assigns) to each point P of the plane a unique point which will be denoted by $F(P)$.
    - So, by definition, the symbol $F(P)$ denotes a single point, unambiguously.
    - The symbol $F(P)$ indicates clearly that F moves P to $F(P)$.
    - The point $F(P)$ will be called the **image of P by F**.
    - We also say F maps P to $F(P)$.
  - The reason for the *image* terminology is that one can, intuitively, think of the plane as a sheet of overhead projector transparency, or as a sheet of paper. A transformation F of the plane is a projection (literally, using a light source) from one sheet of the transparency to a sheet of paper with the two sheets identified as being the same plane. Then, the point $F(P)$ is the image on the sheet of paper when the light source projects the point P from the transparency.



- As to the "map" terminology, think of how you would draw a street map. Drawing a map is a complicated process, but the most mathematically accurate description for the purpose of school mathematics may be that one starts with an *aerial view* of a particular portion of a city. In the picture below we look at the area surrounding the Empire State Building (E.S.B.) in New York City. The picture reduces the 3-dimensional information into two dimensions and then "maps" each point on the street to a point on your paper (the map)[1].

  - A point on the street becomes a point on your paper (the map). So, you are *mapping* each point on the street to your paper.

- Transformations can be defined on spaces of any dimension, but for now we are only concerned with transformations **in the plane**, in the sense that transformations are those that assign a point of the plane to another point of the plane.
- Transformations can be complicated (i.e., the rule in question can be quite convoluted), but for now we will concentrate on only the simplest transformations, namely those that *preserve distance*.
- MP.6 A transformation F **preserves distance**, or is **distance-preserving,** if given any two points P and Q, the distance between the images $F(P)$ and $F(Q)$ is the same as the distance between the original points P and Q.
- An obvious example of this kind of transformation is the *identity transformation,* which assigns each point P of the plane to P itself.
- A main purpose of this module is to introduce many other distance-preserving transformations and show why they are important in geometry.
- A distance-preserving transformation is called a *rigid motion* (or an *isometry*) that, as the name suggests, *moves* the points of the plane around in a *rigid* fashion.

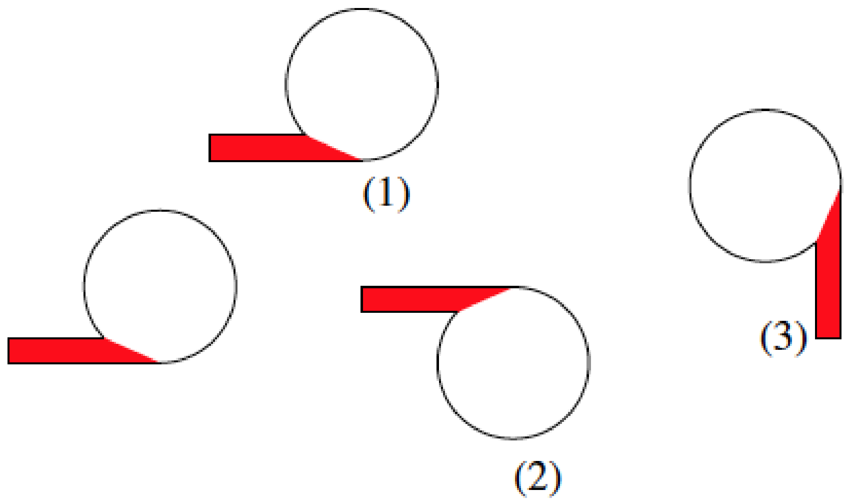
[1] This is done in a way that intuitively *preserves the shape*. The correct terminology here is *similar*, as shown in Module 3.

Exploratory Challenge (15 minutes)

Have students complete Exercise 1 independently and Exercise 2 in small groups. Have students share their responses.

Exploratory Challenge

1. **Describe, intuitively, what kind of transformation will be required to move the figure on the left to each of the figures (1)–(3) on the right. To help with this exercise, use a transparency to copy the figure on the left. Note: Begin by moving the left figure to each of the locations in (1), (2), and (3).**



Slide the original figure to the image (1) until they coincide. Slide the original figure to (2), then flip it so they coincide. Slide the original figure to (3); then, turn it until they coincide.

2. **Given two segments AB and CD, which could be very far apart, how can we find out if they have the same length without measuring them individually? Do you think they have the same length? How do you check? In other words, why do you think we need to move things around on the plane?**

We can trace one of the segments on the transparency and slide it to see if it coincides with the other segment. We move things around in the plane to see if they are exactly the same. This way, we don't have to do any measuring.

Closing (5 minutes)

Summarize, or have students summarize, the lesson.

- We can use a transparency to represent the plane and move figures around.
- We can check to see if one figure is the same as another by mapping one figure onto another and checking to see if they coincide.
- A transformation that preserves distance is known as a rigid motion (the distance between any two corresponding points is the same after the transformation is performed).

Lesson Summary

A transformation of the plane, to be denoted by F, is a rule that assigns to each point P of the plane one and only one (unique) point which will be denoted by $F(P)$.

- **So, by definition, the symbol $F(P)$ denotes a specific single point.**
- **The symbol $F(P)$ shows clearly that F moves P to $F(P)$.**
- **The point $F(P)$ will be called the image of P by F.**
- **We also say F maps P to $F(P)$.**

If given any two points P and Q, the distance between the images $F(P)$ and $F(Q)$ is the same as the distance between the original points P and Q, then the transformation F preserves distance, or is distance-preserving.

- **A distance-preserving transformation is called a rigid motion (or an isometry), and the name suggests that it *moves* the points of the plane around in a *rigid* fashion.**

Exit Ticket (5 minutes)

Name ______________________________ Date ________________

Lesson 1: Why Move Things Around?

Exit Ticket

First, draw a simple figure and name it "Figure W." Next, draw its image under some transformation, (i.e., trace your "Figure W" on the transparency), and then move it. Finally, draw its image somewhere else on the paper.

Describe, intuitively, how you moved the figure. Use complete sentences.

Exit Ticket Sample Solutions

First, draw a simple figure and name it "Figure W." Next, draw its image under some transformation, (i.e., trace your "Figure W" on the transparency), and then move it. Finally, draw its image somewhere else on the paper.

Describe, intuitively, how you moved the figure. Use complete sentences.

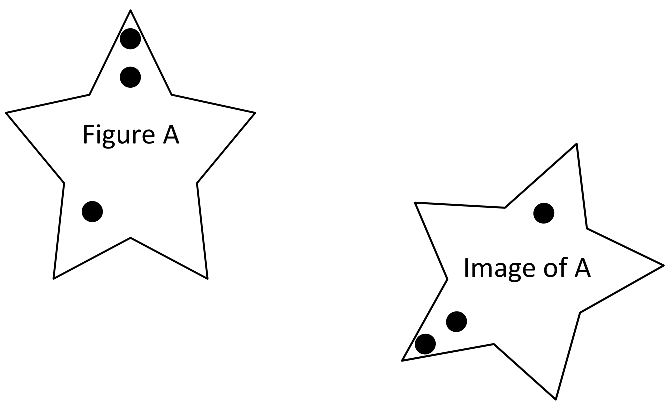
Accept any figure and transformation that is correct. Check for same size and shape. Students should describe the movement of the figure as "sliding" to the left or right, "turning" to the left or right, or "flipping," similar to how they described the movement of figures in the exercises of the lesson.

Problem Set Sample Solutions

1. **Using as much of the new vocabulary as you can, try to describe what you see in the diagram below.**

There was a transformation, F, that moved point A to its image $F(A)$ and point B to its image $F(B)$. Since a transformation preserves distance, the distance between points A and B is the same as the distance between the points $F(A)$ and $F(B)$.

2. **Describe, intuitively, what kind of transformation will be required to move Figure A on the left to its image on the right.**



First, I have to slide Figure A so that the point containing two dots maps onto the Image of A in the same location; next, I have to turn (rotate) it so that Figure A maps onto Image of A; finally, I have to flip the figure over so the part of the star with the single dot maps onto the image.

Lesson 2: Definition of Translation and Three Basic Properties

Student Outcomes

- Students perform translations of figures along a specific vector. Students label the image of the figure using appropriate notation.
- Students learn that a translation maps lines to lines, rays to rays, segments to segments, and angles to angles. Students learn that translations preserve lengths of segments and degrees of angles.

Lesson Notes

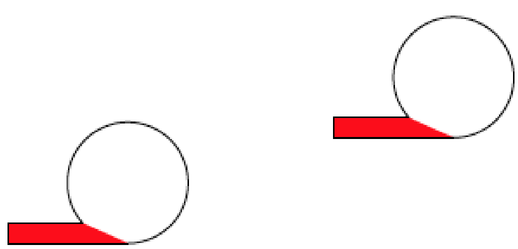
In this lesson, and those that follow, we emphasize learning about translations, reflections, and rotation, by moving a transparency over a piece of paper. At this initial stage of a student's encounter with these basic rigid motions, such an emphasis provides a *tactile* familiarity with these transformations, without the worry of technological malfunctions. We do, however, expect students to employ geometry software for further geometric explorations once they have gone beyond this initial phase of learning. Many versions of such software are available, some of which are open source (e.g., GeoGebra, and Live Geometry), and others that are not (e.g., Cabri, and The Geometer's Sketchpad). Teachers should use discretion about how much technology to incorporate into the lesson, but here are some general guidelines:

1. Students' initial exposure to the basic rigid motions should be made through hands-on activities such as the use of transparencies.
2. Technological tools are just *tools*, and their role in the classroom should be to facilitate learning, but not as an end in itself.
3. Students should be made aware of such software because these tools could be of later use, outside of the classroom.

Classwork

Discussion (2 minutes)

- What is the simplest transformation that would map one of the following figures to the other?



Scaffolding:

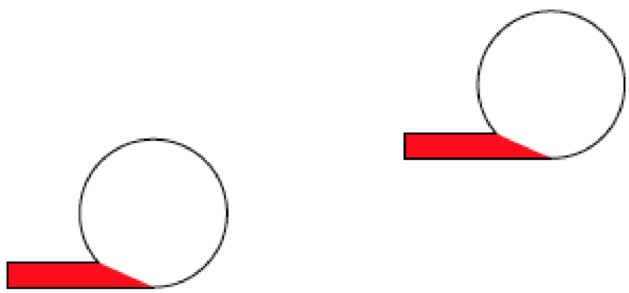
Post new vocabulary words, definitions, and related symbolic notation in a prominent location in the room.

  - *Students will likely answer "a slide," but we want a more precise answer, so we need more information about the topic of transformations to answer this question.*

- In the next few lessons, we will learn about three kinds of simple rigid motions: translation, reflection, and rotation.
- We call these the *basic rigid motions*.
- We use the term "basic" because students will see that every rigid motion can be obtained by a suitable sequence (see: Topic B) of translations, reflections and rotations.[1]
- In the following, we will describe how to move a transparency over a piece of paper to demonstrate the effect each of these basic rigid motions has on the points in the plane.
- We begin with translations or, more precisely, a *translation along a given vector*.

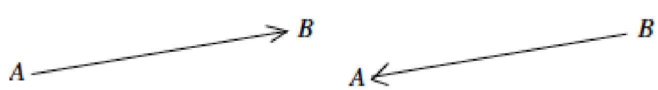
Example 1 (3 minutes)

Before explaining explicitly what *translation along a vector* is, review the question that started the discussion. Then, draw a vector and demonstrate how to translate along the chosen vector to map the lower left figure to the upper right figure.



Scaffolding:
Consider showing several examples of vectors that can be used instead of just one.

- A *vector* is a segment in the plane with direction. One of its two endpoints is designated as a *starting point,* while the other is simply called *the endpoint*.
  - The length of a vector is, by definition, the length of its underlying segment.
  - Visually, we distinguish a vector from its underlying segment by adding an arrow above the symbol. Thus, if the segment is AB (A and B being its endpoints), then the vector with starting point A and endpoint B is denoted by $\overrightarrow{AB}$. Likewise, the vector with starting point B and endpoint A will be denoted by $\overrightarrow{BA}$.
  - Note that the arrowhead on the endpoint of a vector distinguishes it from the starting point. Here, vector $\overrightarrow{AB}$ is on the left, and vector $\overrightarrow{BA}$ is on the right.

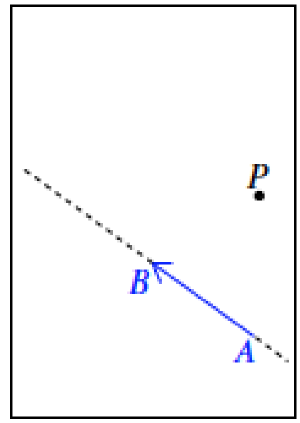


[1] Strictly speaking, all we need are reflections because rotations and translations can be shown to be compositions of reflections. However, for the purpose of fostering geometric intuition, we *should* employ all three.

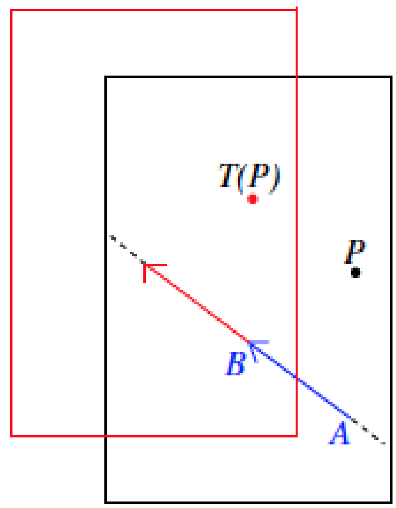
Example 2 (4 minutes)

MP.4

- We are going to describe how to define a translation T along a vector $\overrightarrow{AB}$ by the use of an overhead projector transparency. Let the plane be represented by a piece of paper on which the vector $\overrightarrow{AB}$ has been drawn. Let P be an arbitrary point in the plane (i.e., the paper), as shown. Note that the line containing the vector $\overrightarrow{AB}$, to be denoted by L_{AB}, is represented by the dotted line segment in the following picture. Note, also, that we are using a finite rectangle to represent the plane, which is infinite in all directions, and a finite segment to represent a line, which is infinite in both directions. The rectangle in the picture below represents the border of the piece of paper.



- Now trace $\overrightarrow{AB}$, the line L_{AB}, and the point P exactly on an overhead projector transparency (of exactly the same size as the paper) using a different color, say red. Then, P becomes the red dot on the transparency, and $\overrightarrow{AB}$ becomes a red vector on the transparency; we shall refer to them as the red dot and red vector, respectively, in this example. Keeping the paper fixed, we **slide the transparency along** $\overrightarrow{AB}$, moving the transparency in the direction from A to B, so that the red vector on the transparency stays on the line L_{AB}, until the starting point of the red vector rests on the endpoint B of the vector $\overrightarrow{AB}$, as shown in the picture below. In other words, we slide the transparency along the line AB, in the direction from A to B, for a distance equal to the length of the vector $\overrightarrow{AB}$. The picture shows the transparency after it has been slid along $\overrightarrow{AB}$, and the red rectangle represents the border of the transparency. The point of the plane at the red dot is, by definition, the image $Translation(P)$ of P by the translation T.



- If we need to be precise, we will denote the translation along $\overrightarrow{AB}$ by $Translation_{\overrightarrow{AB}}$. There is some subtlety in this notation: the vector $\overrightarrow{AB}$ has the starting point A and endpoint B, but the vector $\overrightarrow{BA}$ will have starting point B and endpoint A. Thus, $Translation_{\overrightarrow{AB}}$ is *different* from $\text{Translation}_{\overrightarrow{BA}}$. Precisely:

$$Translation_{\overrightarrow{AB}}(A) = B \text{ but, } Translation_{\overrightarrow{BA}}(B) = A. \qquad (1)$$

> *Note to Teacher:*
>
> In reading the descriptions about what translation does, please bear in mind that, in the classroom, a face-to-face demonstration with transparency and paper is far easier to understand than the verbal descriptions given in this lesson.

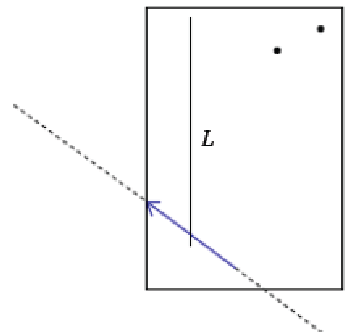
Video Presentation (2 minutes)

The following animation[2] of a translation would be helpful to a beginner: http://www.harpercollege.edu/~skoswatt/RigidMotions/translation.html

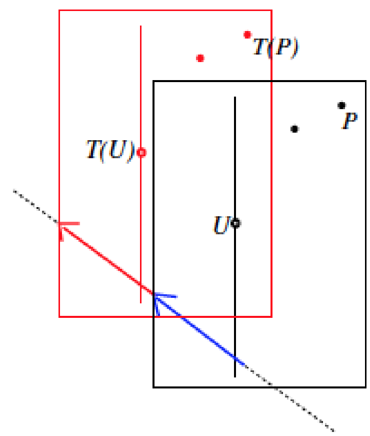
[2] Video developed by Sunil Koswatta.

Example 3 (4 minutes)

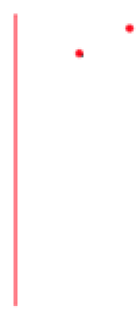
- Suppose we are given a geometric figure consisting of a vertical line L and two points, as shown. We are going to describe the effect on this figure when we translate the plane along the blue vector.



- Again, we copy the whole figure on a transparency in red and then slide the transparency along the blue vector. The whole figure is translated so that the red vertical line and the red dots are in the picture on the right. By definition, the translation T maps the black dots to the red dots, and similarly, T maps a typical point U on the vertical black line (represented by a tiny circle) to the point $T(U)$.



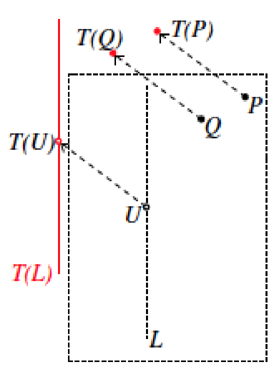
- If we draw the translated figure by itself without reference to the original, it is visually indistinguishable from the original.



Scaffolding:

In this lesson, we try to be clear and, therefore, must define and use terminology, precisely. However, in the classroom, it can sometimes be easier to point to the picture without using any definition, *at least for a while*. Introduce the terminology gradually, repeat often, and remind students of the meaning of the new words.

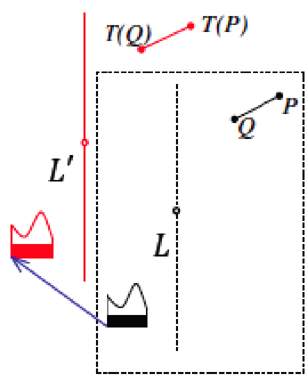
- So, we put back the original black figure as background information to show where the red figure *used to be*. The dashed arrows should suggest the assignment by the translation T.



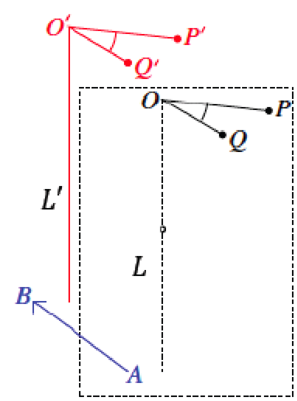
Example 4 (2 minutes)

MP.6

- We now make some observations about the basic properties of translations. We have covered how a translation T along a vector maps some point P to a point $T(P)$. Now, we examine what T does to *the total collection of points in a given figure*. We have actually done that implicitly because the red vertical line in Example 2 *is* the totality of *all* the points $T(U)$ where U is a point on the black vertical line L. For obvious reasons, we denote the red line by $T(L)$ (see picture in Example 3). More formally, if G is a given figure in the plane, then we denote by $T(G)$ the collection of *all* the points $T(P)$, where P is a point in G. We call $T(G)$ the **image of G by T**, and (as in the case of a point) we also say that T **maps G to $T(G)$**.



- The diagram to the right shows a translation of a figure along the blue vector. If G is the black figure, then $T(G)$ is the red figure, as shown.

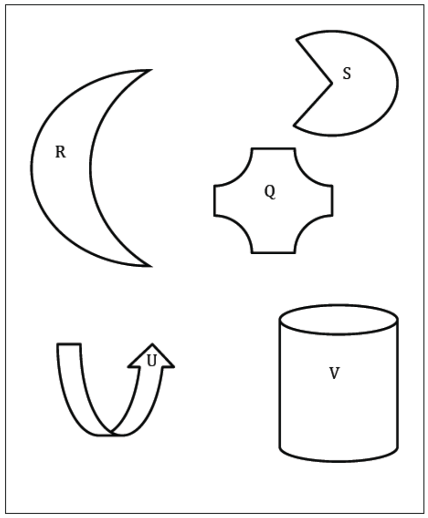


Exercise 1 (5 minutes)

Students complete Exercise 1 in pairs to practice translating a figure along various vectors. Students should describe to their partners what is happening in the translation in order to practice using the vocabulary and notation related to translations. (Note: Students are translating a *curved* figure because figures that consist entirely of line segments can be reproduced elsewhere on the plane by using a ruler and measuring lengths and degrees of angles, without the use of translation. This defeats the purpose of teaching the concept of translation to move figures around the plane.) Circulate to check student work. You may also want to call on students to share their work with the class.

Exercise 1

Draw at least three different vectors, and show what a translation of the plane along each vector will look like. Describe what happens to the following figures under each translation using appropriate vocabulary and notation as needed.



Answers will vary.

Example 5 (2 minutes)

- The notation to represent the image of a point can become cumbersome. However, there are ways to denote the image of the translated points using a more simplified notation. For example, point O after the translation can be denoted by O', said as "O prime;" point P is denoted by P', said as "P prime;" and point Q is denoted by Q', said as "Q prime."

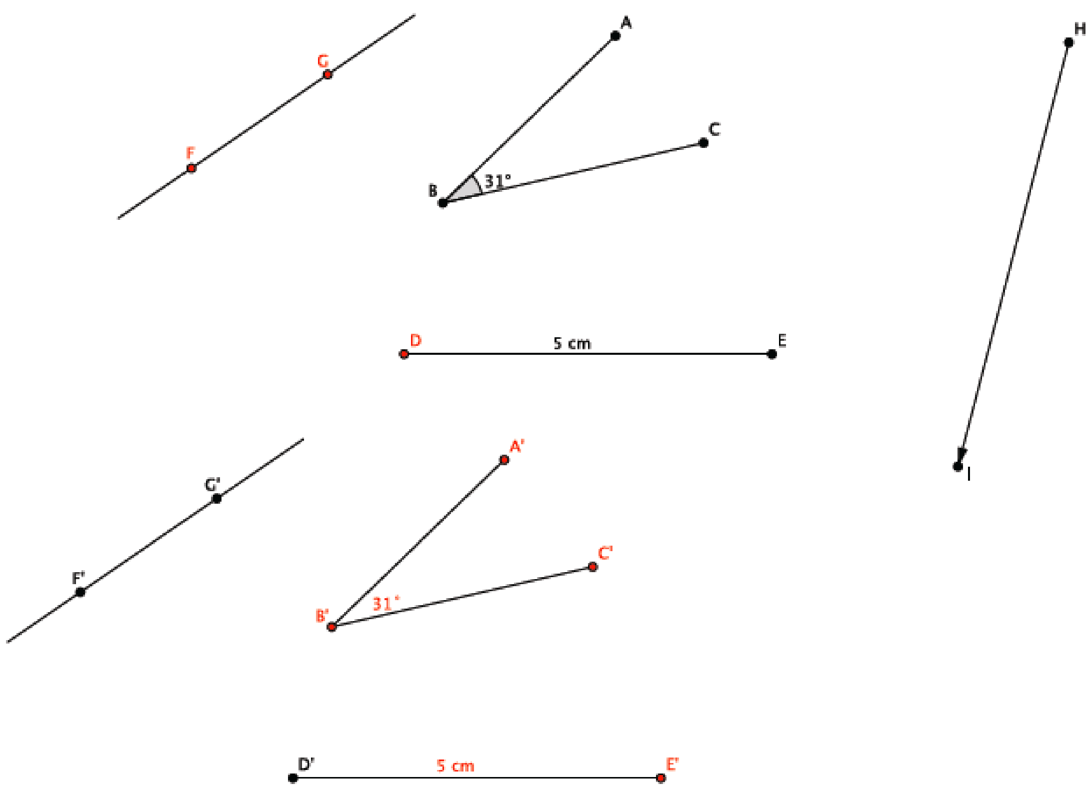
Exercise 2 (4 minutes)

Students will now be translating specific geometric figures (i.e., lines, angles, segments, and points). They should use the new notation, and record their observations as to the lengths of segments and sizes of angles as they complete Exercise 2 independently.

Exercise 2

The diagram below shows figures and their images under a translation along $\overrightarrow{HI}$. Use the original figures and the translated images to fill in missing labels for points and measures.

Solutions are in red, below.



Discussion (10 minutes)

- A translation maps lines to lines, segments to segments, and points to points. Do you believe it?
  - *Yes. After the translation, line FG was still a line, segment DE was still a segment, and the points were still points.*
- A translation preserves lengths of segments. Do you believe it?
  - *Yes. The segment DE in Exercise 2 was the same length after the translation as it was originally.*
- In general, if A and B are any two points in the plane, then the *distance between A and B* is, by definition, the length of the segment joining $Translation(A)$ and $Translation(B)$. Therefore, translations are *distance-preserving* as shown in Lesson 1, and translations are examples of rigid motions.

MP.2 & MP.6

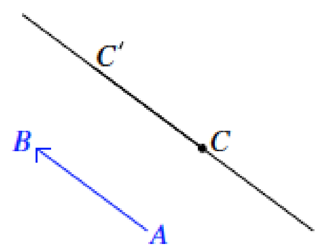
- Translations map angles to angles. Do you believe it?
  - *Yes. After the translation, angle ABC in Exercise 2 was still an angle. Its shape did not change.*
- A translation preserves the degree of an angle. Do you believe it?
  - *Yes. After the translation, angle ABC in Exercise 2 was still 32°. Its size did not change.*
- The following are some basic properties of translations:

  (Translation 1) A translation maps a line to a line, a ray to a ray, a segment to a segment, and an angle to an angle.

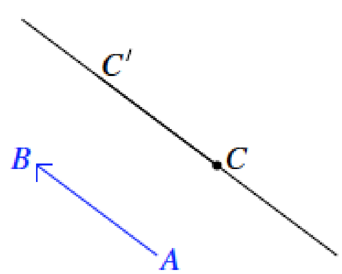
  (Translation 2) A translation preserves lengths of segments.

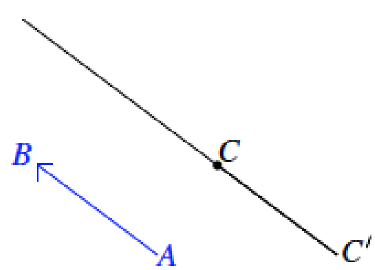
  (Translation 3) A translation preserves degrees of angles.
- These basic properties of translations will be taken for granted in all subsequent discussions of geometry. There are two more points to make about the effect a translation can have on points and lines.
- How can we describe the image of an arbitrary point C by a translation relative to C and the translation itself? In other words, is there a relationship between C, its image C', and the vector C is translated along?
- Let T be the translation along a given vector $\overrightarrow{AB}$. If we slide a transparency along the vector $\overrightarrow{AB}$, it is plausible that C will move to a point C' so that the vector $\overrightarrow{CC'}$ points in the same direction as the vector $\overrightarrow{AB}$, and the length of the segment CC' is the same length as the segment AB. Let's accept these conclusions at this point.

  Here is a pictorial representation of this situation for the case where C does not lie on the line L_{AB}:



To clarify, the statement, "vector $\overrightarrow{CC'}$ points in the same direction as the vector $\overrightarrow{AB}$," means that the point C' is moved as shown in the picture to the left below, rather than as shown in the picture to the right.





Still assuming that C is not a point on line L_{AB}, observe from the definition of a translation (in terms of sliding a transparency) that the line $L_{CC'}$ is parallel to the line L_{AB}. This is because the point C on the transparency is not on the blue line L_{AB} on the transparency. Therefore, as we slide the line L_{AB} on the transparency along the original L_{AB}, the point C will stay away from the original L_{AB}. For this reason, the point C traces out the line $L_{CC'}$ and has no point in common with the original L_{AB}. In other words, $L_{CC'} \parallel L_{AB}$.

If C is a point on the line L_{AB}, then C' will also be a point on the line L_{AB}, and in this case, $L_{CC'}$ coincides with the line L_{AB} rather than being parallel to the line L_{AB}.

- Our preliminary findings can be summarized as follows:

  Let T be the translation along $\overrightarrow{AB}$. Let C be a point in the plane, and let the image $T(C)$ be denoted by C'. Then, the vector $\overrightarrow{CC'}$ points in the same direction as $\overrightarrow{AB}$, and the vectors have the same length. If C lies on line L_{AB}, then so does C'. If C does not lie on L_{AB}, then $L_{CC'} \parallel L_{AB}$.

Closing (3 minutes)

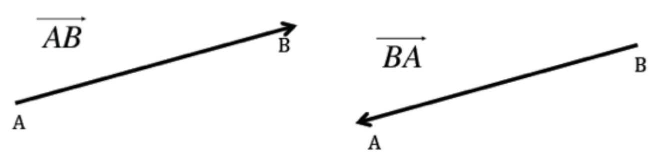
Summarize, or have students summarize, the lesson.

- We now know what a translation of a plane along a vector is.
- We know where a point, a line, or a figure on a plane moves to by a translation along a given vector.
- We know that translations have three basic properties:
  - Translations map lines to lines, segments to segments, rays to rays, and angles to angles.
  - Lengths of segments are preserved.
  - Degrees of measures of angles are preserved.
- We can now use a simplified notation, for example, P' to represent the translated point P.

Lesson Summary

Translation occurs along a given vector:

- **A vector is a segment in the plane with direction. One of its two endpoints is known as a starting point; while the other is known simply as the endpoint.**
- **The length of a vector is, by definition, the length of its underlying segment.**
- **Pictorially note the starting and endpoints:**



A translation of a plane along a given vector is a basic rigid motion of a plane.

The three basic properties of translation are as follows:

(Translation 1) A translation maps a line to a line, a ray to a ray, a segment to a segment, and an angle to an angle.
(Translation 2) A translation preserves lengths of segments.
(Translation 3) A translation preserves measures of angles.

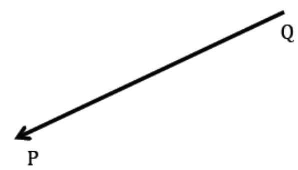
Exit Ticket (4 minutes)

Name ______________________________ Date ______________

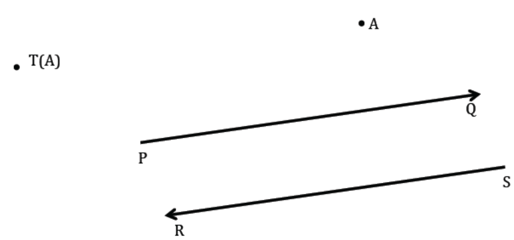
Lesson 2: Definition of Translation and Three Basic Properties

Exit Ticket

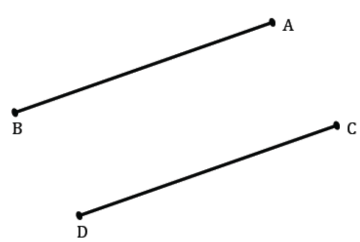
1. Name the vector in the picture below.



2. Name the vector along which a translation of a plane would map point A to its image $T(A)$.



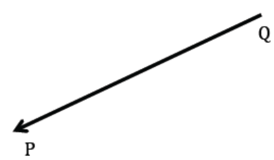
3. Is Maria correct when she says that there is a translation along a vector that will map segment AB to segment CD? If so, draw the vector. If not, explain why not.



4. Assume there is a translation that will map segment AB to segment CD shown above. If the length of segment CD is 8 units, what is the length of segment AB? How do you know?

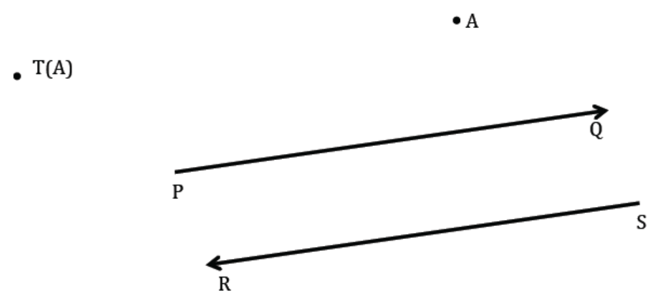
Exit Ticket Sample Solutions

1. **Name the vector in the picture below.**



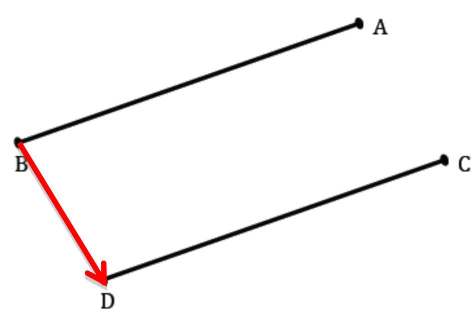
$\overrightarrow{QP}$

2. **Name the vector along which a translation of a plane would map point A to its image $T(A)$.**



$\overrightarrow{SR}$

3. **Is Maria correct when she says that there is a translation along a vector that will map segment AB to segment CD? If so, draw the vector. If not, explain why not.**



Yes. Accept any vector that would translate the segment AB to segment CD. A possible vector is shown in red, above.

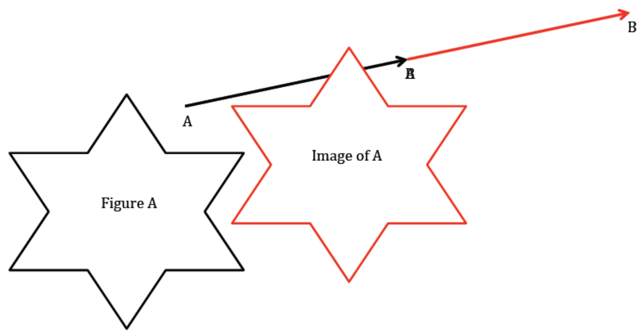
4. **Assume there is a translation that will map segment AB to segment CD shown above. If the length of segment CD is 8 units, what is the length of segment AB? How do you know?**

The length of CD must be 8 units in length because translations preserve the lengths of segments.

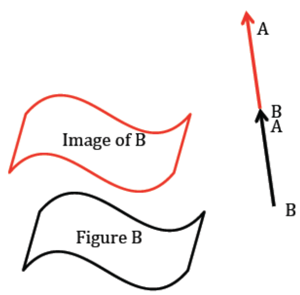
Problem Set Sample Solutions

1. Translate the plane containing Figure A along $\overrightarrow{AB}$. Use your transparency to sketch the image of Figure A by this translation. Mark points on Figure A and label the image of Figure A accordingly.

   Marked points will vary. Verify that students have labeled their points and images appropriately.



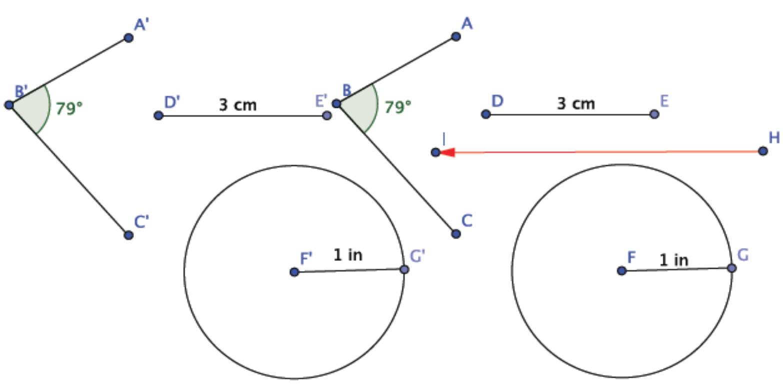
2. Translate the plane containing Figure B along $\overrightarrow{BA}$. Use your transparency to sketch the image of Figure B by this translation. Mark points on Figure B and label the image of Figure B accordingly.



   Marked points will vary. Verify that students have labeled their points and images appropriately.

3. Draw an acute angle (your choice of degree), a segment with length 3 cm, a point, a circle with radius 1 in., and a vector (your choice of length, i.e., starting point and ending point). Label points and measures (measurements do not need to be precise, but your figure must be labeled correctly). Use your transparency to translate all of the figures you have drawn along the vector. Sketch the images of the translated figures and label them.

   Drawings will vary. Note: Drawing is not to scale.



4. **What is the length of the translated segment? How does this length compare to the length of the original segment? Explain.**

The length is 3 cm. The length is the same as the original because translations preserve the lengths of segments.

5. **What is the length of the radius in the translated circle? How does this radius length compare to the radius of the original circle? Explain.**

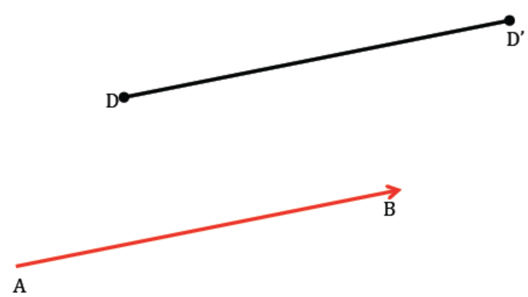
The length is 1 in. The length is the same as the original because translations preserve lengths of segments.

6. **What is the degree of the translated angle? How does this degree compare to the degree of the original angle? Explain.**

Answers will vary based on the original size of the angle drawn. The angles will have the same measure because translations preserve degrees of angles.

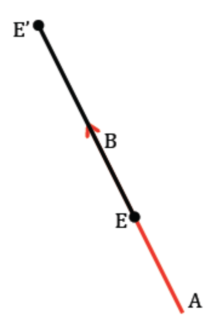
7. **Translate point D along vector $\overrightarrow{AB}$ and label the image D'. What do you notice about the line containing vector $\overrightarrow{AB}$ and the line containing points D and D'? (Hint: Will the lines ever intersect?)**

The lines will be parallel.



8. **Translate point E along vector $\overrightarrow{AB}$ and label the image E'. What do you notice about the line containing vector $\overrightarrow{AB}$ and the line containing points E and E'?**

The lines will coincide.



Lesson 3: Translating Lines

Student Outcomes

- Students learn that when lines are translated, they are either parallel to the given line or they coincide.
- Students learn that translations map parallel lines to parallel lines.

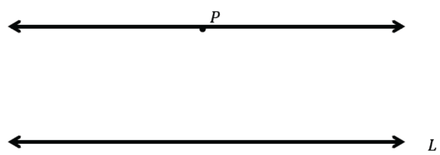
Classwork

Exercise 1 (3 minutes)

Students complete Exercise 1 independently in preparation for the discussion that follows.

Exercises

1. **Draw a line passing through point P that is parallel to line L. Draw a second line passing through point P that is parallel to line L, and that is distinct (i.e., different) from the first one. What do you notice?**



Students should realize that they can only draw one line through point P that is parallel to L.

Discussion (3 minutes)

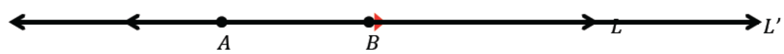
Bring out a fundamental assumption about the plane (as observed in Exercise 1):

- Given a line L and a point P not lying on L, there is at most one line passing through P and parallel to L.
  - Based on what we have learned up to now, we can cannot prove or explain this, so we have to simply agree that this is one of the starting points in the study of the plane.
  - This idea plays a key role in everything we do in the plane. A first consequence is that given a line L and a point P not lying on L, we can now refer to the line (because we agree there is only one) passing through P and parallel to L.

Exercises 2–4 (9 minutes)

Students complete Exercises 2–4 independently in preparation for the discussion that follows.

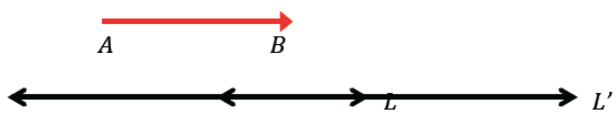
2. **Translate line L along the vector $\overrightarrow{AB}$. What do you notice about L and its image L'?**



L and L' coincide. $L = L'$.

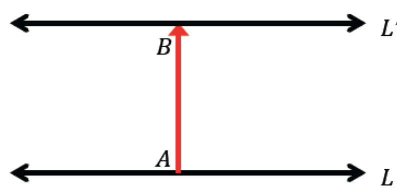
> *Scaffolding:*
>
> - Refer to Exercises 2–4 throughout the discussion and in the summary of findings about translating

3. **Line L is parallel to vector $\overrightarrow{AB}$. Translate line L along vector $\overrightarrow{AB}$. What do you notice about L and its image, L'?**



L and L' coincide, again. $L = L'$.

4. **Translate line L along the vector $\overrightarrow{AB}$. What do you notice about L and its image, L'?**



$L \parallel L'$.

Discussion (15 minutes)

- Now we examine the effect of a translation on a *line*. Thus, let line L be given. Again, let the translation be along a given $\overrightarrow{AB}$ and let L' denote the image line of the translated L. We want to know what L' is relative to AB and line L.
- If $L = L_{AB}$, or $L \parallel L_{AB}$, then $L' = L$.
  - If $L = L_{AB}$, then this conclusion follows directly from the work in Lesson 2, which says if C is on L_{AB}, then so is C'; therefore, $L' = L_{AB}$ and $L = L'$ (Exercise 2).
  - If $L \parallel L_{AB}$ and C is on L, then it follows from the work in Lesson 2, which says that C' lies on the line l passing through C and parallel to L_{AB}. However, L is given as a line passing through C and parallel to L_{AB}, so the fundamental assumption that there is just one line passing through a point, parallel to a line (Exercise 1), implies $l = L$. Therefore, C' lies on L after all, and the translation maps every point of L to a point of L'. Therefore, $L = L'$ again (Exercise 3).

> *Note to Teacher:*
>
> We use the notation $Translation(L)$ as a precursor to the notation students will encounter in Grade 10, i.e., $T(L)$. We want to make clear the basic rigid motion that is being performed, so the notation: $Translation(L)$ is written to mean "the translation of L along the specified vector."

- Caution: One must not over-interpret the equality $Translation(L) = L$ (which is the same as $L = L'$).
- All the equality says is that the two lines L and L' coincide completely. It is easy (but wrong) to infer from the equality $Translation(L) = L$ that for any point P on L, $Translation(P) = P$. Suppose the vector $\overrightarrow{AB}$ lying on L is not the zero vector (i.e., assume $A \neq B$). Trace the line L on a transparency to obtain a red line L, and now slide the transparency along $\overrightarrow{AB}$. Then, the red line, as a line, coincides with the original L, but clearly every point on L has been moved by the slide (the translation). Indeed, as we saw in Example 2 of Lesson 2, $Translation(A) = B \neq A$. Therefore, the equality $L' = L$ only says that for any point C on L, $Translation(C)$ is also a point on L, but as long as $\overrightarrow{AB}$ is not a zero vector, $Translation(C) \neq C$.

MP.6

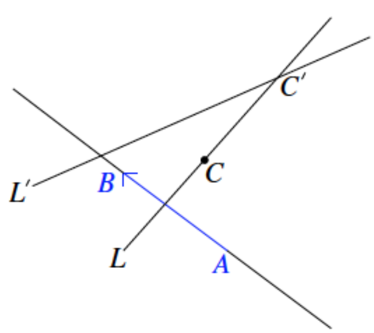
Note to Teacher: Strictly speaking, we have not completely proved $L = L'$ in either case. To explain this, let us *define* what it means for two geometric figures F and G to be equal, i.e., $F = G$: it means each point of F is also a point of G; conversely, each point of G is also a point of F. In this light, all we have shown above is that if every point C' of L' belongs to L, then Q is also a point of L'. To show the latter, we have to show that this Q is equal to $Translation(P)$ for some P on L. This will then complete the reasoning.

However, at this point of students' education in geometry, it may be prudent not to bring up such a sticky point because they already will be challenged with all of the new ideas and definitions. Simply allow the preceding reasoning to stand for now, and clarify later in the school year when students are more comfortable with the geometric environment.

- Next, if L is neither L_{AB} nor parallel to L_{AB}, then $L' \parallel L$.
- If we use a transparency to see this translational image of L by the stated translation, then the pictorial evidence is clear: the line L moves in a parallel manner along $\overrightarrow{AB}$ and a typical point C of L is translated to a point C' of L'. The fact that $L' \parallel L$ is unmistakable, as shown. In the classroom, students should be convinced by the pictorial evidence. If so, we will leave it at that (Exercise 4).

MP.2 & MP.7

Note to Teacher: Here is a simple proof, but if you are going to present it in class, begin by asking students *how* they would prove that two lines are parallel. Ensure students understand that they have no tools in their possession to accomplish this goal. It is only then that they see the need for invoking a proof by contradiction (see discussion above). *If there are no obvious ways to do something, then you just have to do the best you can by trying to see what happens if you assume the opposite is true*. Thus, if L' is not parallel to L, then they intersect at a point C'. Since C' lies on L', it follows from the definition of L' (as the image of L under the translation T) that there is a point C on L so that $Translation(C) = C'$.



It follows from Lesson 2 that $L_{CC'} \parallel L_{AB}$. However, both C and C' lie on L, so $L_{CC'} \parallel L$, and we get $L \parallel L_{AB}$. This contradicts the assumption that L is not parallel to L_{AB}, so L could not possibly intersect L'. Therefore, $L' \parallel L$.

- Note that a translation maps parallel lines to parallel lines. More precisely, consider a translation T along a vector $\overrightarrow{AB}$. Then:

  If L_1 and L_2 are parallel lines, so are $Translation(L_1)$ and $Translation(L_2)$.

- The reasoning is the same as before: Copy L_1 and L_2 onto a transparency and then translate the transparency along $\overrightarrow{AB}$. If L_1 and L_2 do not intersect, then their red replicas on the transparency will not intersect either, no matter what $\overrightarrow{AB}$ is used. So, $Translation(L_1)$ and $Translation(L_2)$ are parallel.
- We summarize these findings as follows:

  Given a translation T along a vector $\overrightarrow{AB}$, let L be a line, and let L' denote the image of L by T.

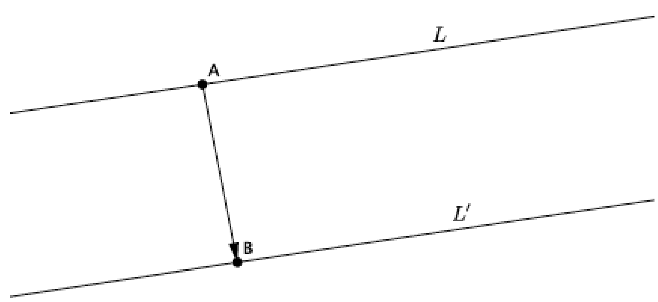
MP.6

  - If $L \parallel L_{AB}$ or $L = L_{AB}$, then $L' \parallel L$.
  - If L is neither parallel to L_{AB} nor equal to L_{AB}, then $L' \parallel L$.

Exercises 5–6 (5 minutes)

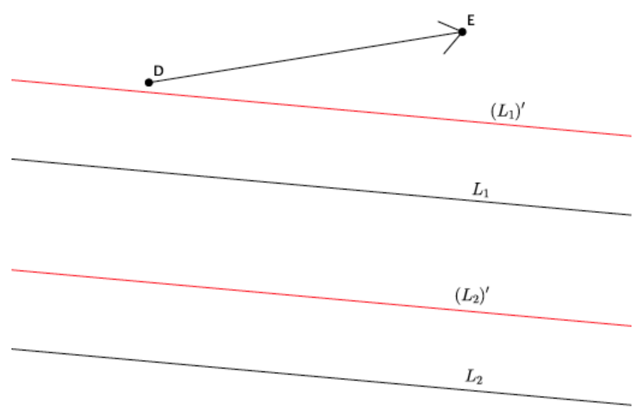
Students complete Exercises 5 and 6 in pairs or small groups.

5. Line L has been translated along vector $\overrightarrow{AB}$ resulting in L'. What do you know about lines L and L'?



$L \parallel T(L)$

6. Translate L_1 and L_2 along vector $\overrightarrow{DE}$. Label the images of the lines. If lines L_1 and L_2 are parallel, what do you know about their translated images?



Since $L_1 \parallel L_2$, then $(L_1)' \parallel (L_2)'$.

Closing (5 minutes)

Summarize, or have students summarize, the lesson.

- We know that there exists just one line, parallel to a given line and through a given point not on the line.
- We know that translations map parallel lines to parallel lines.
- We know that when lines are translated, they are either parallel to the given line or they coincide.

> Lesson Summary
>
> - **Two lines are parallel if they do not intersect.**
> - **Translations map parallel lines to parallel lines.**
> - **Given a line L and a point P not lying on L, there is at most one line passing through P and parallel to L.**

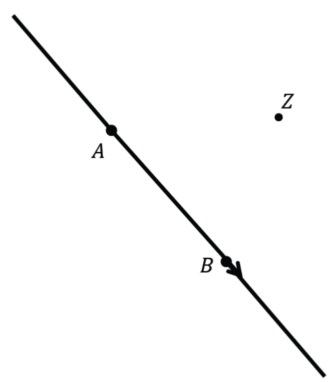
Exit Ticket (5 minutes)

Name ______________________________ Date ____________

Lesson 3: Translating Lines

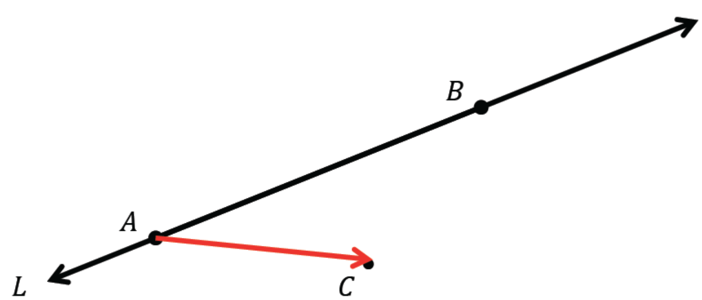
Exit Ticket

1. Translate point Z along vector $\overrightarrow{AB}$. What do you know about the line containing vector $\overrightarrow{AB}$ and the line formed when you connect Z to its image Z'?



2. Using the above diagram, what do you know about the lengths of segments ZZ' and AB?

3. Let points A and B be on line L, and the vector $\overrightarrow{AC}$ be given, as shown below. Translate line L along vector $\overrightarrow{AC}$. What do you know about line L and its image, L'? How many other lines can you draw through point C that have the same relationship as L and L'? How do you know?



Exit Ticket Sample Solutions

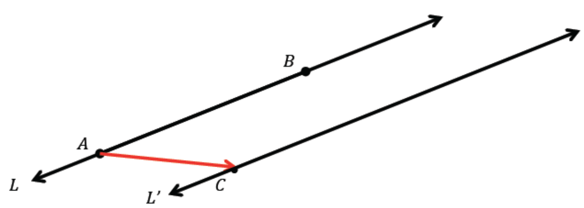
1. Translate point Z along vector $\overrightarrow{AB}$. What do you know about the line containing vector $\overrightarrow{AB}$ and the line formed when you connect Z to its image Z'?

   The line containing vector $\overrightarrow{AB}$ and ZZ' is parallel.

2. Using the above diagram, what do you know about the lengths of segment ZZ' and segment AB?

   The lengths are equal: $|ZZ'| = |AB|$.

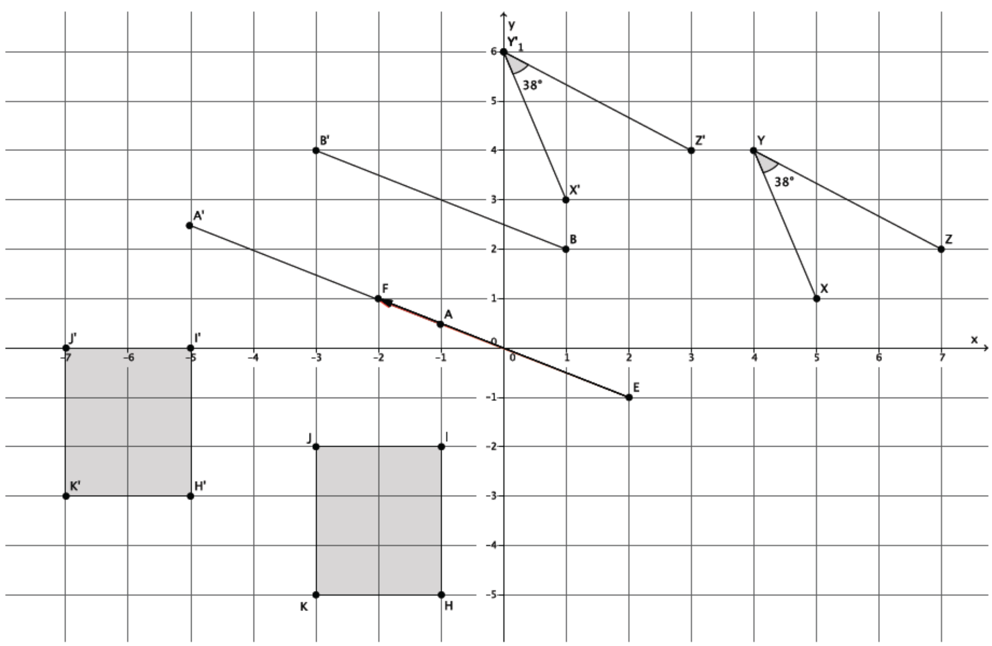
3. Let points A and B be on line L, and the vector $\overrightarrow{AC}$ be given, as shown below. Translate line L along vector $\overrightarrow{AC}$. What do you know about line L and its image, L'? How many other lines can you draw through point C that have the same relationship as L and L'? How do you know?



   L and L' are parallel. There is only one line parallel to line L that goes through point C. The fact that there is only one line through a point parallel to a given line guarantees it.

Problem Set Sample Solutions

1. Translate $\angle XYZ$, point A, point B, and rectangle $HIJK$ along vector $\overrightarrow{EF}$. Sketch the images and label all points using prime notation.



2. What is the measure of the translated image of $\angle XYZ$. How do you know?

   The measure is $38°$. Translations preserve angle measure.

3. Connect B to B'. What do you know about the line formed by BB' and the line containing the vector $\overrightarrow{EF}$?

   $\boldsymbol{BB' \parallel \overrightarrow{EF}}$.

4. Connect A to A'. What do you know about the line formed by AA' and the line containing the vector $\overrightarrow{EF}$?

   AA' and $\overrightarrow{EF}$ coincide.

5. Given that figure $HIJK$ is a rectangle, what do you know about lines HI and JK and their translated images? Explain.

   Since $HIJK$ is a rectangle, I know that $HI \parallel JK$. Since translations map parallel lines to parallel lines, then $H'I' \parallel J'K'$.

Lesson 4: Definition of Reflection and Basic Properties

Student Outcomes

- Students know the definition of reflection and perform reflections across a line using a transparency.
- Students show that reflections share some of the same fundamental properties with translations (e.g., lines map to lines, angle- and distance-preserving motion, etc.). Students know that reflections map parallel lines to parallel lines.
- Students know that for the reflection across a line L, then every point P, not on L, L is the bisector of the segment joining P to its reflected image P'.

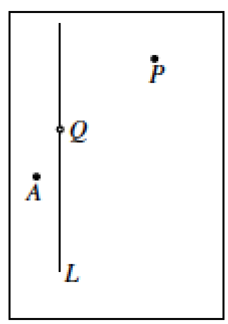
Classwork

Example 1 (5 minutes)

MP.6

The reflection across a line L is defined by using the following example.

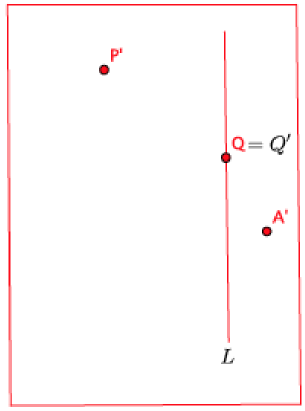
- Let L be a vertical line, and let P and A be two points not on L, as shown below. Also, let Q be a point on L. (The black rectangle indicates the border of the paper.)



- The following is a description of how the reflection moves the points P, Q, and A by making use of the transparency.
- Trace the line L and three points onto the transparency exactly, using red. (Be sure to use a transparency that is the same size as the paper.)
- Keeping the paper fixed, flip the transparency across the vertical line (interchanging left and right) while keeping the vertical line and point Q on top of their black images.
- The position of the red figures on the transparency now represents the reflection of the original figure. $Reflection(P)$ is the point represented by the red dot to the left of L, $Reflection(A)$ is the red dot to the right of L, and point $Reflection(Q)$ is point Q itself.
- Note that point Q is unchanged by the reflection.
- The red rectangle in the picture on the next page represents the border of the transparency.

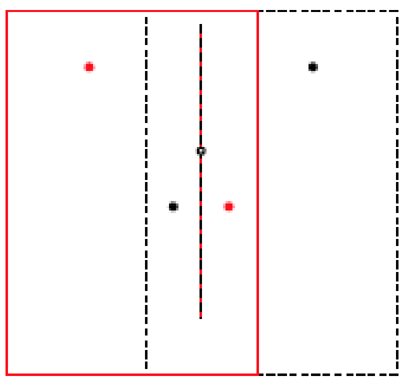
Scaffolding:

- There are manipulatives, such as MIRA and Georeflector, which facilitate the learning of reflections by producing a reflected image.



- In the picture above, you see that the reflected image of the points is noted similar to how we represented translated images in Lesson 2. That is, the reflected point P is P'. More importantly, note that the line L and point Q have reflected images in exactly the same location as the original; hence, $Reflection(L) = L$ and $Reflection(Q) = Q$, respectively.

- The figure and its reflected image are shown together, below.



- Pictorially, reflection moves all of the points in the plane by *reflecting* them across L as if L were a mirror. The line L is called the *line of reflection.* A reflection across line L may also be noted as $Reflection_L$.

Video Presentation (2 minutes)

The following animation[1] of a reflection will be helpful to beginners.

http://www.harpercollege.edu/~skoswatt/RigidMotions/reflection.html

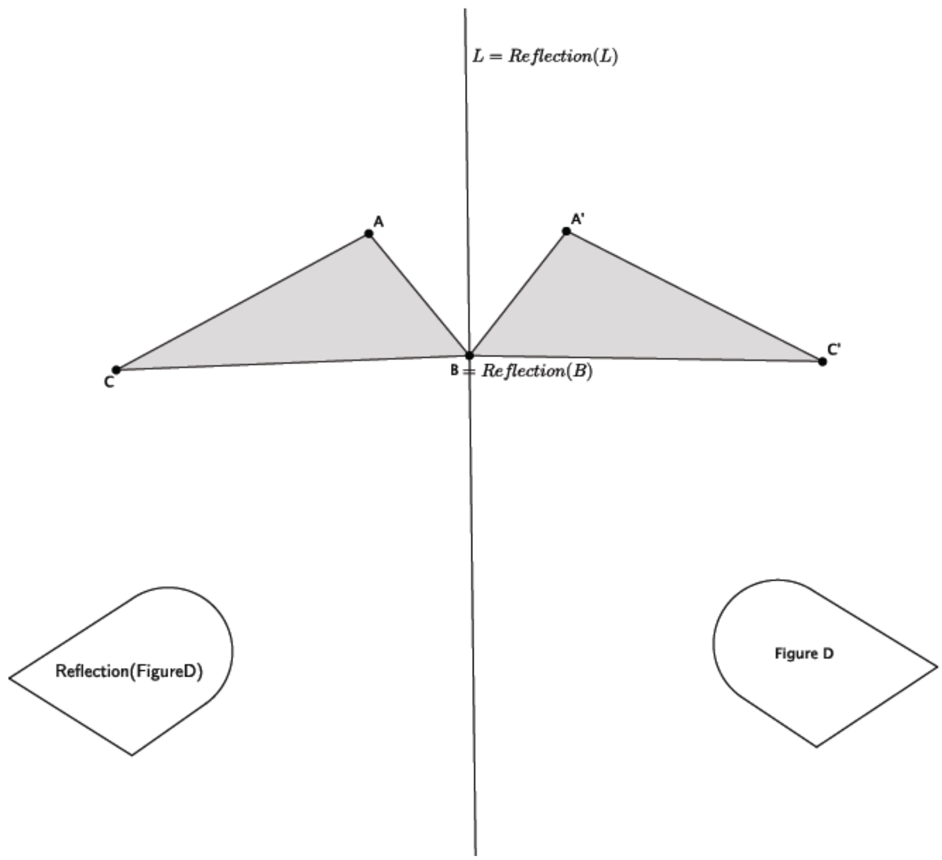
[1] Animation developed by Sunil Koswatta.

Exercises 1–2 (3 minutes)

Students complete Exercises 1 and 2 independently.

Exercises

1. **Reflect $\triangle ABC$ and Figure D across line L. Label the reflected images.**



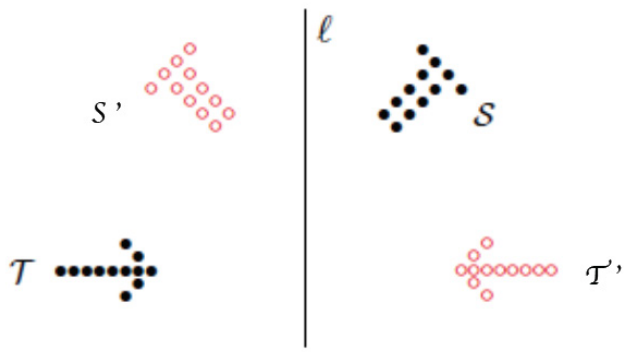
2. **Which figure(s) were not moved to a new location on the plane under this transformation?**

   Point B and line L were not moved to a new location on the plane under this reflection.

Example 2 (3 minutes)

Now we look at some features of reflected geometric figures in the plane.

- If we reflect across a vertical line l, then the reflected image of right-pointing figures, such as T below, will be left-pointing. Similarly, the reflected image of a right-leaning figure, such as S below, will become left-leaning.

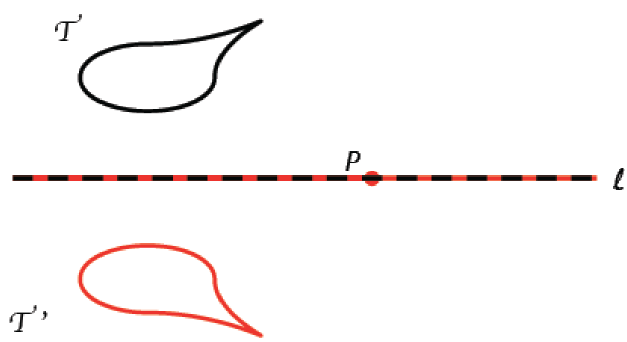


- Observe that *up* and *down* do not change in the reflection across a vertical line. Also observe that the horizontal figure T remains horizontal. This is similar to what a real mirror does.

Example 3 (2 minutes)

A line of reflection can be any line in the plane. In this example, we look at a horizontal line of reflection.

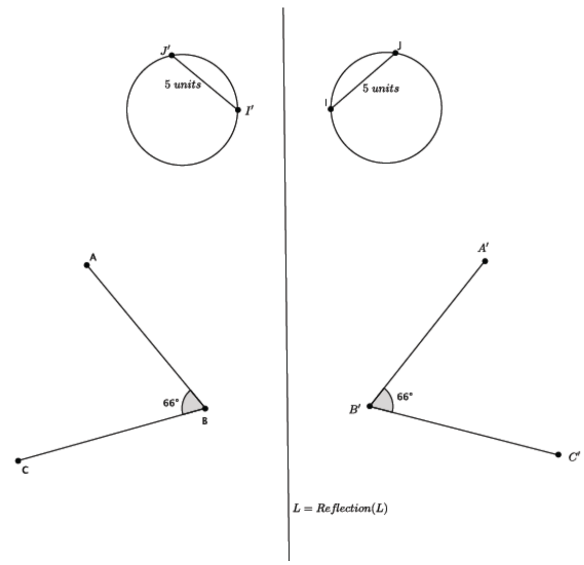
- Let l be the horizontal line of reflection, P be a point off of line l, and T be the figure above the line of reflection.
- Just as before, if we trace everything in red on the transparency and reflect across the horizontal line of reflection, we see the reflected images in red, as shown below.



Exercises 3–5 (5 minutes)

Students complete Exercises 3–5 independently.

3. **Reflect the images across line L. Label the reflected images.**



4. **Answer the questions about the previous image.**
   a. **Use a protractor to measure the reflected $\angle ABC$. What do you notice?**

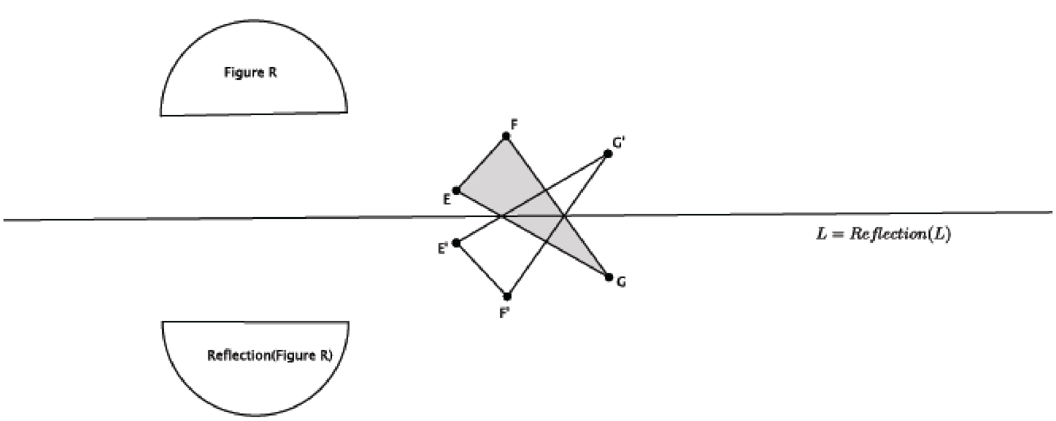
      The measure of the reflected image of $\angle ABC$ is $66°$.

   b. **Use a ruler to measure the length IJ and the length of the image of IJ after the reflection. What do you notice?**

      The length of the reflected segment is the same as the original segment, 5 units.

      Note: This is not something students are expected to know, but it is a preview for what is to come later in this lesson.

5. **Reflect Figure R and $\triangle EFG$ across line L. Label the reflected images.**



Discussion (3 minutes)

As with translation, a reflection has the same properties as (Translation 1)–(Translation 3) of Lesson 2. Precisely, lines, segments, angles, etc., are moved by a reflection by moving their *exact* replicas (on the transparency) to another part of the plane. Therefore, distances and degrees are preserved.

(Reflection 1) A reflection maps a line to a line, a ray to a ray, a segment to a segment, and an angle to an angle.

(Reflection 2) A reflection preserves lengths of segments.

(Reflection 3) A reflection preserves degrees of angles.

These basic properties of reflections will be taken for granted in all subsequent discussions of geometry.

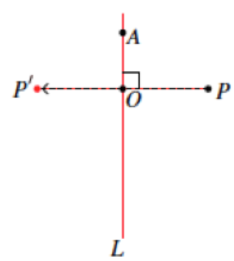
Basic Properties of Reflections:

(Reflection 1) A reflection maps a line to a line, a ray to a ray, a segment to a segment, and an angle to an angle.

(Reflection 2) A reflection preserves lengths of segments.

(Reflection 3) A reflection preserves measures of angles.

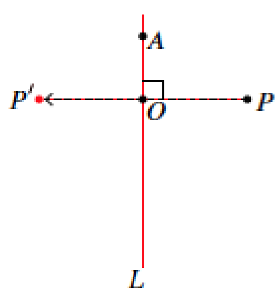
If the reflection is across a line L and P is a point not on L, then L bisects the segment PP', joining P to its reflected image P'. That is, the lengths of OP and OP' are equal.



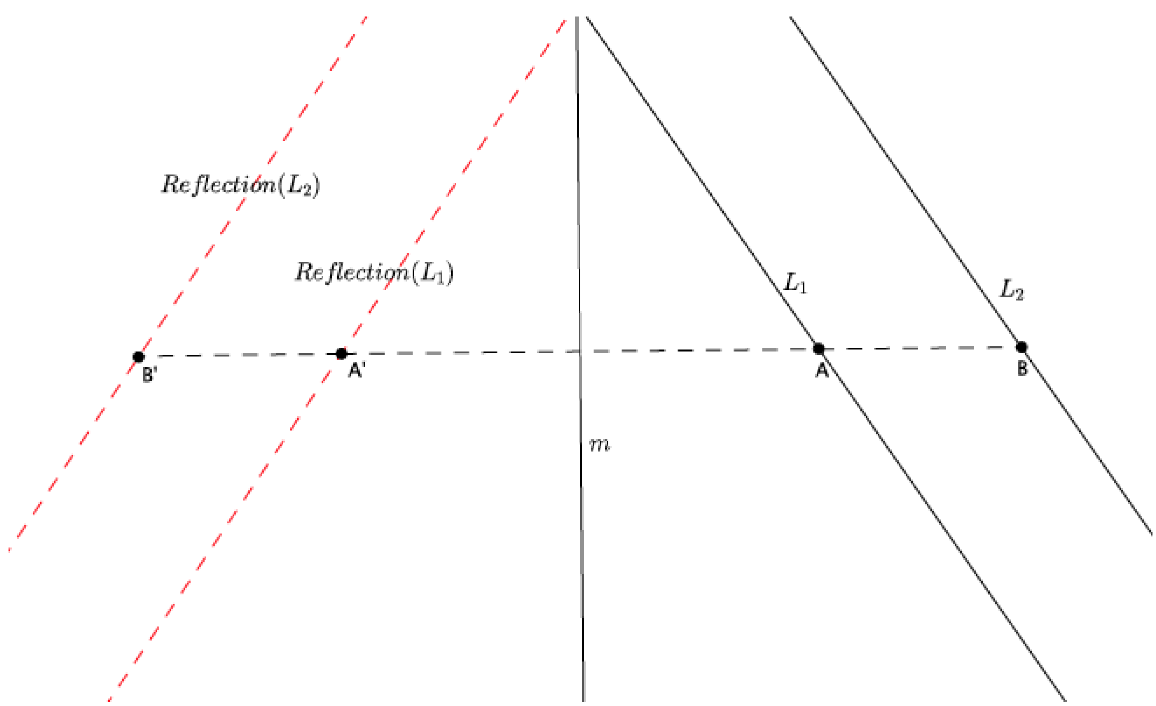
Example 4 (7 minutes)

A simple consequence of (Reflection 2) is that it gives a more precise description of the position of the reflected image of a point.

- Let there be a reflection across line L, let P be a point not on line L, and let P' represent $Reflection(P)$. Let the line PP' intersect L at O, and let A be a point on L distinct from O, as shown.



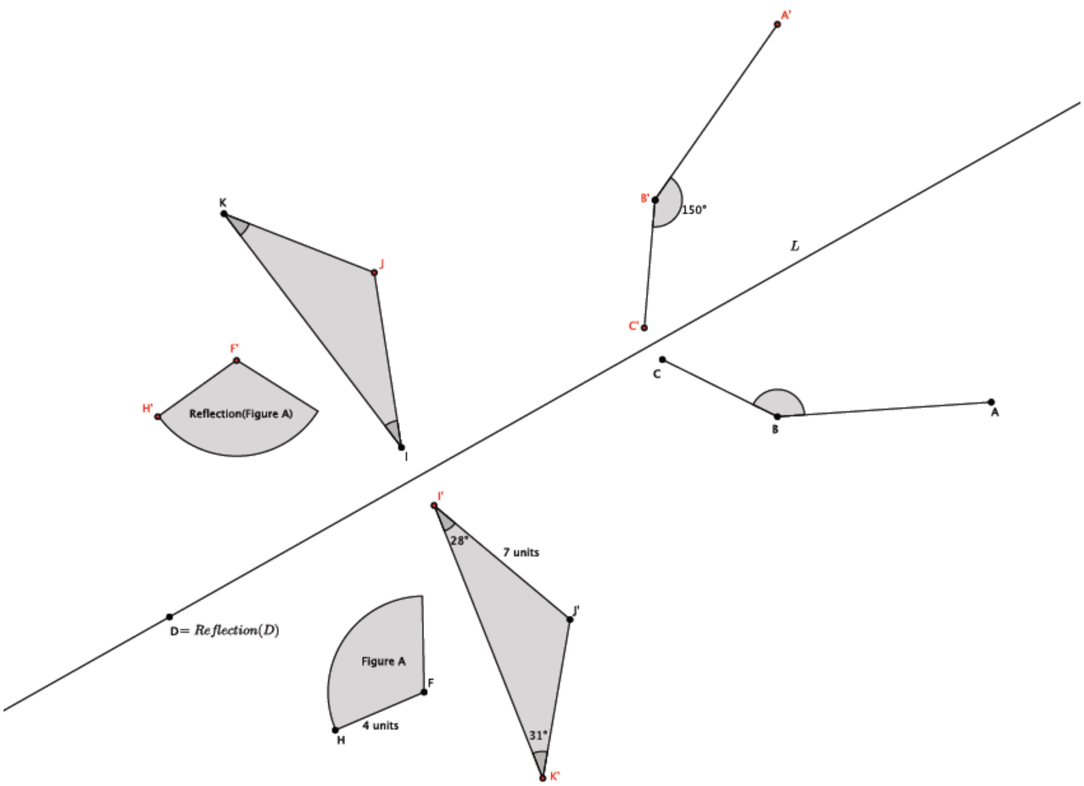
- What can we say about segments PO and OP'?
  - *Because* $Reflection(PO) = P'O$, *(Reflection 2) guarantees that segments* PO *and* $P'O$ *have the same length.*
- In other words, O is the *midpoint* (i.e., the point equidistant from both endpoints) of PP'.
- In general, the line passing through the midpoint of a segment is said to *bisect* the segment.
- What happens to point A under the reflection?
  - *Because the line of reflection maps to itself, then point A remains unmoved, i.e.,* $A = A'$.
- As with translations, reflections map parallel lines to parallel lines. (i.e., If $L_1 \parallel L_2$, and there is a reflection across a line, then $Reflection(L_1) \parallel Reflection(L_2)$.)
- Let there be a reflection across line m. Given $L_1 \parallel L_2$, then $Reflection(L_1) \parallel Reflection(L_2)$. The reason is that any point A on line L_1 will be reflected across m to a point A' on $Reflection(L_1)$. Similarly, any point B on line L_2 will be reflected across m to a point B' on $Reflection(L_2)$. Since $L_1 \parallel L_2$, no point A on line L_1 will ever be on L_2, and no point B on L_2 will ever be on L_1. The same can be said for the reflections of those points. Then, since $Reflection(L_1)$ shares no points with $Reflection(L_2)$, $Reflection(L_1) \parallel Reflection(L_2)$.



Exercises 6–9 (7 minutes)

Students complete Exercises 6–9 independently.

Use the picture below for Exercises 6–9.



6. **Use the picture to label the unnamed points.**

   Points labeled in red, above.

7. **What is the measure of $\angle JKI$? $\angle KIJ$? $\angle ABC$? How do you know?**

   $m\angle JKI = 31°$, *and* $m\angle KIJ = 28°$, *and* $m\angle ABC = 150°$. *Reflections preserve angle measures.*

8. **What is the length of segment $Reflection(FH)$? IJ? How do you know?**

   $|Reflection(FH)| = 4$ *units, and* $IJ = 7$ *units. Reflections preserve lengths of segments.*

9. **What is the location of $Reflection(D)$? Explain.**

   Point D and its image are in the same location on the plane. Point D was not moved to another part of the plane because it is on the line of reflection. The image of any point on the line of reflection will remain in the same location as the original point.

Closing (4 minutes)

Summarize, or have students summarize, the lesson.

- We know that a reflection across a line is a basic rigid motion.
- Reflections have the same basic properties as translations; reflections map lines to lines, rays to rays, segments to segments and angles to angles.
- Reflections have the same basic properties as translations because they, too, are distance- and angle-preserving.
- The line of reflection L is the bisector of the segment that joins a point not on L to its image.

> Lesson Summary
>
> - **A reflection is another type of basic rigid motion.**
> - **Reflections occur across lines. The line that you reflect across is called the line of reflection.**
> - **When a point, P, is joined to its reflection, P', the line of reflection bisects the segment, PP'.**

Exit Ticket (4 minutes)

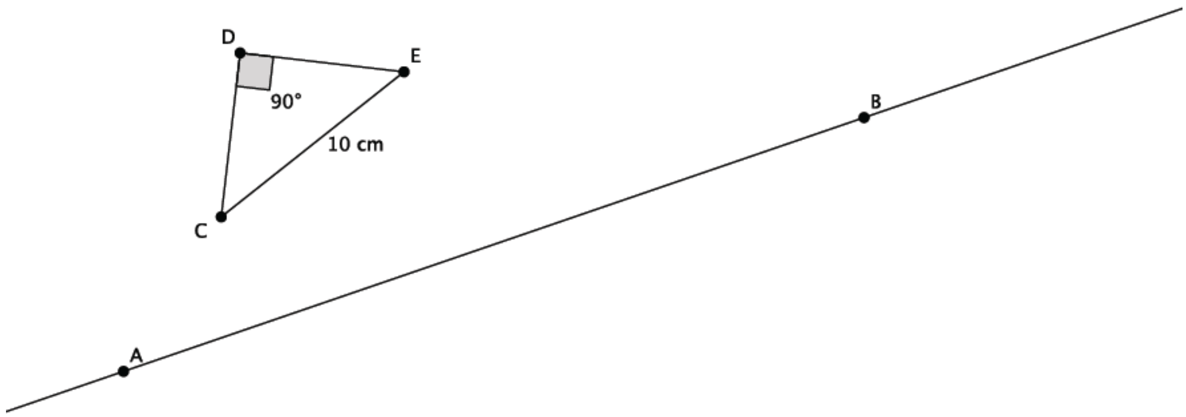
Name ______________________________ Date ______________

Lesson 4: Definition of Reflection and Basic Properties

Exit Ticket

1. Let there be a reflection across line L_{AB}. Reflect $\triangle\, CDE$ and label the reflected image.

Picture not drawn to scale.



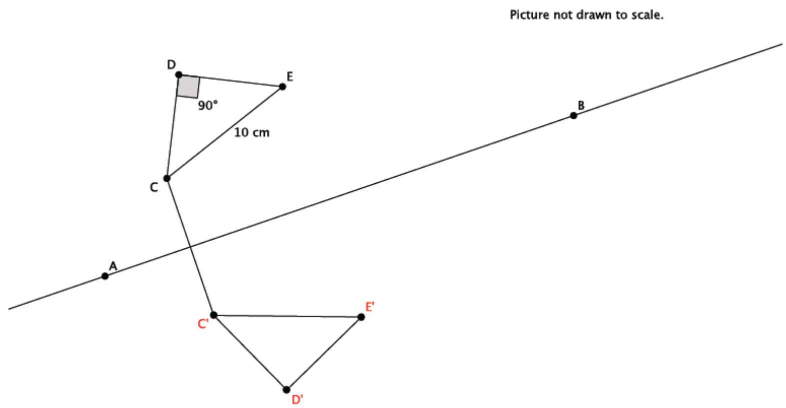
2. Use the diagram above to state the measure of $Reflection(\angle CDE)$. Explain.

3. Use the diagram above to state the length of segment $Reflection(CE)$. Explain.

4. Connect point C to its image in the diagram above. What is the relationship between line L_{AB} and the segment that connects point C to its image?

Exit Ticket Sample Solutions

1. Let there be a reflection across line L_{AB}. Reflect $\triangle CDE$ across line L_{AB}. Label the reflected image.



2. Use the diagram above to state the measure of $Reflection(\angle CDE)$. Explain.

   The measure of $Reflection(\angle CDE) = 90°$ because reflections preserve degrees of measures of angles.

3. Use the diagram above to state the length of segment $Reflection(CE)$. Explain.

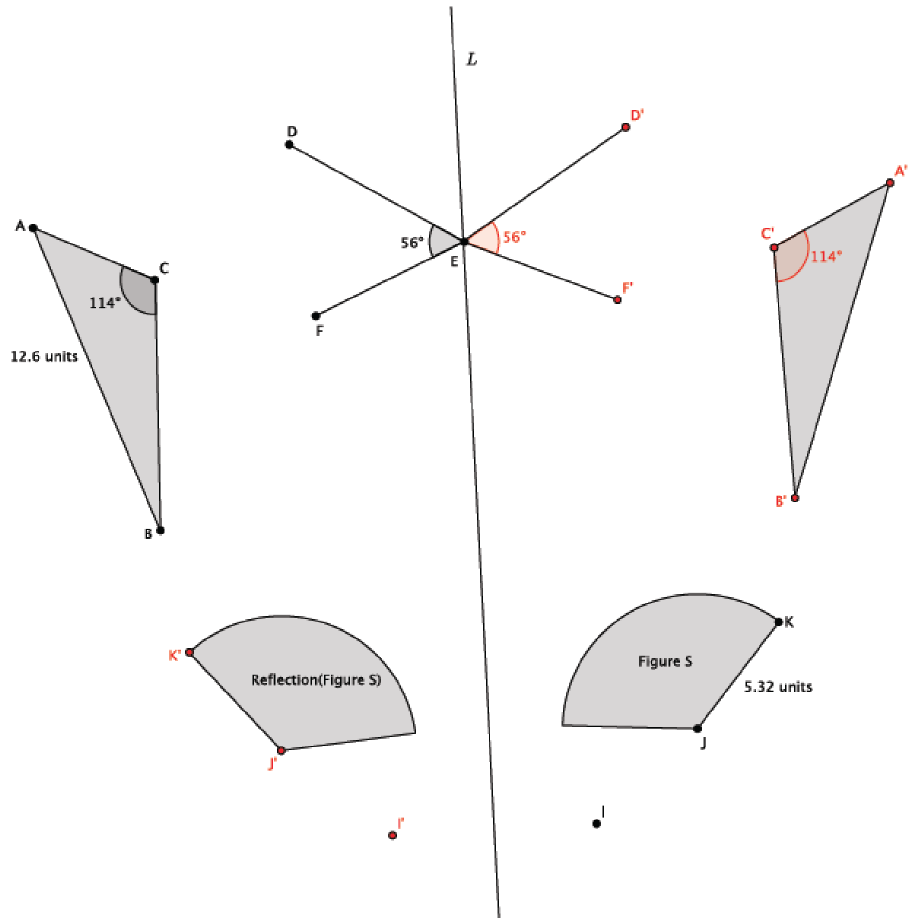
   The length of $Reflection(CE)$ is 10 cm because reflections preserve segment lengths.

4. Connect point C to its image in the diagram above. What is the relationship between line L_{AB} and the segment that connects point C to its image?

   The line of reflection bisects the segment that connects C to its image.

Problem Set Sample Solutions

1. In the picture below, $\angle DEF = 56°$, $\angle ACB = 114°$, $AB = 12.6$ units, $JK = 5.32$ units, point E is on line L, and point I is off of line L. Let there be a reflection across line L. Reflect and label each of the figures, and answer the questions that follow.



2. What is the measure of $Reflection(\angle DEF)$? Explain.

   The measure of $Reflection(\angle DEF) = 56°$. Reflections preserve degrees of angles.

3. What is the length of $Reflection(JK)$? Explain.

   The length of $Reflection\ (JK) = 5.32$ units. Reflections preserve lengths of segments.

4. What is the measure of $Reflection(\angle ACB)$?

   The measure of $Reflection(\angle ACB) = 114°$.

5. What is the length of $Reflection(AB)$?

   The length of $Reflection(AB) = 12.6$ units.

6. Two figures in the picture were not moved under the reflection. Name the two figures and explain why they were not moved.

   Point E and line L were not moved. All of the points that make up the line of reflection remain in the same location when reflected. Since point E is on the line of reflection, it is not moved.

7. Connect points I and I'. Name the point of intersection of the segment with the line of reflection point Q. What do you know about the lengths of segments IQ and QI'?

   Segments IQ and QI' are equal in length. The segment II' connects point I to its image, I'. The line of reflection will go through the midpoint, or bisect, the segment created when you connect a point to its image.

Lesson 5: Definition of Rotation and Basic Properties

Student Outcomes

- Students know how to rotate a figure a given degree around a given center.
- Students know that rotations move lines to lines, rays to rays, segments to segments, and angles to angles. Students know that rotations preserve lengths of segments and degrees of measures of angles. Students know that rotations move parallel lines to parallel lines.

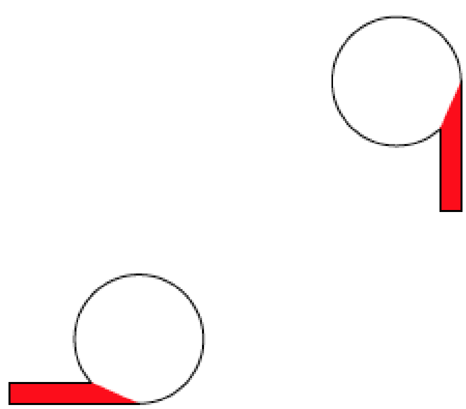
Lesson Notes

In general, students are not required to rotate a certain degree nor identify the degree of rotation. The only exceptions are when the rotations are multiples of $90°$. For this reason, it is recommended in the discussion following the video presentation that you show students how to use the transparency to rotate in multiples of $90°$, i.e., turn transparency one quarter turn for each $90°$ rotation.

Classwork

Discussion (8 minutes)

- What is the simplest transformation that would map one of the following figures to the other?



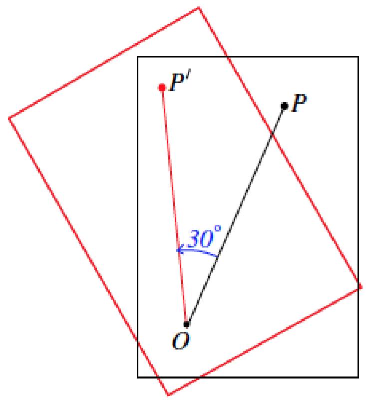
- Would a translation work? Would a reflection work?
  - *Because there seems to be no known simple transformation that would do the job, we will learn about a new transformation called rotation. Rotation is the transformation needed to map one of the figures onto the other.*

Let O be a point in the plane and let d be a number between –360 and 360, or, in the usual notation, $-360 < d < 360$.

- Why do you think the numbers –360 and 360 are used in reference to rotation?
  - *Rotating means that we are moving in a circular pattern, and circles have* $360°$*.*

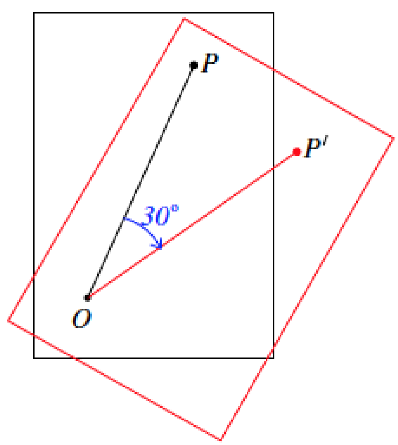
The rotation of d degrees with center O is defined by using transparencies. On a piece of paper, fix a point O as the center of rotation, let P be a point in the plane, and let the ray $\overrightarrow{OP}$ be drawn. Let d be a number between -360 and 360.

MP.6 **Definition.** If there is a rotation of d degrees with center O, the image $Rotation(P)$ is the point described as follows. On a piece of transparency, trace O, P, and $\overrightarrow{OP}$ in red. Now, use a pointed object (e.g., the leg-with-spike of a compass) to pin the transparency at the point O. First, suppose $d \geq 0$. Then, holding the paper in place, rotate the transparency counterclockwise so that if we denote the final position of the rotated red point (that was P) by P', then the $\angle P'OP$ is d degrees. For example, if $d = 30$, we have the following picture:



As before, the red rectangle represents the border of the rotated transparency. Then, by definition, $Rotation(P)$ is the point P'.

If, however, $d < 0$, then holding the paper in place, we would now rotate the transparency clockwise so that if we denote the position of the red point (that was P) by P', then the angle $\angle POP'$ is d degrees. For example, if $d = -30$, we have the following picture:



Again, we define $Rotation(P)$ to be P' in this case. Notice that the rotation moves the center of rotation O to itself, i.e., $Rotation(O) = O$.

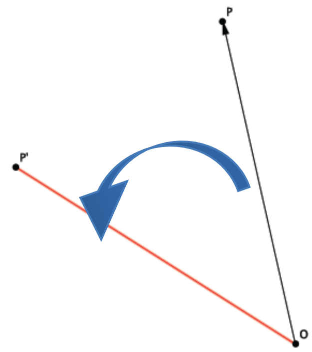
Exercises 1–4 (4 minutes)

Students complete Exercises 1–4 independently.

Exercises

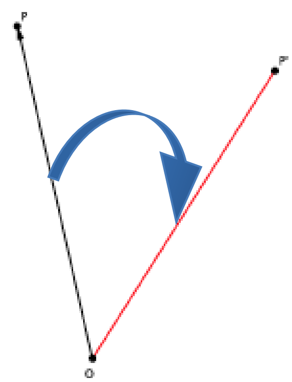
1. **Let there be a rotation of d degrees around center O. Let P be a point other than O. Select d so that $d \geq 0$. Find P' (i.e., the rotation of point P) using a transparency.**

   Verify that students have rotated around center O in the counterclockwise direction.



2. **Let there be a rotation of d degrees around center O. Let P be a point other than O. Select d so that $d < 0$. Find P' (i.e., the rotation of point P) using a transparency.**

   Verify that students have rotated around center O in the clockwise direction.



3. **Which direction did the point P rotate when $d \geq 0$?**

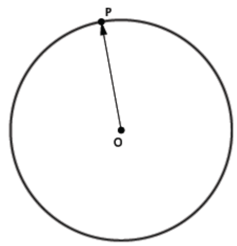
   It rotated counterclockwise, or to the left of the original point.

4. **Which direction did the point P rotate when $d < 0$?**

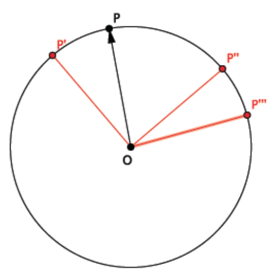
   It rotated clockwise, or to the right of the original point.

Discussion (5 minutes)

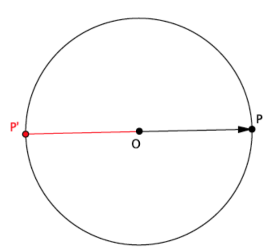
Observe that, with O as the center of rotation, the points P and $Rotation(P)$ lie on a circle whose center is O and whose radius is $\overrightarrow{OP}$.



- Assume we rotate the plane d degrees around center O. Let P be a point other than O. Where do you think P' will be located?
  - *The point P and P' will be equidistant from O; that is, P' is on the circumference of the circle with center O and radius OP. The point P' would be clockwise from P if the degree of rotation is negative. The point P' would be counterclockwise from P if the degree of rotation is positive.*
- If we rotated P, d degrees around center O several times, where would all of the images of P be located?
  - *All images of P will be on the circumference of the circle with radius OP.*

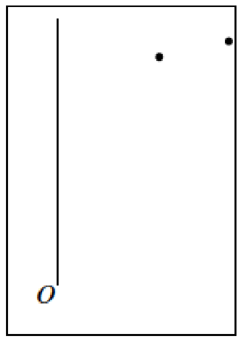


- Why do you think this happens?
  - *Because, like translations and reflections, rotations preserve lengths of segments. The segments of importance here are the segments that join the center O to the images of P. Each segment is the radius of the circle. We will discuss this more, later in the lesson.*
- Consider a rotation of point P, around center O, 180 degrees and -180 degrees. Where do you think the images of P will be located?
  - *Both rotations, although they are in opposite directions, will move the point P to the same location, P'. Further, the points P, O, and P' will always be collinear (i.e., they will lie on one line, for any point P). This concept will be discussed in more detail in Lesson 6.*

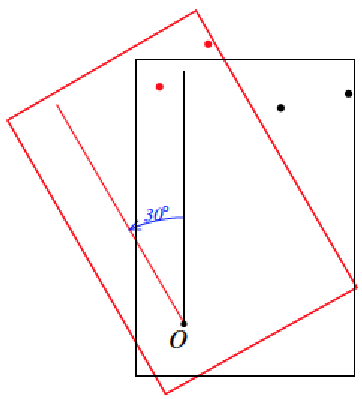


Concept Development (3 minutes)

- Now that we know how a point gets moved under a rotation, let us look at how a geometric figure gets moved under a rotation. Let S be the figure consisting of a vertical segment (not a line) and two points. Let the center of rotation be O, the lower endpoint of the segment, as shown.



- Then, the rotation of 30 degrees with center O moves the point represented by the left black dot to the lower red dot, the point represented by the right black dot to the upper red dot, and the vertical black segment to the red segment to the left at an angle of 30 degrees, as shown.



Video Presentation (2 minutes)

The following two videos[1] show how a rotation of 35 degrees and -35 degrees with center B, respectively, rotates a geometric figure consisting of three points and two line segments.

http://www.harpercollege.edu/~skoswatt/RigidMotions/rotateccw.html

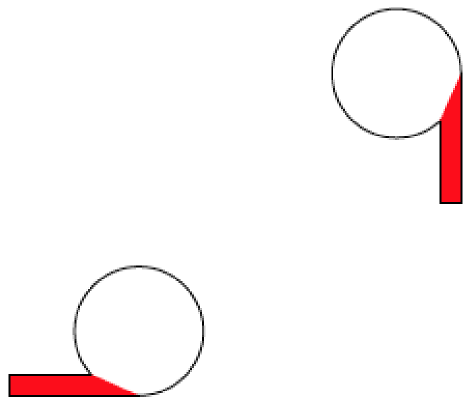
http://www.harpercollege.edu/~skoswatt/RigidMotions/rotatecw.html

[1] The videos were developed by Sunil Koswatta.

Discussion (2 minutes)

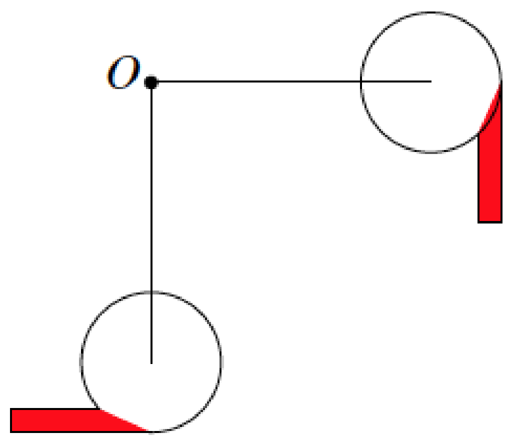
Revisit the question posed at the beginning of the lesson and ask students:

- What is the simplest transformation that would map one of the following figures to the other?



  - *We now know that the answer is a rotation.*

Show students how a rotation of approximately 90 degrees around a point O, chosen on the perpendicular bisector [⊥ bisector] of the segment joining the centers of the two circles in the figures, would map the figure on the left to the figure on the right. Similarly, a rotation of -90 degrees would map the figure on the right to the figure on the left.



Note to Teacher:

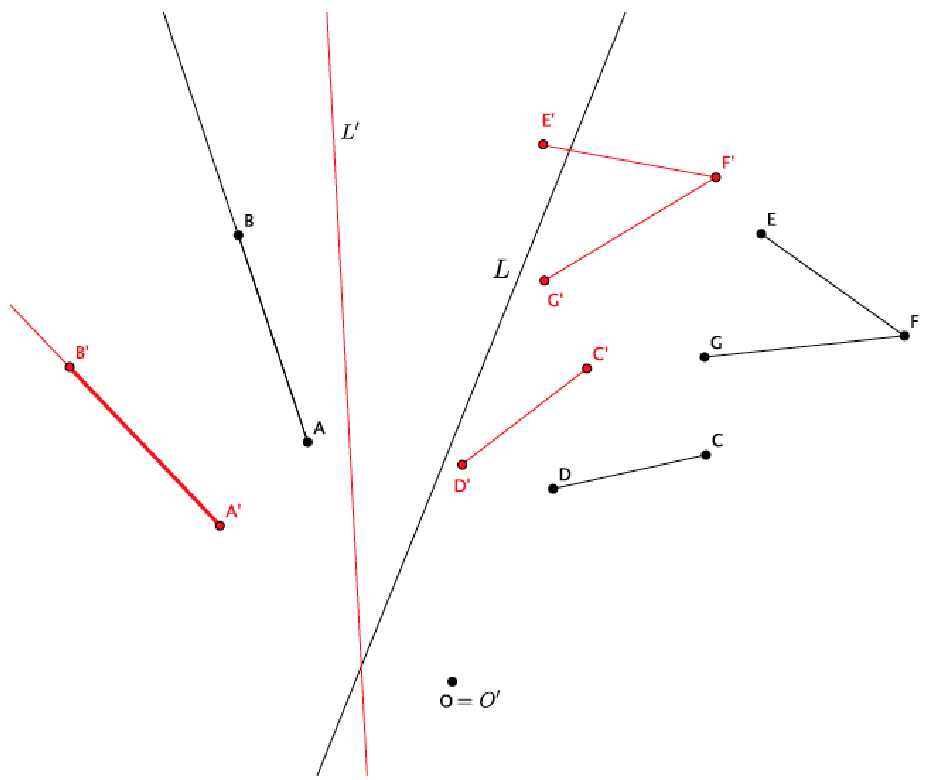
Continue to remind students that a positive degree of rotation moves the figure counterclockwise, and a negative degree of rotation moves the figure clockwise.

Exercises 5–6 (4 minutes)

Students complete Exercises 5 and 6 independently.

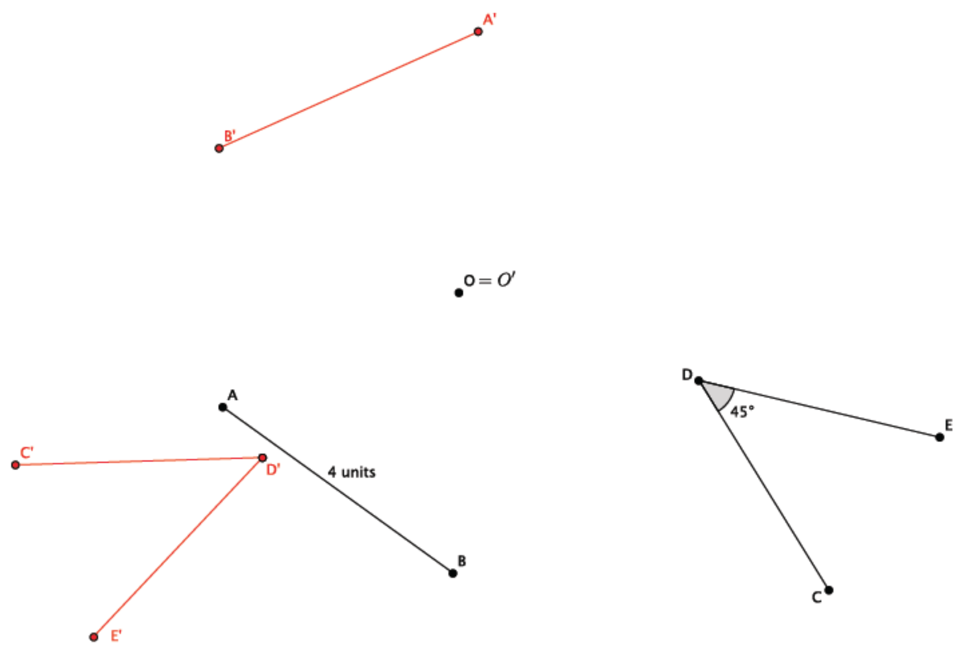
5. **Let L be a line, $\overrightarrow{AB}$ be a ray, CD be a segment, and $\angle EFG$ be an angle, as shown. Let there be a rotation of d degrees around point O. Find the images of all figures when $d \geq 0$.**

   Verify that students have rotated around center O in the counterclockwise direction.



6. Let $\overline{AB}$ be a segment of length 4 units and $\angle CDE$ be an angle of size $45°$. Let there be a rotation by d degrees, where $d < 0$, about O. Find the images of the given figures. Answer the questions that follow.

Verify that students have rotated around center O in the clockwise direction.



a. What is the length of the rotated segment $Rotation(AB)$?

The length of the rotated segment is 4 units.

b. What is the degree of the rotated angle $Rotation\ (\angle CDE)$?

The degree of the rotated angle is $45°$.

Concept Development (4 minutes)

Based on the work completed during the lesson, and especially in Exercises 5 and 6, we can now states that rotations have properties similar to translations with respect to (Translation 1)–(Translation 3) of Lesson 2 and reflections with respect to (Reflection 1)–(Reflection 3) of Lesson 4:

(Rotation 1) A rotation maps a line to a line, a ray to a ray, a segment to a segment, and an angle to an angle.

(Rotation 2) A rotation preserves lengths of segments.

(Rotation 3) A rotation preserves measures of angles.

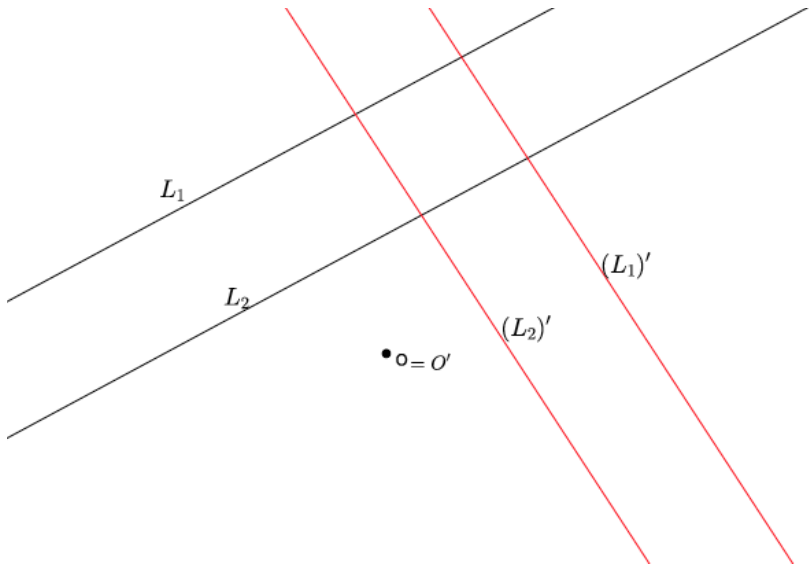
Also, as with translations and reflections, if L_1 and L_2 are parallel lines and if there is a rotation, then the lines $Rotation(L_1)$ and $Rotation(L_2)$ are also parallel. However, if there is a rotation of degree $d \neq 180$ and L is a line, L and $Rotation(L)$ are not parallel. (Note to teacher: Exercises 7 and 8 will illustrate these two points.)

Exercises 7–8 (4 minutes)

Students complete Exercises 7 and 8 independently.

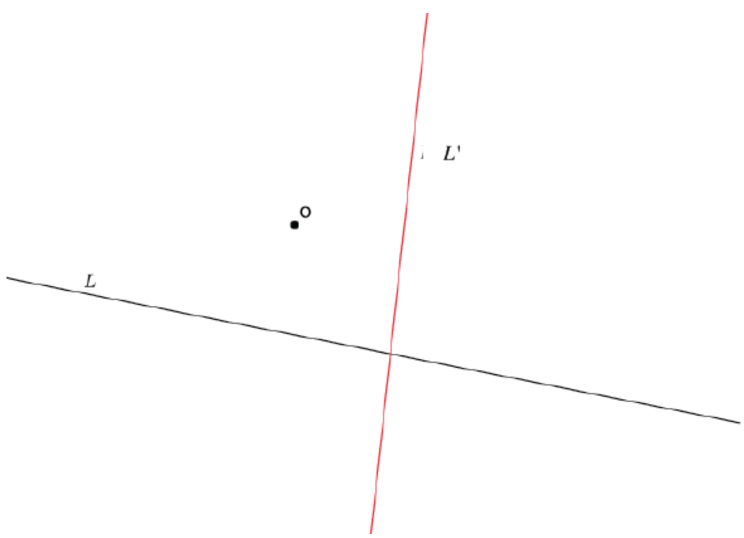
7. Let L_1 and L_2 be parallel lines. Let there be a rotation by d degrees, where $-360 < d < 360$, about O. Is $(L_1)' \parallel (L_2)'$?

   Verify that students have rotated around center O in either direction. Students should respond that $(L_1)' \parallel (L_2)'$.



8. Let L be a line and O be the center of rotation. Let there be a rotation by d degrees, where $d \neq 180$ about O. Are the lines L and L' parallel?

   Verify that students have rotated around center O in either direction any degree other than 180. Students should respond that L and L' are not parallel.



Closing (3 minutes)

Summarize, or have students summarize, what we know of rigid motions to this point:

- We now have definitions for all three rigid motions: translations, reflections, and rotations.
- Rotations move lines to lines, rays to rays, segments to segments, angles to angles, and parallel lines to parallel lines, similar to translations and reflections.
- Rotations preserve lengths of segments and degrees of measures of angles similar to translations and reflections.
- Rotations require information about the center and degree of rotation, whereas translations require only a vector, and reflections require only a line of reflection.

Lesson Summary

Rotations require information about the center of rotation and the degree in which to rotate. Positive degrees of rotation move the figure in a counterclockwise direction. Negative degrees of rotation move the figure in a clockwise direction.

Basic Properties of Rotations:

- **(Rotation 1) A rotation maps a line to a line, a ray to a ray, a segment to a segment, and an angle to an angle.**
- **(Rotation 2) A rotation preserves lengths of segments.**
- **(Rotation 3) A rotation preserves measures of angles.**

When parallel lines are rotated, their images are also parallel. A line is only parallel to itself when rotated exactly $180°$.

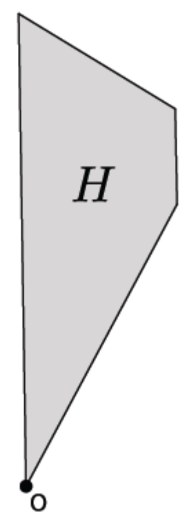
Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 5: Definition of Rotation and Basic Properties

Exit Ticket

1. Given the figure H, let there be a rotation by d degrees, where $d \geq 0$, about O. Let $Rotation(H)$ be H'.

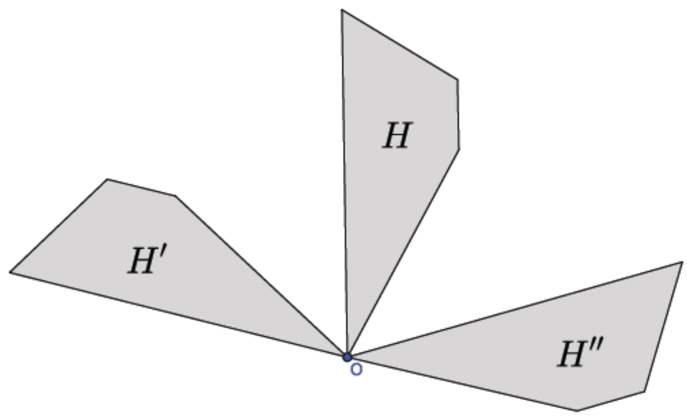


2. Using the drawing above, let $Rotation_1$ be the rotation d degrees with $d < 0$, about O. Let $Rotation_1(H)$ be H''.

Exit Ticket Sample Solutions

1. Given the figure H, let there be a rotation by d degrees, where $d \geq 0$, about O. Let $Rotation(H)$ be H'.

   Sample rotation shown below. Verify that the figure H' has been rotated counterclockwise with center O.



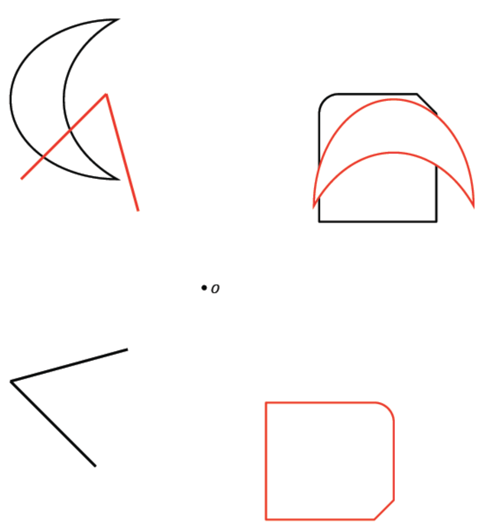
2. Using the drawing above, let $Rotation_1$ be the rotation d degrees with $d < 0$, about O. Let $Rotation_1(H)$ be H''.

   Sample rotation shown above. Verify that the figure H'' has been rotated clockwise with center O.

Problem Set Sample Solutions

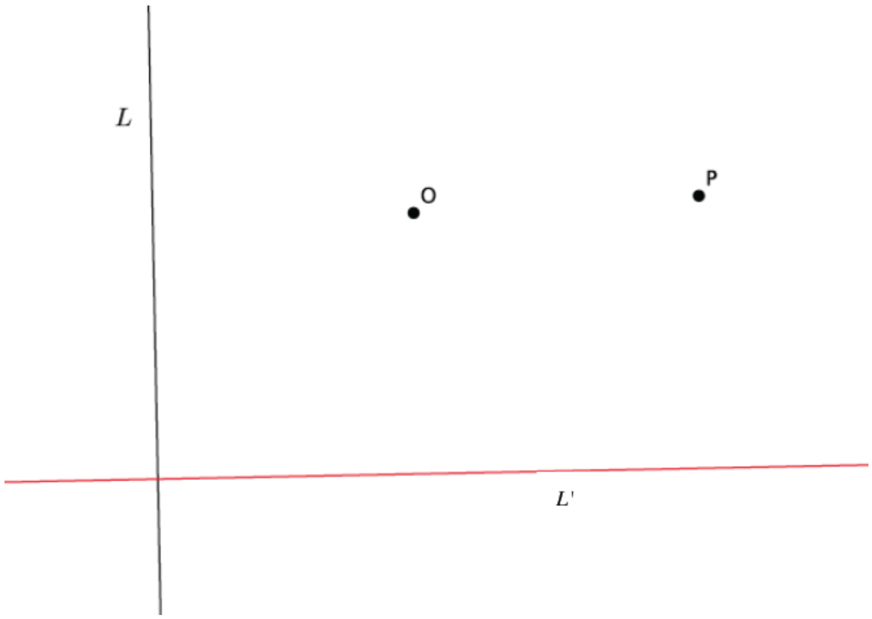
1. Let there *be a* rotation by $-90°$ around the center O.

   Rotated figures are shown in red.



2. **Explain why a rotation of 90 degrees around any point O never maps a line to a line parallel to itself.**

A 90 degree rotation around point O will move a given line L to L'. Parallel lines never intersect, so it is obvious that a 90 degree rotation in either direction does not make lines L and L' parallel. Additionally, we know that there exists just one line parallel to the given line L that goes through a point not on L. If we let P be a point not on L, the line L' must go through it in order to be parallel to L. L' does not go through point P; therefore, L and L' are not parallel lines. Assume we rotate line L first and then place a point P on line L' to get the desired effect (a line through P). This contradicts our definition of parallel (i.e., parallel lines never intersect); so, again, we know that line L is not parallel to L'.



3. **A segment of length 94 cm has been rotated d degrees around a center O. What is the length of the rotated segment? How do you know?**

The rotated segment will be 94 cm in length. (Rotation 2) states that rotations preserve lengths of segments, so the length of the rotated segment will remain the same as the original.

4. **An angle of size $124°$ has been rotated d degrees around a center O. What is the size of the rotated angle? How do you know?**

The rotated angle will be $124°$. (Rotation 3) states that rotations preserve the degrees of angles, so the rotated angle will be the same size as the original.

Lesson 6: Rotations of 180 Degrees

Student Outcomes

- Students learn that a rotation of 180 degrees moves a point on the coordinate plane (a, b) to $(-a, -b)$.
- Students learn that a rotation of 180 degrees around a point, not on the line, produces a line parallel to the given line.

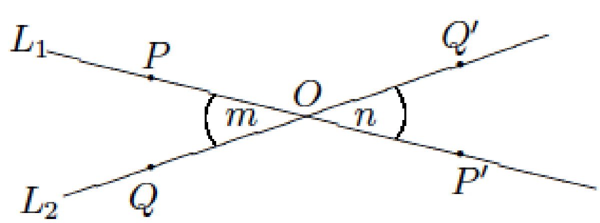
Classwork

Example 1 (5 minutes)

- Rotations of 180 degrees are special. Recall, a rotation of 180 degrees around O is a rigid motion so that if P is any point in the plane, P, O, and $Rotation(P)$ are collinear (i.e., they lie on the same line).
- Rotations of 180 degrees occur in many situations. For example, the frequently cited fact that vertical angles [vert. $\angle s$] at the intersection of two lines are equal, follows immediately from the fact that 180-degree rotations are angle-preserving. More precisely, let two lines L_1 and L_2 intersect at O, as shown:

Example 1

The picture below shows what happens when there is a rotation of $180°$ around center O.



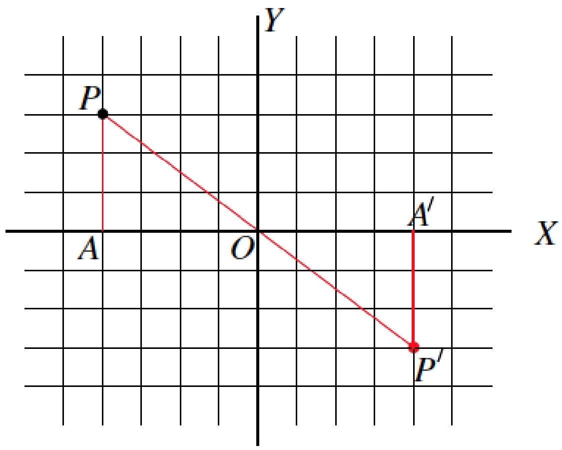
- We want to show that the vertical angles [vert. $\angle s$], $\angle m$ and $\angle n$, are equal (i.e., $\angle m = \angle n$). If we let Rotation_0 be the 180-degree rotation around O, then $Rotation_0$ maps $\angle m$ to $\angle n$. More precisely, if P and Q are points on L_1 and L_2, respectively (as shown above), let $Rotation_0(P) = P'$ and $Rotation_0(Q) = Q'$. Then, $Rotation_0$ maps $\angle POQ$ ($\angle m$) to $\angle Q'OP'$ ($\angle n$), and since $Rotation_0$ is angle-preserving, we have $\angle m = \angle n$.

Example 2 (5 minutes)

- Let's look at a 180-degree rotation, $Rotation_0$ around the origin O of a coordinate system. If a point P has coordinates (a, b), it is generally said that $Rotation_0(P)$ is the point with coordinates $(-a, -b)$.
- Suppose the point P has coordinates $(-4, 3)$; we will show that the coordinates of $Rotation_0(P)$ are $(4, -3)$.

Example 2

The picture below shows what happens when there is a rotation of $180°$ around center O, the origin of the coordinate plane.



- Let $P' = Rotation_0(P)$. Let the *vertical line* (i.e., the line parallel to the y-axis) through P meet the x-axis at a point A. Because the coordinates of P are $(-4, 3)$, the point A has coordinates $(-4, 0)$ by the way coordinates are defined. In particular, A is of distance 4 from O, and since $Rotation_0$ is length-preserving, the point $A' = Rotation_0(A)$ is also of distance 4 from O. However, $Rotation_0$ is a 180-degree rotation around O, so A' also lies on the x-axis but on the opposite side of the x-axis from A. Therefore, the coordinates of A' are $(4,0)$. Now, $\angle PAO$ is a right angle and—since $Rotation_0$ maps it to $\angle P'A'O$, and also preserves degrees—we see that $\angle P'A'O$ is also a right angle. This means that A' is the point of intersection of the vertical line through P' and the x-axis. Since we already know that A' has coordinates of $(4, 0)$, then the x-coordinate of P' is 4, by definition.
- Similarly, the y-coordinate of P being 3 implies that the y-coordinate of P' is -3. Altogether, we have proved that the 180-degree rotation of a point of coordinates $(-4, 3)$ is a point with coordinates $(4, -3)$.

The reasoning is perfectly general: The same logic shows that the 180-degree rotation around the origin of a point of coordinates (a, b) is the point with coordinates $(-a, -b)$, as desired.

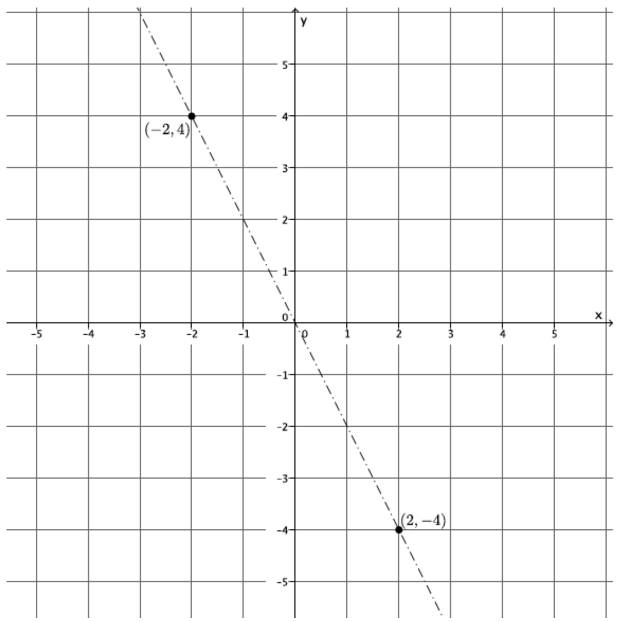
Exercises 1–9 (16 minutes)

Students complete Exercises 1–2 independently. Check solutions. Then, let students work in pairs on Exercises 3–4. Students complete Exercises 5–9 independently in preparation for the example that follows.

Exercises 1–9

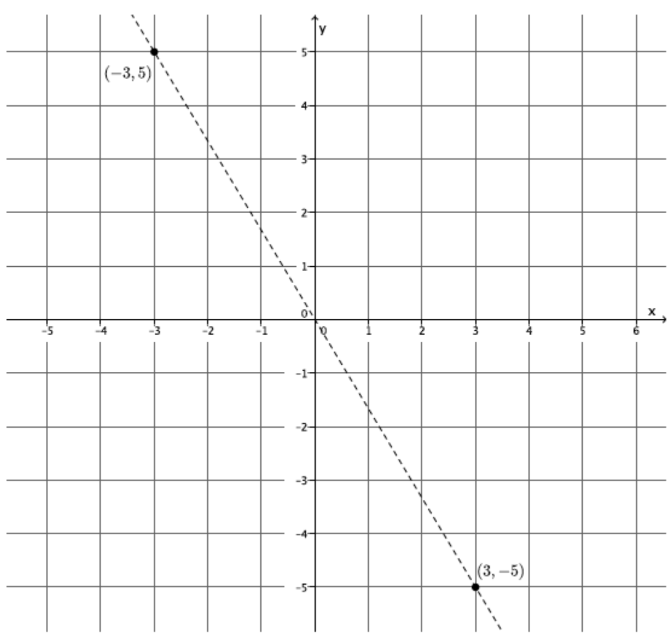
1. **Using your transparency, rotate the plane 180 degrees, about the origin. Let this rotation be $Rotation_0$. What are the coordinates of $Rotation_0(2,-4)$?**

   $Rotation_0(2,-4) = (-2,4)$.

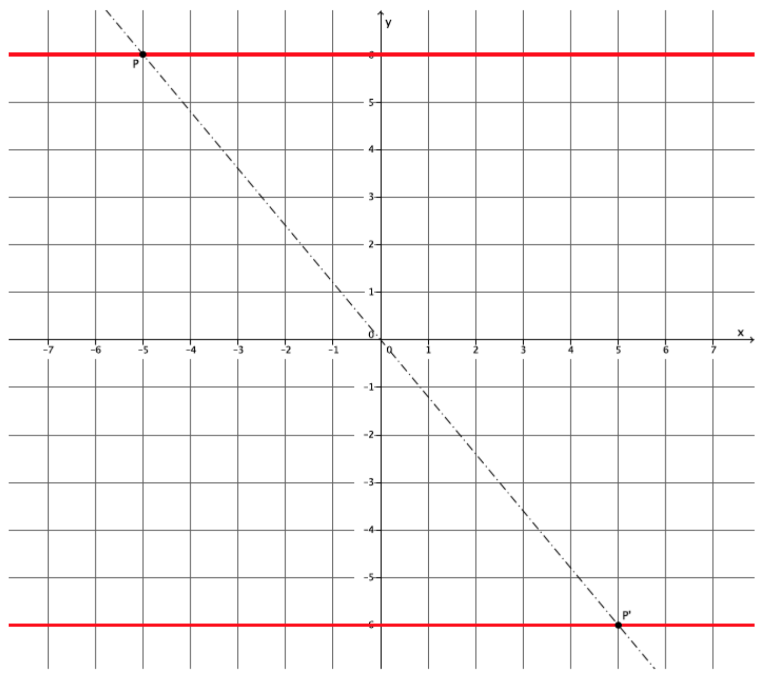


2. **Let $Rotation_0$ be the rotation of the plane by 180 degrees, about the origin. Without using your transparency, find $Rotation_0(-3,5)$.**

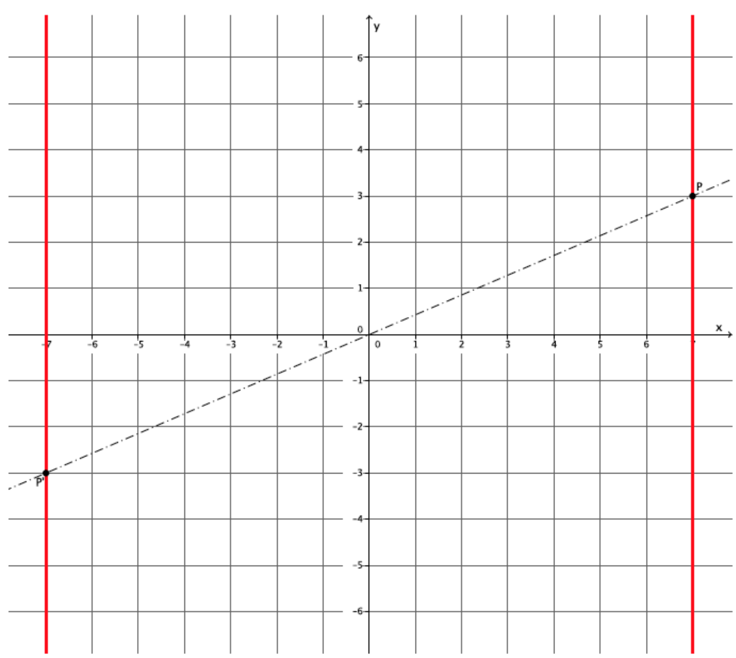
   $Rotation_0(-3,5) = (3,-5)$.



3. Let $Rotation_0$ be the rotation of 180 degrees around the origin. Let L be the line passing through $(-6, 6)$ parallel to the x-axis. Find $Rotation_0(L)$. Use your transparency if needed.

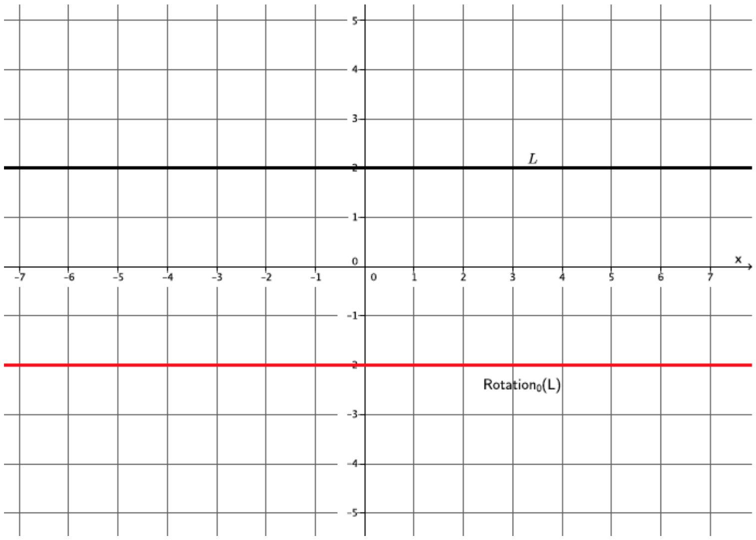


4. Let $Rotation_0$ be the rotation of 180 degrees around the origin. Let L be the line passing through $(7, 0)$ parallel to the y-axis. Find $Rotation_0(L)$. Use your transparency if needed.



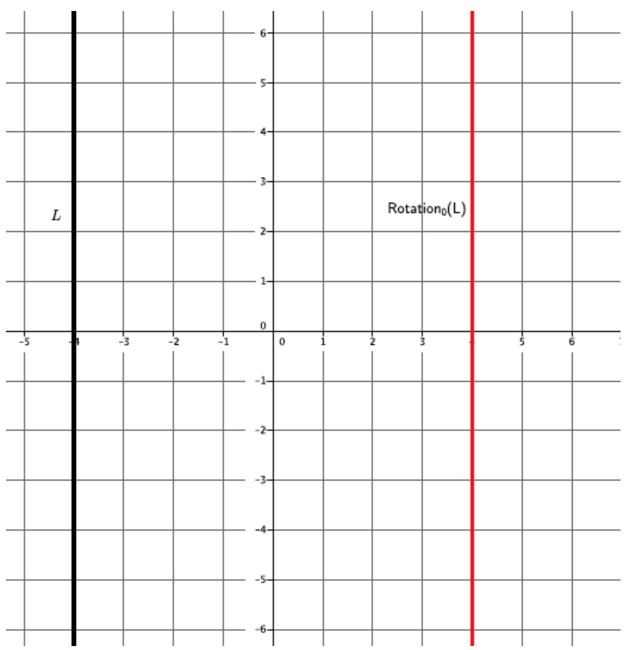
5. Let $Rotation_0$ be the rotation of 180 degrees around the origin. Let L be the line passing through $(0, 2)$ parallel to the x-axis. Is L parallel to $Rotation_0(L)$?

Yes, $L \parallel Rotation_0(L)$.



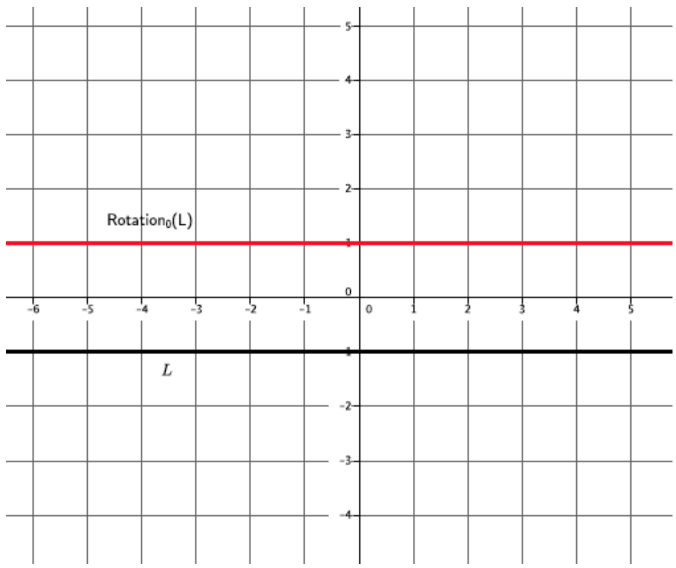
6. Let $Rotation_0$ be the rotation of 180 degrees around the origin. Let L be the line passing through $(4, 0)$ parallel to the y-axis. Is L parallel to $Rotation_0(L)$?

Yes, $L \parallel Rotation_0(L)$.



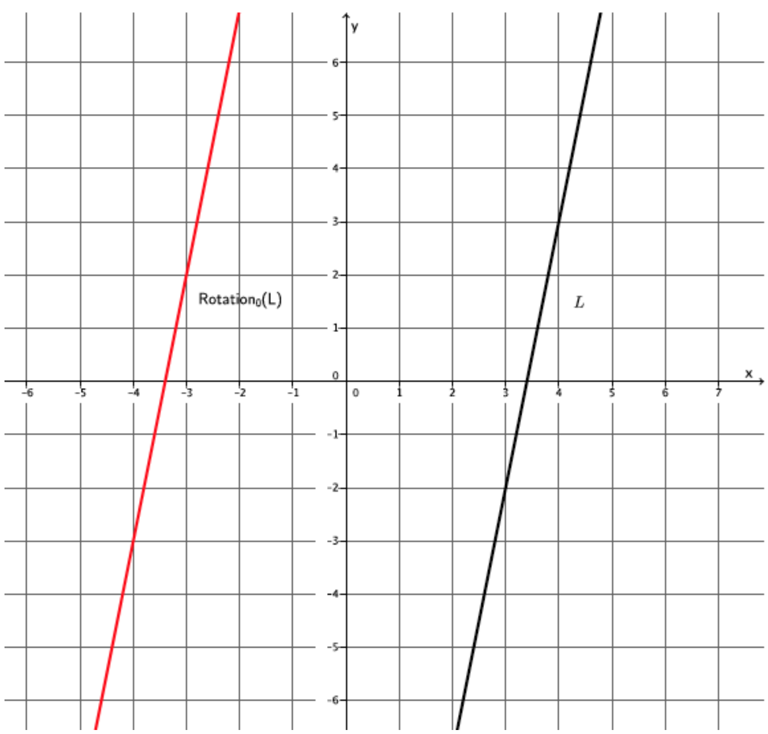
7. Let $Rotation_0$ be the rotation of 180 degrees around the origin. Let L be the line passing through $(0, -1)$ parallel to the x-axis. Is L parallel to $Rotation_0(L)$?

Yes, $L \parallel Rotation_0(L)$.



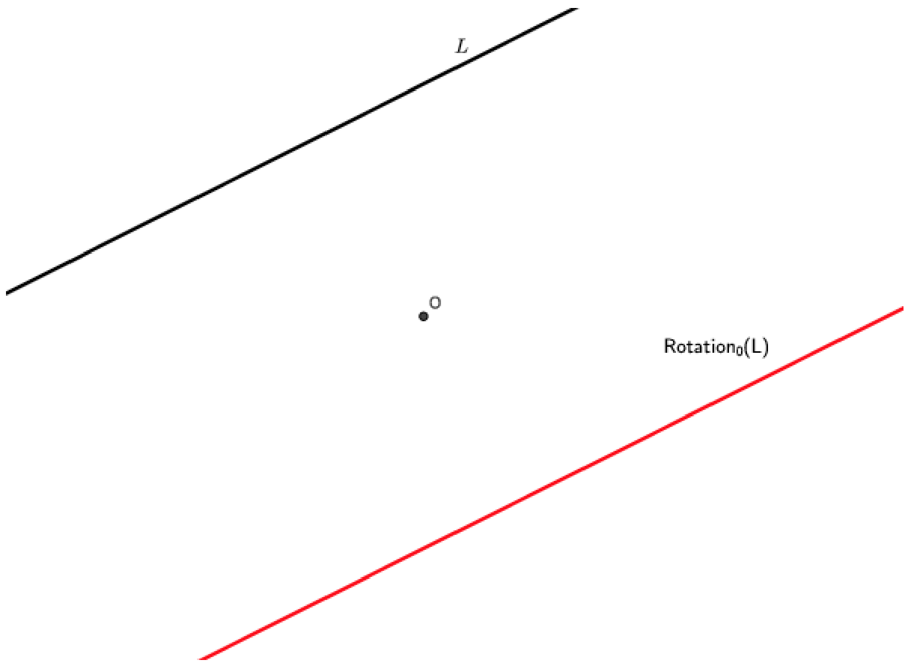
8. Let $Rotation_0$ be the rotation of 180 degrees around the origin. Is L parallel to $Rotation_0(L)$? Use your transparency if needed.

Yes, $L \parallel Rotation_0(L)$.



9. Let $Rotation_0$ be the rotation of 180 degrees around the origin. Is L parallel to $Rotation_0(L)$? Use your transparency if needed.

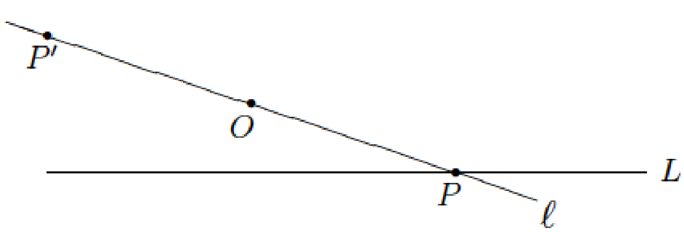
Yes, $L \parallel Rotation_0(L)$.



Example 3 (5 minutes)

MP.2 **Theorem**. Let O be a point not lying on a given line L. Then, the 180-degree rotation around O maps L to a line parallel to L.

Proof: Let $Rotation_0$ be the 180-degree rotation around O, and let P be a point on L. As usual, denote $Rotation_0(P)$ by P'. Since $Rotation_0$ is a 180-degree rotation, P, O, P' lie on the same line (denoted by ℓ).



Scaffolding:

After completing Exercises 5–9, students should be convinced that the theorem is true. Make it clear that their observations can be proven (by contradiction) if we assume something different will happen (e.g., the lines will intersect).

We want to investigate whether P' lies on L or not. Keep in mind that we want to show that the 180-degree rotation maps L to a line parallel to L. If the point P' lies on L, then at some point, the line L and $Rotation_0(L)$ intersect, meaning they are not parallel. If we can eliminate the possibility that P' lies on L, then we have to conclude that P' does not lie on L (rotations of 180 degrees make points that are collinear). If P' lies on L, then ℓ is a line that joins two points, P' and P, on L. However L is already a line that joins P' and P, so ℓ and L must be the same line (i.e., $\ell = L$). This is trouble because we know O lies on ℓ, so $\ell = L$ implies that O lies on L. Look at the hypothesis of the theorem: "Let O be a point not lying on a given line L." We have a contradiction. So, the possibility that P' lies on L is nonexistent. As we said, this means that P' does not lie on L.

What we have proved is that no matter which point P we take from L, we know $Rotation_0(P)$ does not lie on L. But $Rotation_0(L)$ consists of *all* the points of the form $Rotation_0(P)$ where P lies on L, so what we have proved is that no point of $Rotation_0(L)$lies on L. In other words, L and $Rotation_0(L)$ have no point in common (i.e., $L \parallel Rotation_0(L)$). The theorem is proved.

Closing (5 minutes)

Summarize, or have students summarize, the lesson.

- Rotations of 180 degrees are special:
  - A point, P, that is rotated 180 degrees around a center O, produces a point P' so that P, O, P' are collinear.
  - When we rotate around the origin of a coordinate system, we see that the point with coordinates (a, b) is moved to the point $(-a, -b)$.
- We now know that when a line is rotated 180 degrees around a point not on the line, it maps to a line parallel to the given line.

Lesson Summary

- **A rotation of 180 degrees around O is the rigid motion so that if P is any point in the plane, P, O, and $Rotation(P)$ are *collinear* (i.e., lie on the same line).**
- **Given a 180-degree rotation, R_0 around the origin O of a coordinate system, and a point P with coordinates (a, b), it is generally said that $R_0(P)$ is the point with coordinates $(-a, -b)$.**

Theorem: Let O be a point not lying on a given line L. Then, the 180-degree rotation around O maps L to a line parallel to L.

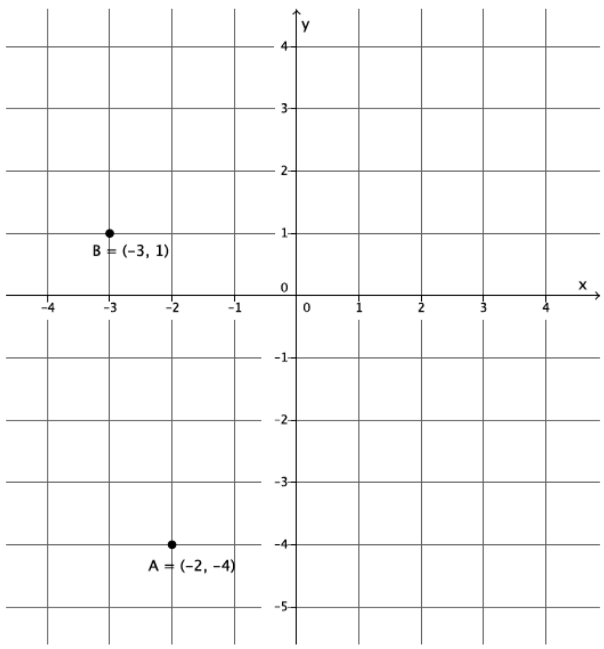
Exit Ticket (5 minutes)

Name ______________________________ Date ____________

Lesson 6: Rotations of 180 Degrees

Exit Ticket

Let there be a rotation of 180 degrees about the origin. Point A has coordinates $(-2, -4)$, and point B has coordinates $(-3, 1)$, as shown below.



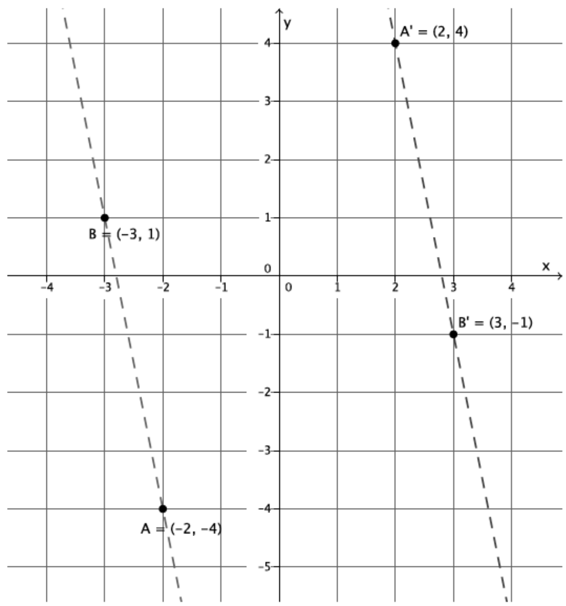
1. What are the coordinates of $Rotation(A)$? Mark that point on the graph so that $Rotation(A) = A'$. What are the coordinates of $Rotation(B)$? Mark that point on the graph so that $Rotation(B) = B'$.

2. What can you say about the points A, A', and O? What can you say about the points B, B', and O?

3. Connect point A to point B to make the line L_{AB}. Connect point A' to point B' to make the line $L_{A'B'}$. What is the relationship between L_{AB} and $L_{A'B'}$?

Exit Ticket Sample Solutions

Let there be a rotation of 180 degrees about the origin. Point A has coordinates $(-2,-4)$, and point B has coordinates $(-3,1)$, as shown below.



1. What are the coordinates of $Rotation(A)$? Mark that point on the graph so that $Rotation(A) = A'$. What are the coordinates of $Rotation(B)$? Mark that point on the graph so that $Rotation(B) = B'$.

   $A' = (2,4)$, $B' = (3,-1)$

2. What can you say about the points A, A', and O? What can you say about the points B, B', and O?

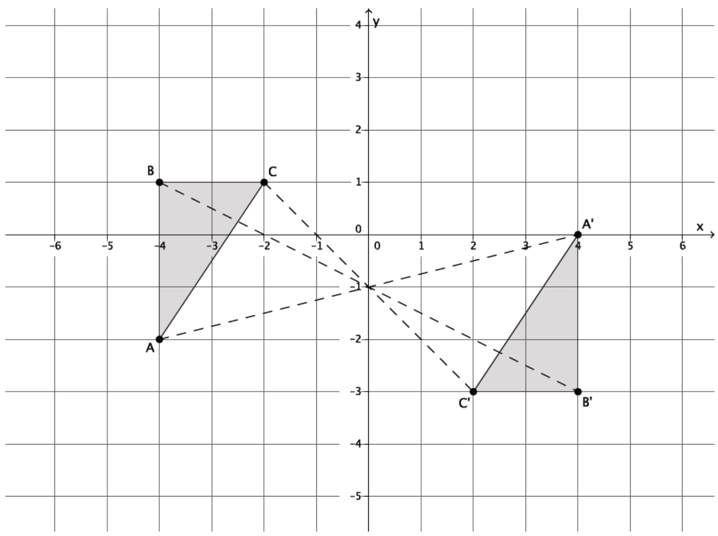
   The points A, A', and O are collinear. The points B, B', and O are collinear.

3. Connect point A to point B to make the line L_{AB}. Connect point A' to point B' to make the line $L_{A'B'}$. What is the relationship between L_{AB} and $L_{A'B'}$?

   $L_{AB} \parallel L_{A'B'}$.

Problem Set Sample Solutions

Use the following diagram for Problems 1–5. Use your transparency as needed.



1. **Looking only at segment BC, is it possible that a $180°$ rotation would map BC onto $B'C'$? Why or why not?**

   It is possible because the segments are parallel.

2. **Looking only at segment AB, is it possible that a $180°$ rotation would map AB onto $A'B'$? Why or why not?**

   It is possible because the segments are parallel.

3. **Looking only at segment AC, is it possible that a $180°$ rotation would map AC onto $A'C'$? Why or why not?**

   It is possible because the segments are parallel.

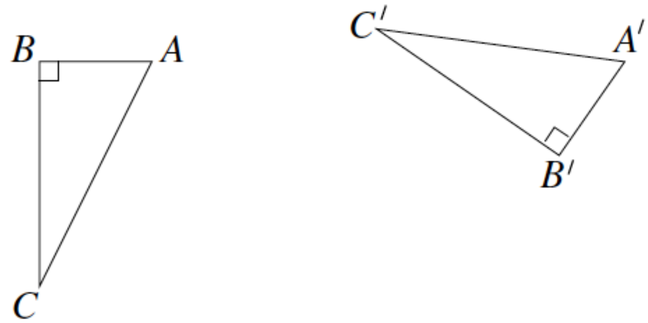
4. **Connect point B to point B', point C to point C', and point A to point A'. What do you notice? What do you think that point is?**

   All of the lines intersect at one point. The point is the center of rotation, I checked by using my transparency.

5. **Would a rotation map triangle ABC onto triangle $A'B'C'$? If so, define the rotation (i.e., degree and center). If not, explain why not.**

   Let there be a rotation $180°$ around point $(0,-1)$. Then, $Rotation(\Delta ABC) = \Delta A'B'C'$.

6. The picture below shows right triangles ABC and $A'B'C'$, where the right angles are at B and B'. Given that $AB = A'B' = 1$, and $BC = B'C' = 2$, and that AB is not parallel to $A'B'$, is there a $180°$ rotation that would map ΔABC onto $\triangle A'B'C'$? Explain.



No, because a $180°$ rotation of a segment will map to a segment that is parallel to the given one. It is given that AB is not parallel to $A'B'$; therefore, a rotation of $180°$ will not map $\triangle ABC$ onto $\triangle A'B'C'$.

Topic B:

Sequencing the Basic Rigid Motions

8.G.A.2

Focus Standard:	8.G.A.2	Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.
Instructional Days:	4	
Lesson 7:	Sequencing Translations (E)[1]	
Lesson 8:	Sequencing Reflections and Translations (S)	
Lesson 9:	Sequencing Rotations (E)	
Lesson 10:	Sequences of Rigid Motions (P)	

Topic B focuses on the first part of **8.G.A.2** in the respect that students learn how to sequence rigid motions. Lesson 7 begins with the concept of composing translations and introduces the idea that translations can be undone. In Lesson 8, students explore images of figures under a sequence of reflections and translations. In Lesson 9, students explore with sequences of rotations around the same center and rotations around different centers. In each of the Lessons 7–9, students verify that the basic properties of individual rigid motions remain intact and describe the sequences using precise language. In Lesson 10, students perform sequences of translations, rotations, and reflections as a prelude to learning about congruence.

[1] Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson

Lesson 7: Sequencing Translations

Student Outcomes

- Students learn about the sequence of transformations (one move on the plane followed by another) and that a sequence of translations enjoy the same properties as a single translation with respect to lengths of segments and degrees of angles.
- Students learn that a translation along a vector followed by another translation along a vector of the same length in the opposite direction can move all points of a plane back to its original position.

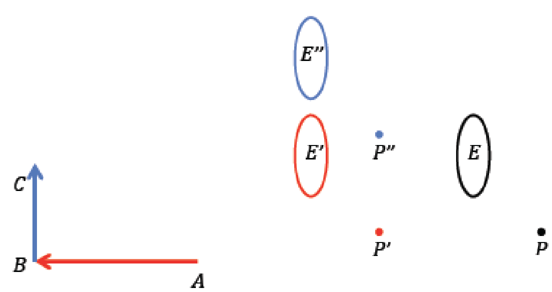
Classwork

Discussion (5 minutes)

- Is it possible to translate a figure more than one time? That is, translating a figure along one vector, then taking that figure and translating it along another vector?
  - *The simple answer is yes. It is called a* ***sequence of transformations*** *or, more specifically, a sequence of translations.*

MP.6

- Suppose we have two transformations of the plane, F and G. A point P, under transformation F, will be assigned to a new location, *Transformation* $F(P)$ denoted by P'. Transformation G will assign P' to a new location, *Transformation* $G(P')$ denoted by P''. This is true for every point P in the plane.
- In the picture below, the point P and ellipse E in black have undergone a sequence of transformations, first along the red vector where images are shown in red, and then along the blue vector where images are shown in blue.



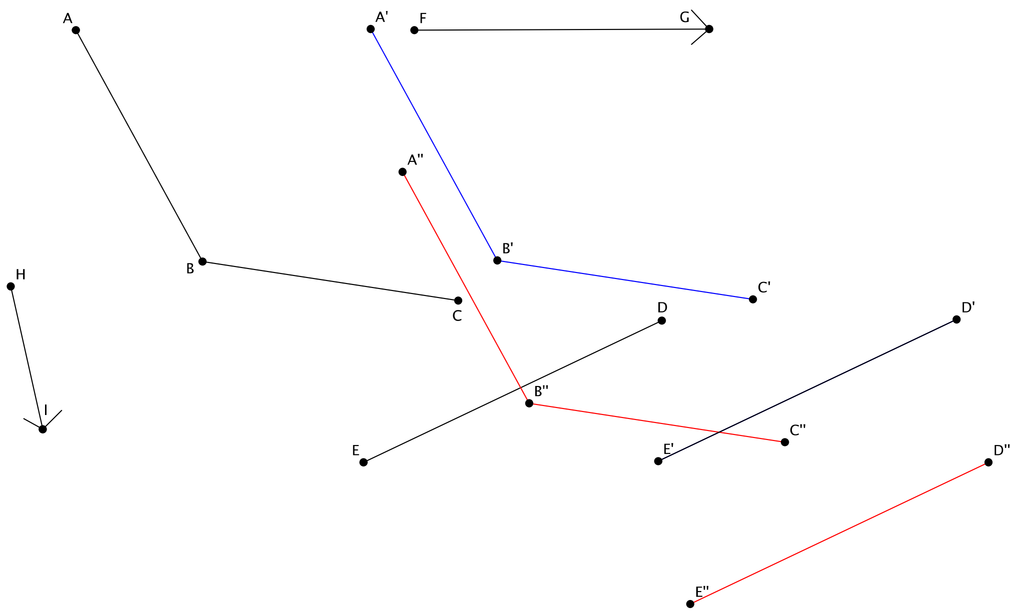
Exploratory Challenge

Exercises 1–4 (22 minutes)

Students complete Exercises 1–4 individually or in pairs.

Exploratory Challenge

1.



a. Translate $\angle ABC$ and segment ED along vector $\overrightarrow{FG}$. Label the translated images appropriately, i.e., $A'B'C'$ and $E'D'$.

Images shown above in blue.

b. Translate $\angle A'B'C'$ and segment $E'D'$ along vector $\overrightarrow{HI}$. Label the translated images appropriately, i.e., $A''B''C''$ and $E''D''$.

Images shown above in red.

c. How does the size of $\angle ABC$ compare to the size of $\angle A''B''C''$?

The measure of $\angle ABC$ is equal to the size of $\angle A''B''C''$.

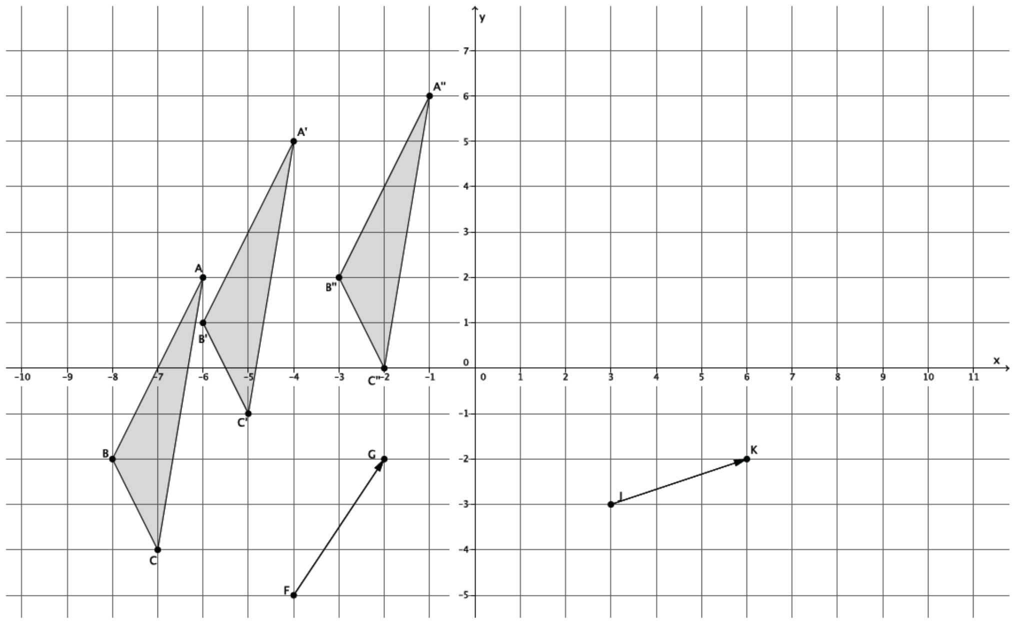
d. How does the length of segment ED compare to the length of the segment $E''D''$?

The length of ED is equal to the length of $E''D''$.

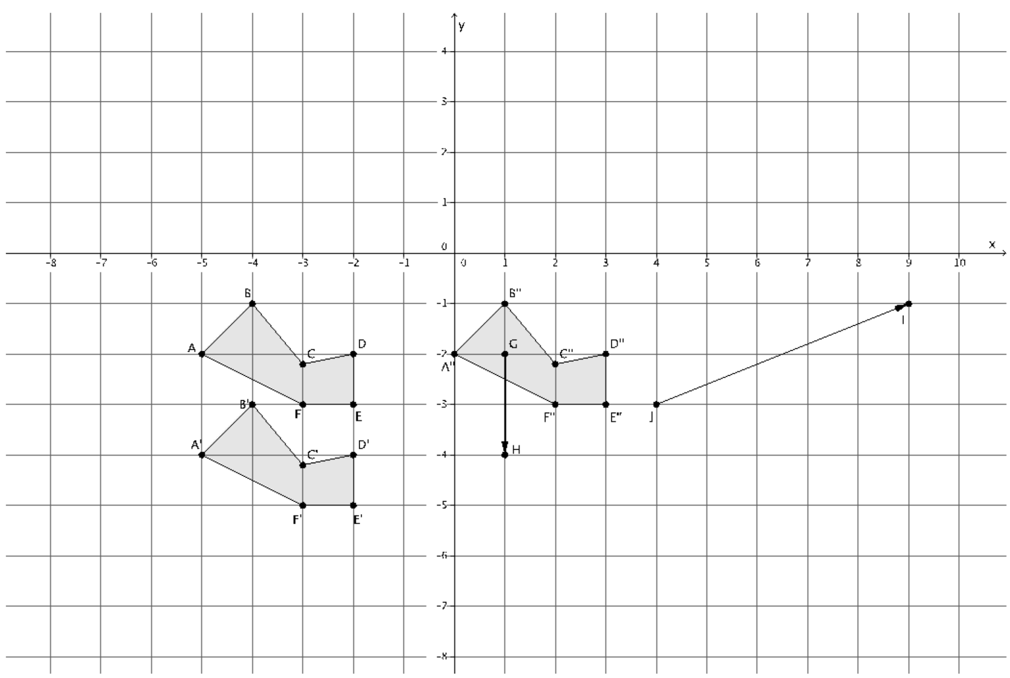
e. Why do you think what you observed in parts (d) and (e) were true?

One translation of the plane moved the angle and the segment to a new location. We know that translations preserve lengths of segments and degrees of measures of angles. The second translation moved the images to another new location, also preserving the length of the segment and the measure of the angle. Therefore, performing two translations, or a sequence of translations, keeps lengths of segments and size of angles rigid.

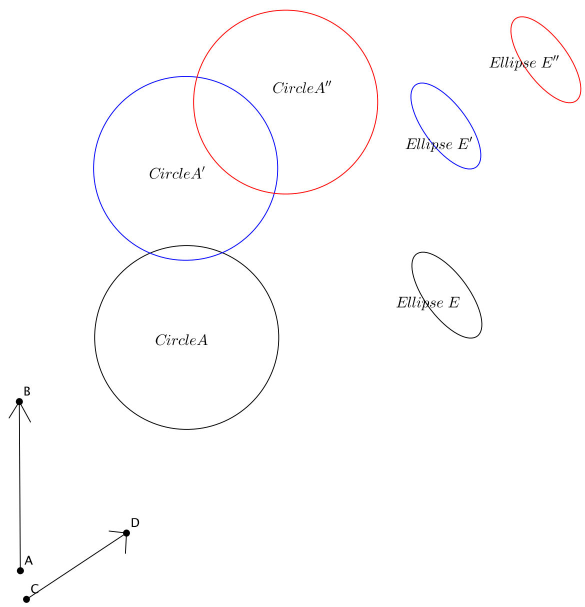
2. Translate $\triangle ABC$ along vector $\overrightarrow{FG}$ and then translate its image along vector $\overrightarrow{JK}$. Be sure to label the images appropriately.



3. Translate figure $ABCDEF$ along vector $\overrightarrow{GH}$. Then translate its image along vector $\overrightarrow{JI}$. Label each image appropriately.



4.



a. **Translate Circle A and Ellipse E along vector $\overrightarrow{AB}$. Label the images appropriately.**

Images shown above in blue.

b. **Translate Circle A' and Ellipse E' along vector $\overrightarrow{CD}$. Label each image appropriately.**

Images shown above in red.

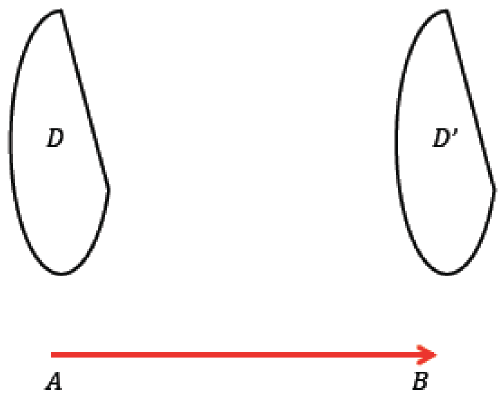
c. **Did the size or shape of either figure change after performing the sequence of translations? Explain.**

The circle and the ellipse remained the same in size and shape after the sequence of translations. Since translations are a basic rigid motion, a sequence of translations will maintain the shape and size of the figures rigid and unchanged.

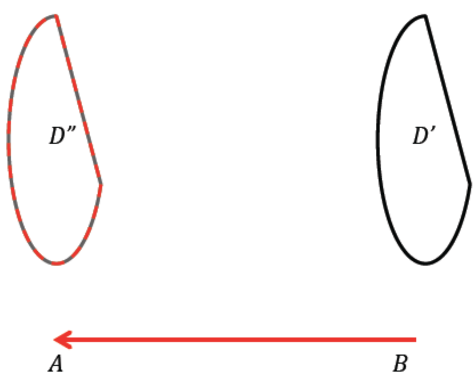
Discussion (5 minutes)

- What need is there for sequencing transformations?
- Imagine life without an undo button on your computer or smartphone. If we move something in the plane, it would be nice to know we can move it back to its original position.
- Specifically, if a figure undergoes two transformations F and G, and ends up in the same place as it was originally, then the figure has been mapped onto itself.

- Suppose we translate figure D along vector $\overrightarrow{AB}$.



- How do we undo this move? That is, what translation of figure D along vector $\overrightarrow{AB}$ that would bring D' back to its original position?
  - *We translate D' (the image of D) along the vector $\overrightarrow{BA}$.*



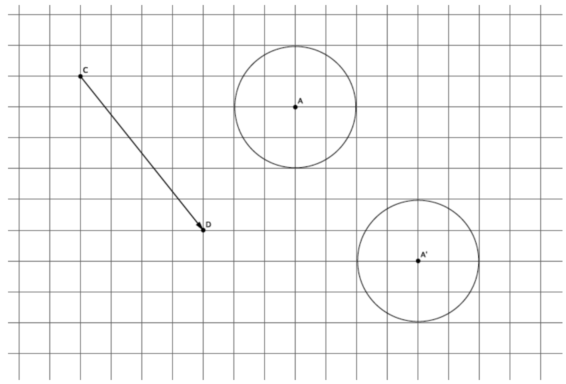
- All of the points in D were translated to D', and then translated again to D''. Because all of the points in D (shown in grey under the dashed red lines of D'') are also in D'' (shown as the figure with the dashed red lines), we can be sure that we have performed a sequence of translations that map the figure back onto itself.
- The ability to undo something or put it back in its original place is obviously very desirable. We will see in the next few lessons that every basic rigid motion can be undone. That is one of the reasons we want to learn about basic rigid motions and their properties.

Exercises 5–6 (3 minutes)

Students continue with the Exploratory Challenge to complete Exercises 5 and 6 independently.

5. The picture below shows the translation of Circle A along vector $\overrightarrow{CD}$. Name the vector that will map the image of Circle A back to its original position.

$\overrightarrow{DC}$



6. If a figure is translated along vector $\overrightarrow{QR}$, what translation takes the figure back to its original location?

A translation along vector $\overrightarrow{RQ}$ would take the figure back to its original location.

Closing (5 minutes)

Summarize, or have students summarize, the lesson.

- We know that we can sequence translations and the figure remains rigid, i.e., lengths of segments and degrees of measures of angles are preserved.
- Any translation of the plane can be undone, and figures can be mapped onto themselves.

Lesson Summary

- **Translating a figure along one vector then translating its image along another vector is an example of a sequence of transformations.**
- **A sequence of translations enjoys the same properties as a single translation. Specifically, the figures' lengths and degrees of angles are preserved.**
- **If a figure undergoes two transformations, F and G, and is in the same place it was originally, then the figure has been mapped onto itself.**

Exit Ticket (5 minutes)

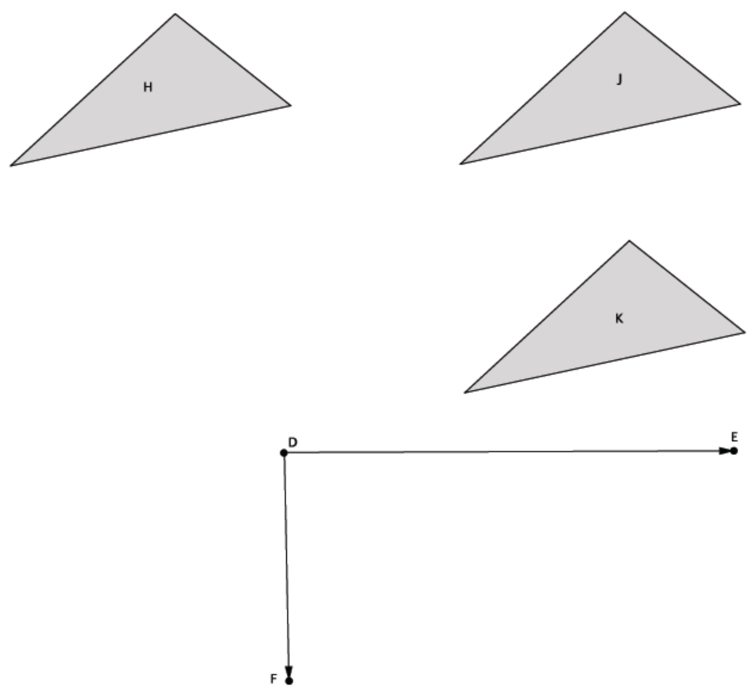
T

Name ____ Date ________________

Lesson 7: Sequencing Translations

Exit Ticket

Use the picture below to answer Problems 1 and 2.

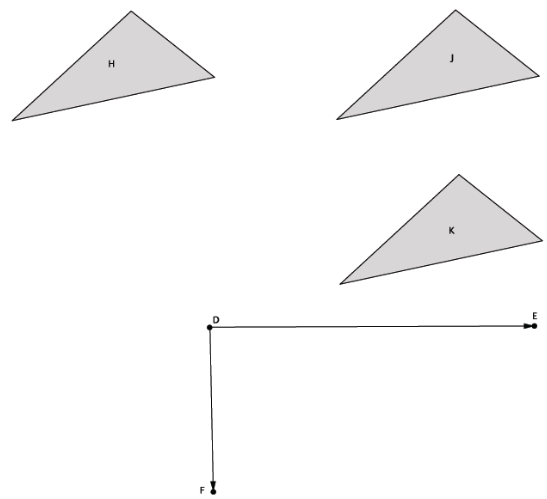


1. Describe a sequence of translations that would map Figure H onto Figure K.

2. Describe a sequence of translations that would map Figure J onto itself.

Exit Ticket Sample Solutions

Use the picture below to answer Problems 1 and 2.



1. Describe a sequence of translations that would map Figure H onto Figure K.

   Translate Figure H along vector $\overrightarrow{DE}$, and then translate the image along vector $\overrightarrow{DF}$.

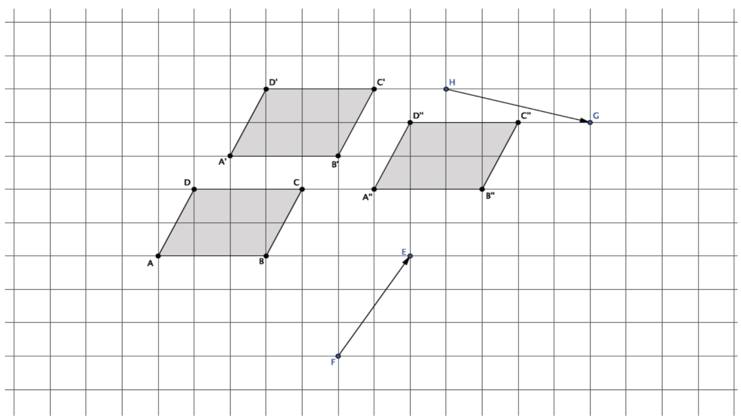
2. Describe a sequence of translations that would map Figure J onto itself.

   Translate Figure J along vector $\overrightarrow{DE}$, and then translate the image along vector $\overrightarrow{ED}$.

   Translate Figure J along vector $\overrightarrow{DF}$, and then translate the image along vector $\overrightarrow{FD}$.

Problem Set Sample Solutions

1. Sequence translations of parallelogram $ABCD$ (a quadrilateral in which both pairs of opposite sides are parallel) along vectors $\overrightarrow{HG}$ and $\overrightarrow{FE}$. Label the translated images.



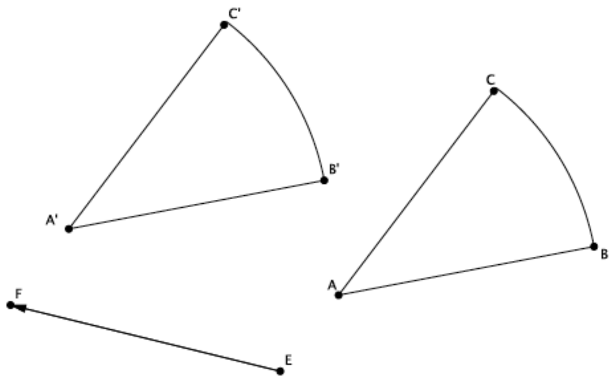
2. What do you know about AD and BC compared with $A'D'$ and $B'C'$? Explain.

   By the definition of a parallelogram, $AD \parallel BC$. Since translations map parallel lines to parallel lines, I know that $A'D' \parallel B'C'$.

3. Are $A'B'$ and $A''B''$ equal in length? How do you know?

   Yes, $|A'B'| = |A''B''|$. Translations preserve lengths of segments.

4. **Translate the curved shape ABC along the given vector. Label the image.**



5. **What vector would map the shape $A'B'C'$ back onto ABC?**

Translating the image along vector $\overrightarrow{FE}$ would map the image back onto its original position.

Lesson 8: Sequencing Reflections and Translations

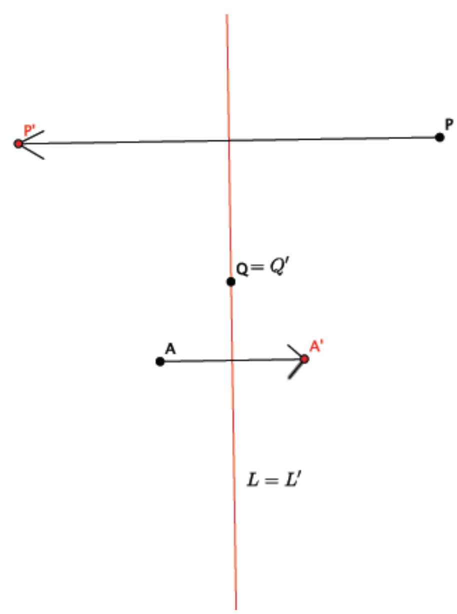
Student Outcomes

- Students learn that the reflection is its own inverse transformation.
- Students understand that a sequence of a reflection followed by a translation is not necessarily equal to a translation followed by a reflection.

Classwork

Discussion (10 minutes)

- Lesson 7 was an introduction to sequences of translations. It was clear that when a figure was translated along a vector, we could undo the move by translating along the same vector, but in the opposite direction, creating an inverse transformation.
- Note that not all transformations can be undone. For that reason, we will investigate sequences of reflections.
- Let there be a reflection across line L. What would undo this action? What is the inverse of this transformation?
  - *A reflection is always its own inverse.*
- Consider the picture below of a reflection across a vertical line L.



- Trace this picture of the line L and the points P, A, and Q as shown. Create a reflection across line L followed by another reflection across line L. Is the transformation corresponding to flipping the transparency once across L, and then flipping it once more across L? Obviously, the red figure on the transparency would be right back on top of the original black figure. *Everything stays the same*.

Let us take this opportunity to reason through the preceding fact without a transparency.

- For a point P not on line L, what would the refletion of the reflection of point P be?
- The picture shows $Reflection(P) = P'$ is a point to the left of L, and if we reflect the point P' across L, clearly we get back to P itself. Thus the reflection of the reflection of point P is P itself. The same holds true for A: the reflection of the reflection of point A is A.
- For point Q on the line L, what would the reflection of the reflection of point Q be?
- The lesson on reflection showed us that a point on the line of reflection is equal to itself, i.e., $Reflection(Q) = Q$. Then, the reflection of the reflection of point Q is Q. No matter how many times a point on the line of reflection is reflected, it will be equal to itself.
- Based on the last two statements, we can say that the reflection of the reflection of P is P for any point P in the plane. Further,

  $$\text{reflection of } P \text{ followed by the reflection of } P = I, \tag{4}$$

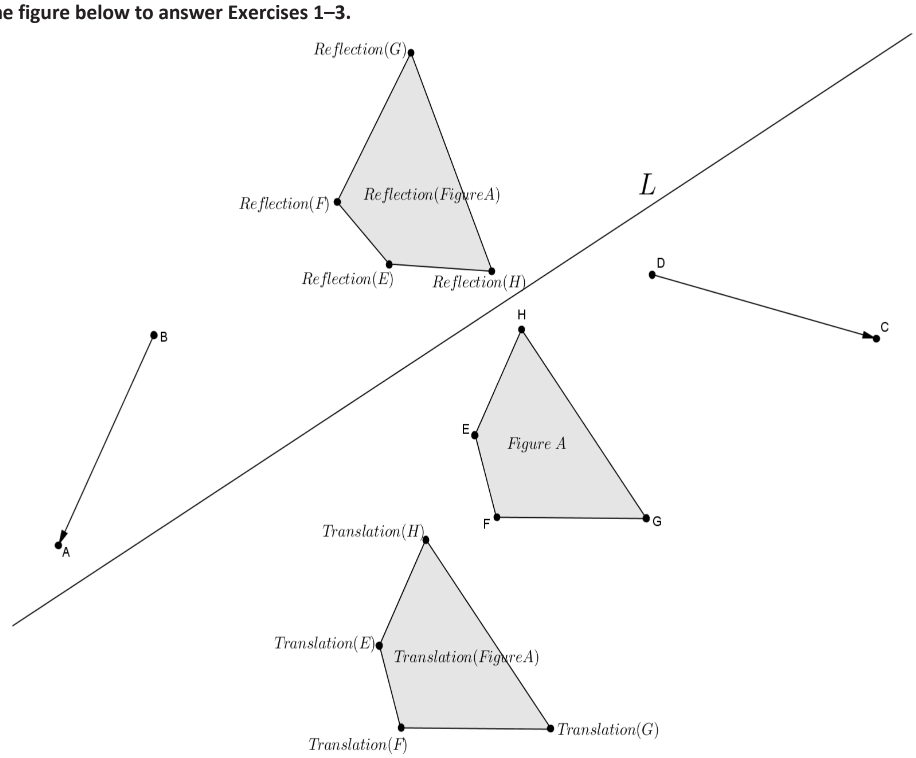
  where I denotes the identity transformation (Lesson 1). In terms of transparencies, equation (4) says that if we flip the transparency (on which we have traced a given figure in red) across the line of reflection L, then flipping it once more across L brings the red figure to coincide completely with the original figure. In this light, equation (4) is hardly surprising!

Exercises 1–3 (3 minutes)

Students complete Exercises 1–3 independently.

Exercises 1–3

Use the figure below to answer Exercises 1–3.



1. **Figure A was translated along vector $\overrightarrow{BA}$ resulting in $Translation(Figure\ A)$. Describe a sequence of translations that would map Figure A back onto its original position.**

   Translate Figure A along vector $\overrightarrow{BA}$; then, translate the image of Figure A along vector $\overrightarrow{AB}$.

2. **Figure A was reflected across line L resulting in $Reflection(Figure\ A)$. Describe a sequence of reflections that would map Figure A back onto its original position.**

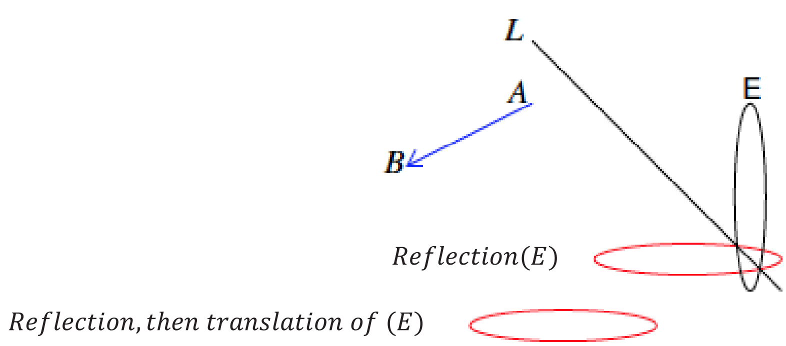
   Reflect Figure A across line L; then, reflect Figure A across line L again.

3. **Can $Translation_{\overrightarrow{BA}}$ of Figure A undo the transformation of $Translation_{\overrightarrow{DC}}$ of Figure A? Why or why not?**

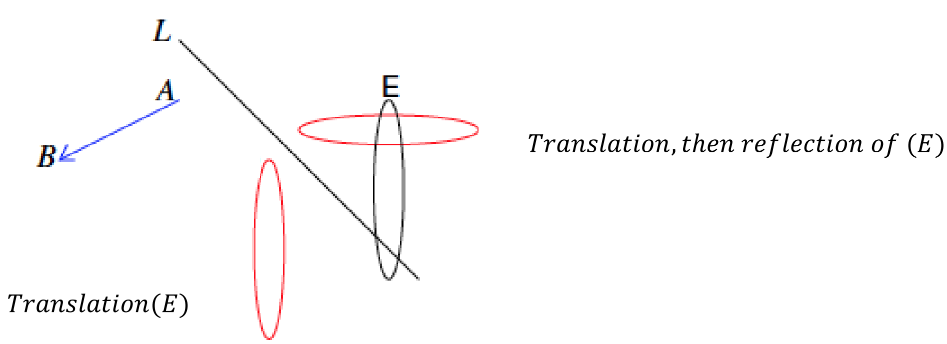
   No. To undo the transformation, you would need to move the image of Figure A after the translations back to Figure A. The listed translations do not do that.

Discussion (10 minutes)

- Does the order in which we sequence rigid motions really matter?
- Consider a reflection followed by a translation. Would a figure be in the same final location if the translation was done first then followed by the reflection?
- Let there be a reflection across line L and let T be the translation along vector $\overrightarrow{AB}$. Let E represent the ellipse. The following picture shows the reflection of E followed by the translation of E.
- Before showing the picture, ask students which transformation happens first: the reflection or the translation?
  - *Reflection*



- Ask students again if they think the image of the ellipse will be in the same place if we translate first and then reflect. The following picture shows a translation of E followed by the reflection of E.



- It must be clear now that the order in which the rigid motions are performed matters. In the above example, we saw that the reflection followed by the translation of E is not the same as the translation followed by the reflection of E; therefore, a translation followed by a reflection and a reflection followed by a translation are not equal.

Video Presentation (2 minutes)

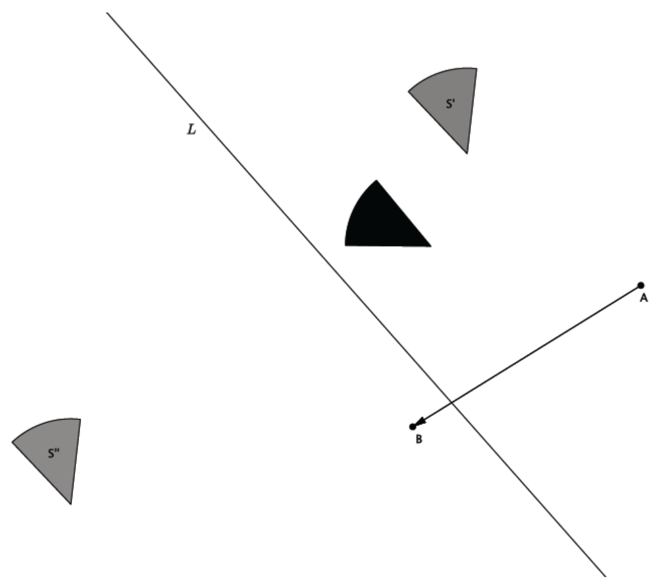
Show the video on the sequence of basic rigid motions located at http://youtu.be/O2XPy3ZLU7Y. Note that this video makes use of rotation, which will not be defined until Lesson 9. The video, however, does clearly convey the general idea of sequencing.

Exercises 4–7 (10 minutes)

Students complete Exercises 4, 5, and 7 independently. Students complete Exercise 6 in pairs.

Exercises 4–7

Let S be the black figure.



4. **Let there be the translation along vector $\overrightarrow{AB}$ and a reflection across line L.**
   Use a transparency to perform the following sequence: Translate figure S; then, reflect figure S. Label the image S'.
   Solution on the diagram above.

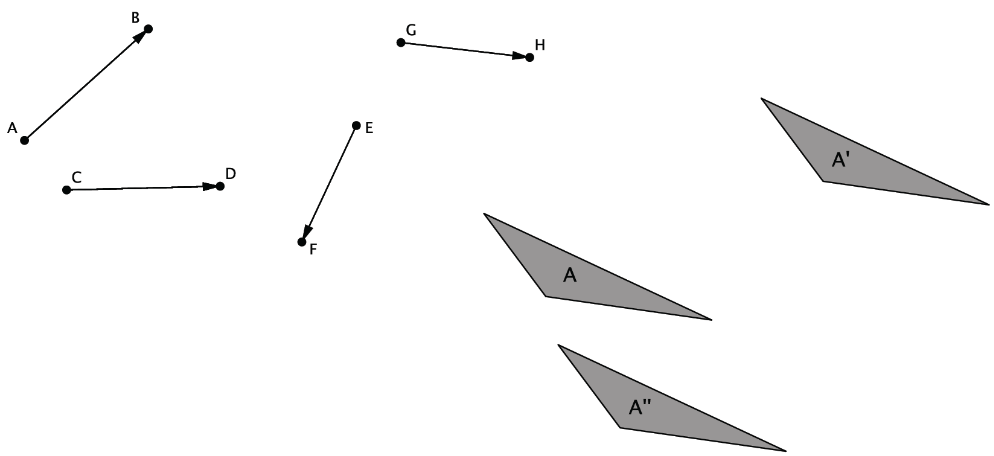
5. **Let there be the translation along vector $\overrightarrow{AB}$ and a reflection across line L.**
   Use a transparency to perform the following sequence: Reflect figure S; then, translate figure S. Label the image S''.
   Solution on the diagram above.

6. Using your transparency, show that under a sequence of any two translations, $Translation$ and $Translation_0$ (along different vectors), that the sequence of the $Translation$ followed by the $Translation_0$ is equal to the sequence of the $Translation_0$ followed by the $Translation$. That is, draw a figure, A, and two vectors. Show that the translation along the first vector, followed by a translation along the second vector, places the figure in the same location as when you perform the translations in the reverse order. (This fact will be proven in high school). Label the transformed image A'. Now, draw two new vectors and translate along them just as before. This time, label the transformed image A''. Compare your work with a partner. Was the statement "the sequence of the $Translation$ followed by the $Translation_0$ is equal to the sequence of the $Translation_0$ followed by the $Translation$" true in all cases? Do you think it will always be true?

Sample Student Work:

First, let T be the translation along vector $\overrightarrow{AB}$, and let T_0 be the translation along vector $\overrightarrow{CD}$.

Then, let T be the translation along vector $\overrightarrow{EF}$, and let T_0 be the translation along vector $\overrightarrow{GH}$.



7. Does the same relationship you noticed in Exercise 6 hold true when you replace one of the translations with a reflection. That is, is the following statement true: A translation followed by a reflection is equal to a reflection followed by a translation?

No. The translation followed by a reflection would put a figure in a different location in the plane when compared to the same reflection followed by the same translation.

Closing (5 minutes)

Summarize, or have students summarize, the lesson.

- We can sequence rigid motions.
- We have notation related to sequences of rigid motions.
- The sequence of a reflection followed by the same reflection is the identity transformation, and the order in which we sequence rigid motions matters.

Lesson Summary

- A reflection across a line followed by a reflection across the same line places all figures in the plane back onto their original position.
- A reflection followed by a translation does not place a figure in the same location in the plane as a translation followed by a reflection. The order in which we perform a sequence of rigid motions matters.

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 8: Sequencing Reflections and Translations

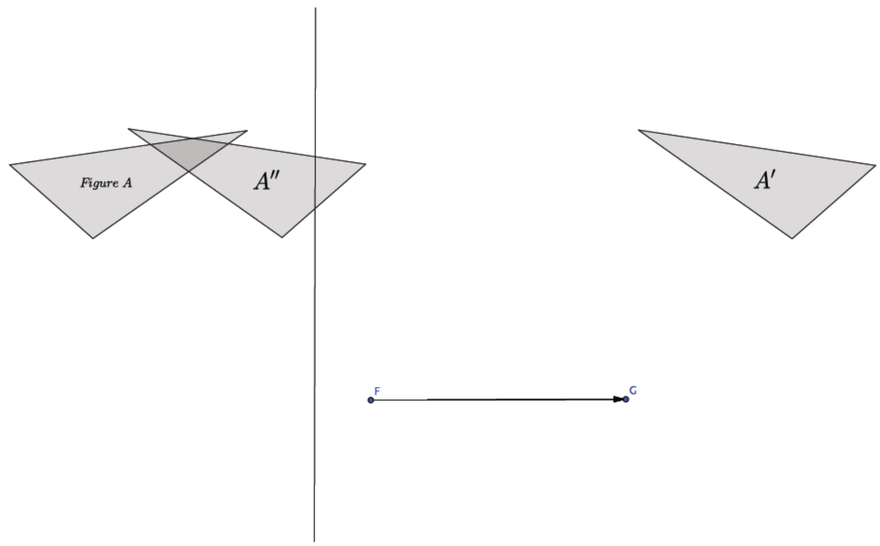
Exit Ticket

Draw a figure, A, a line of reflection, L, and a vector $\overrightarrow{FG}$ in the space below. Show that under a sequence of a translation and a reflection that the sequence of the reflection followed by the translation is not equal to the translation followed by the reflection. Label the figure as A' after finding the location according to the sequence reflection followed by the translation, and label the figure A'' after finding the location according to the composition translation followed by the reflection. If A' is not equal to A'', then we have shown that the sequence of the reflection followed by a translation is not equal to the sequence of the translation followed by the reflection. (This will be proven in high school.)

Exit Ticket Sample Solutions

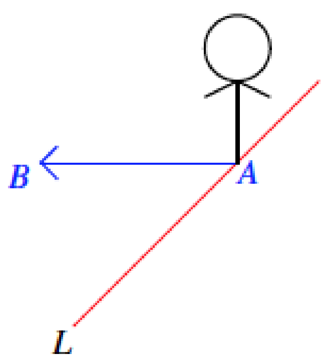
Draw a figure, A, a line of reflection, L, and a vector $\overrightarrow{FG}$ in the space below. Show that under a sequence of a translation and a reflection, that the sequence of the reflection followed by the translation is not equal to the translation followed by the reflection. Label the figure as A' after finding the location according to the sequence reflection followed by the translation, and label the figure A''after finding the location according to the composition translation followed by the reflection. If A' is not equal to A'', then we have shown that the sequence of the reflection followed by a translation is not equal to the sequence of the translation followed by the reflection. (This will be proven in high school.)

Sample student drawing:



Problem Set Sample Solutions

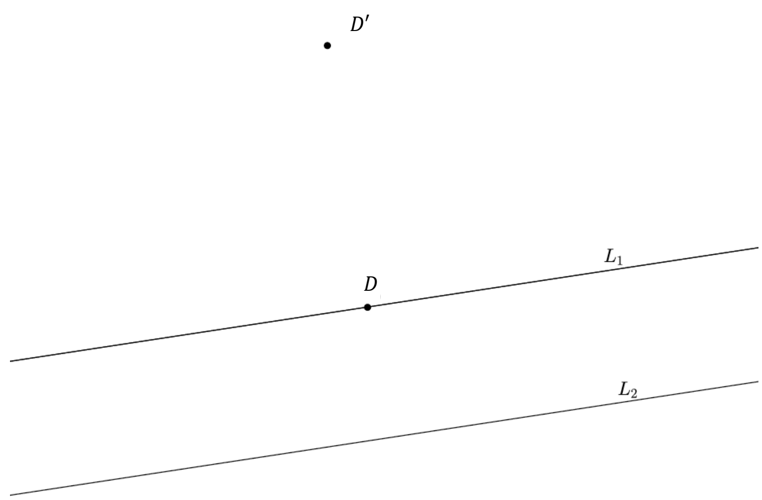
1. Let there be a reflection across line L, and let there be a translation along vector $\overrightarrow{AB}$, as shown. If S denotes the black figure, compare the translation of S followed by the reflection of S with the reflection of S followed by the translation of S.



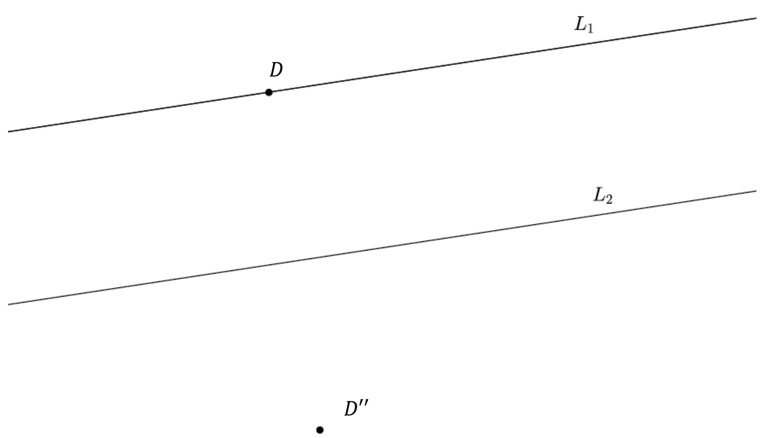
Students should notice that the two sequences place figure S in different locations in the plane.

2. Let L_1 and L_2 be parallel lines, and let $Reflection_1$ and $Reflection_2$ be the reflections across L_1 and L_2, respectively (in that order). Show that a $Reflection_2$ followed by $Reflection_1$ is not equal to a $Reflection_1$ followed by $Reflection_2$. (Hint: Take a point on L_1 and see what each of the sequences does to it.)

Let D be a point on L_1, as shown, and let $D' = Reflection_2$ followed by $Reflection_1$. Notice where D' is.



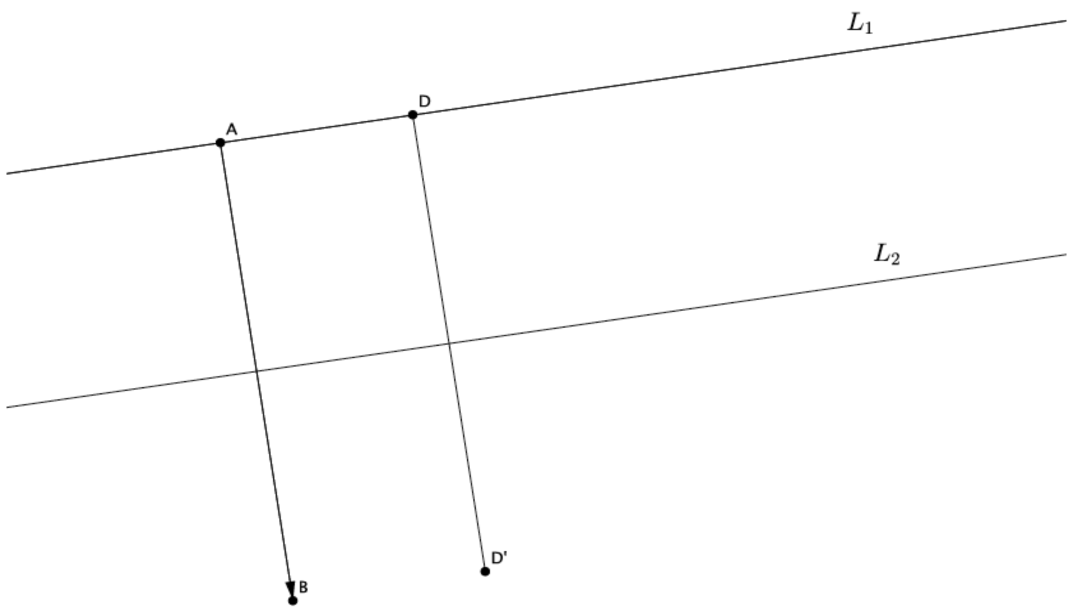
Let $D'' = Reflection_1$ followed by $Reflection_2$. Notice where the D'' is.



Since $D' \neq D''$, the sequences are not equal.

3. Let L_1 and L_2 be parallel lines, and let $Reflection_1$ and $Reflection_2$ be the reflections across L_1 and L_2, respectively (in that order). Can you guess what $Reflection_1$ followed by $Reflection_2$ is? Give as persuasive an argument as you can. (Hint: Examine the work you just finished for the last problem.)

The sequence $Reflection_1$ followed by $Reflection_2$ is just like the translation along a vector $\overrightarrow{AB}$, as shown below, where $AB \perp L_1$. The length of AB is equal to twice the distance between L_1 and L_2.



Lesson 9: Sequencing Rotations

Student Outcomes

- Students learn that sequences of rotations preserve lengths of segments as well as degrees of measures of angles.
- Students describe a sequence of rigid motions that would map a triangle back to its original position after being rotated around two different centers.

Classwork

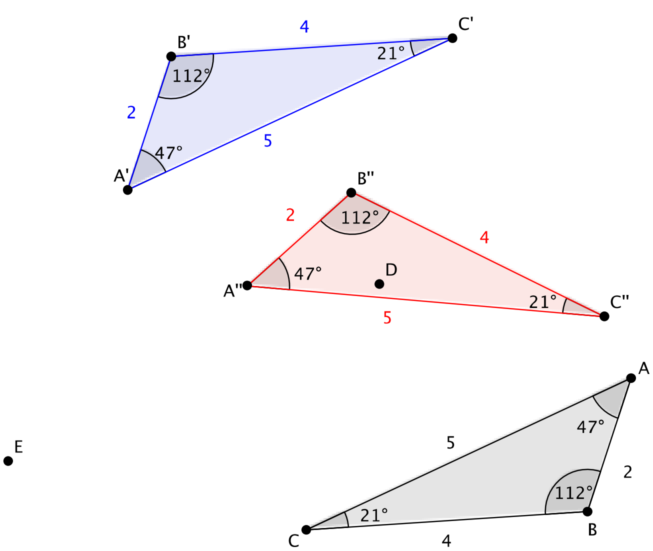
Exploratory Challenge (35 minutes)

Scaffolding:

It may be beneficial to check student work periodically throughout the exploratory challenge. As necessary, provide teacher modeling.

Exploratory Challenge

1.



a. **Rotate $\triangle ABC$ d degrees around center D. Label the rotated image as $\triangle A'B'C'$.**

Sample student work shown in blue.

b. **Rotate $\triangle A'B'C'$ d degrees around center E. Label the rotated image as $\triangle A''B''C''$.**

Sample student work shown in red.

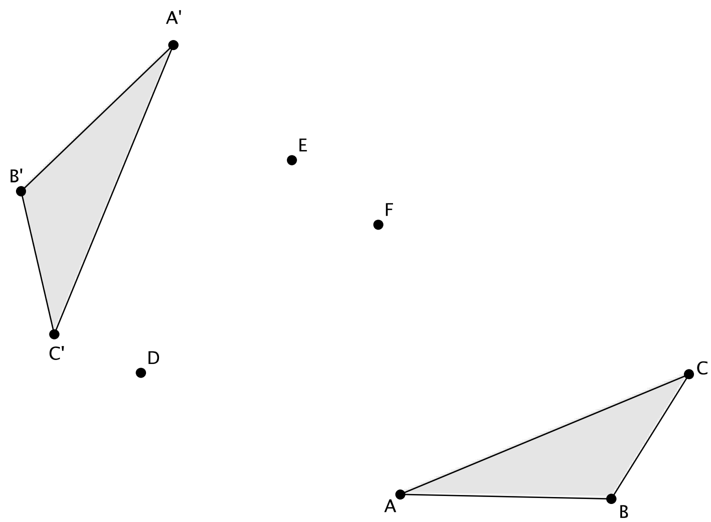
c. **Measure and label the angles and side lengths of $\triangle ABC$. How do they compare with the images $\triangle A'B'C'$ and $\triangle A''B''C''$?**

Measures of corresponding sides and measures of corresponding angles of three triangles are equal.

d. **How can you explain what you observed in part (c)? What statement can you make about properties of sequences of rotations as they relate to a single rotation?**

We already knew that a single rotation would preserve the lengths of segments and degrees of angles. Performing one rotation after the other will not change the lengths of segments or degrees of angles. That means that sequences of rotations enjoy the same properties as a single rotation.

2.



a. **Rotate $\triangle ABC$ d degrees around center D, and then rotate again d degrees around center E. Label the image as $\triangle A'B'C'$ after you have completed both rotations.**

Possible student solution shown in diagram as $\triangle A'B'C'$.

b. **Can a single rotation around center D map $\triangle A'B'C'$ onto $\triangle ABC$?**

No, a single rotation around center D will not map $\triangle A'B'C'$ onto $\triangle ABC$.

c. **Can a single rotation around center E map $\triangle A'B'C'$ onto $\triangle ABC$?**

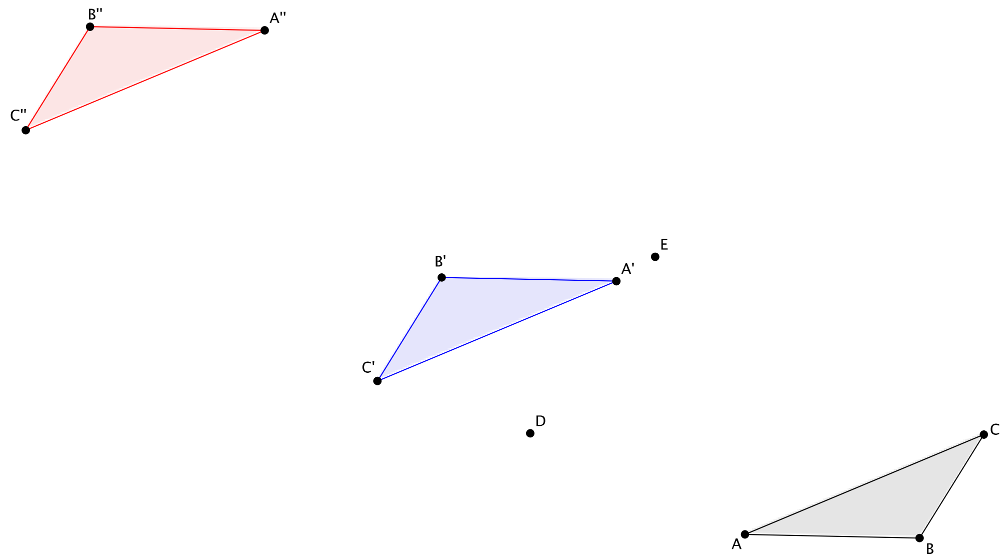
No, a single rotation around center E will not map $\triangle A'B'C'$ onto $\triangle ABC$.

d. **Can you find a center that would map $\triangle A'B'C'$ onto $\triangle ABC$ in one rotation? If so, label the center F.**

Yes, a d-degree rotation around center F will map $\triangle A'B'C'$ onto $\triangle ABC$.

Note: Students can only find the center F through trial and error at this point. Finding the center of rotation for two congruent figures is a skill that will be formalized in high school Geometry.

3.



a. Rotate $\triangle ABC$ 90° (counterclockwise) around center D, and then rotate the image another 90° (counterclockwise) around center E. Label the image $\triangle A'B'C'$.

Sample student work shown in blue.

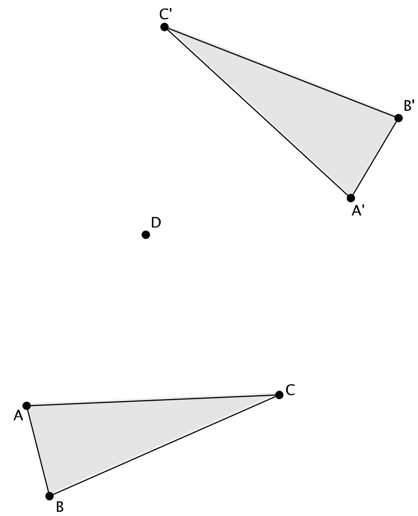
b. Rotate $\triangle ABC$ 90° (counterclockwise) around center E, and then rotate the image another 90° (counterclockwise) around center D. Label the image $\triangle A''B''C''$.

Sample student work shown in red.

c. What do you notice about the locations of $\triangle A'B'C'$ and $\triangle A''B''C''$? Does the order in which you rotate a figure around different centers have an impact on the final location of the figure's image?

The triangles are in two different locations. Yes, the order in which we rotate a figure around two different centers must matter because the triangles are not in the same location after rotating around center D and then center E, compared to center E and then center D.

4.



a. Rotate $\triangle ABC$ $90°$ (counterclockwise) around center D, and then rotate the image another $45°$ (counterclockwise) around center D. Label the $\triangle A'B'C'$.

Rotated triangle shown above.

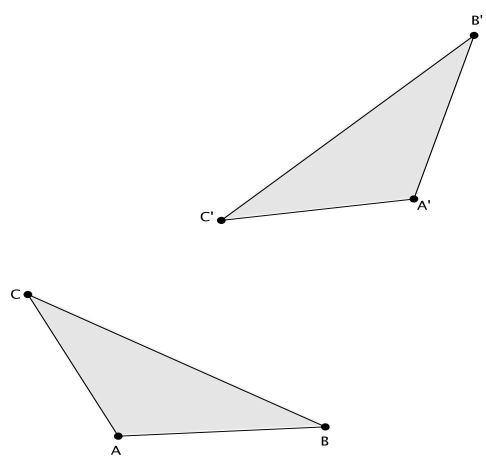
b. Rotate $\triangle ABC$ $45°$ (counterclockwise) around center D, and then rotate the image another $90°$ (counterclockwise) around center D. Label the $\triangle A''B''C''$.

Rotated triangle shown above.

c. What do you notice about the locations of $\triangle A'B'C'$ and $\triangle A''B''C''$? Does the order in which you rotate a figure around the same center have an impact on the final location of the figure's image?

The triangles are in the same location. This indicates that when a figure is rotated twice around the same center, it does not matter in which order you perform the rotations.

5. $\triangle ABC$ has been rotated around two different centers, and its image is $\triangle A'B'C'$. Describe a sequence of rigid motions that would map $\triangle ABC$ onto $\triangle A'B'C'$.



Translate $\triangle ABC$ along vector $\overrightarrow{CC'}$. Then, rotate $\triangle ABC$ around point C' until $\triangle ABC$ maps onto $\triangle A'B'C'$.

Closing (5 minutes)

Summarize, or have students summarize, the lesson.

- Sequences of rotations enjoy the same properties as single rotations. That is, a sequence of rotations preserves lengths of segments and degrees of measures of angles.
- The order in which a sequence of rotations around two different centers is performed matters. The order in which a sequence of rotations around the same center is performed does not matter.
- When a figure is rotated around two different centers, we can describe a sequence of rigid motions that would map the original figure onto the resulting image.

Lesson Summary

- **Sequences of rotations have the same properties as a single rotation:**
  - **A sequence of rotations preserves degrees of measures of angles.**
  - **A sequence of rotations preserves lengths of segments.**
- **The order in which a sequence of rotations around different centers is performed matters with respect to the final location of the image of the figure that is rotated.**
- **The order in which a sequence of rotations around the same center is performed does not matter. The image of the figure will be in the same location.**

Exit Ticket (5 minutes)

Name ____ Date ____________________

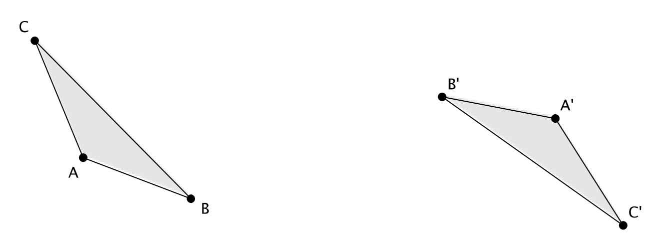
Lesson 9: Sequencing Rotations

Exit Ticket

1. Let $Rotation_1$ be the rotation of a figure d degrees around center O. Let $Rotation_2$ be the rotation of the same figure d degrees around center P. Does the $Rotation_1$ of the figure followed by the $Rotation_2$ equal a $Rotation_2$ of the figure followed by the $Rotation_1$? Draw a picture if necessary.

2. Angle ABC underwent a sequence of rotations. The original size of $\angle ABC = 37°$. What was the size of the angle after the sequence of rotations? Explain.

3. Triangle ABC underwent a sequence of rotations around two different centers. Its image is $\triangle A'B'C'$. Describe a sequence of rigid motions that would map $\triangle ABC$ onto $\triangle A'B'C'$.



Exit Ticket Sample Solutions

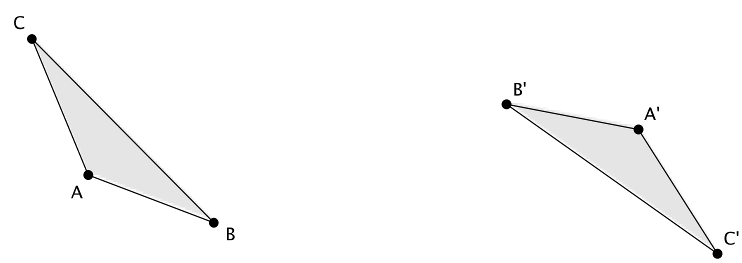
1. Let $Rotation_1$ be the rotation of a figure d degrees around center O. Let $Rotation_2$ be the rotation of the same figure d degrees around center P. Does the $Rotation_1$ of the figure followed by the $Rotation_2$ equal a $Rotation_2$ of the figure followed by the $Rotation_1$? Draw a picture if necessary.

   No. If the sequence of rotations were around the same center, then it would be true. However, when the sequence involves two different centers, the order in which they are performed matters because the images will not be in the same location in the plane.

2. Angle ABC underwent a sequence of rotations. The original size of $\angle ABC = 37°$. What was the size of the angle after the sequence of rotations? Explain.

   Since sequences of rotations enjoy the same properties as a single rotation, then the measure of any image of $\angle ABC$ under any sequence of rotations will remain $37°$. Rotations and sequences of rotations preserve the measure of degrees of angles.

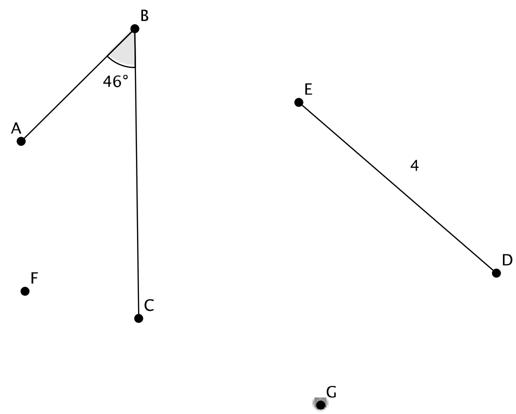
3. Triangle ABC underwent a sequence of rotations around two different centers. Its image is $\triangle A'B'C'$. Describe a sequence of rigid motions that would map $\triangle ABC$ onto $\triangle A'B'C'$.



   Translate $\triangle ABC$ along vector $\overrightarrow{BB'}$. Then, rotate $\triangle ABC$ d degrees around point B' until $\triangle ABC$ maps onto $\triangle A'B'C'$.

Problem Set Sample Solutions

1. Refer to the figure below.



a. Rotate $\angle ABC$ and segment DE d degrees around center F, then d degrees around center G. Label the final location of the images as $\angle A'B'C'$ and $D'E'$.

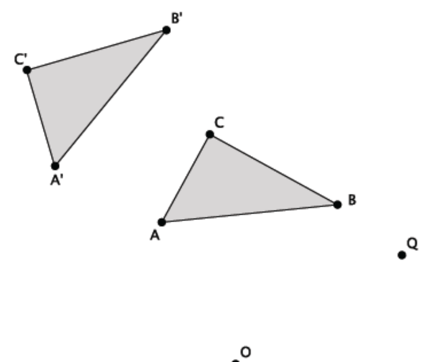
b. What is the size of $\angle ABC$, and how does it compare to the size of $\angle A'B'C'$? Explain.

The measure of $\angle ABC = 46°$. The measure of $\angle A'B'C' = 46°$. The angles are equal in measure because a sequence of rotations will preserve the degrees of an angle.

c. What is the length of segment DE, and how does it compare to the length of segment $D'E'$? Explain.

The length of segment DE is 4 cm. The length of segment $D'E'$ is also 4 cm. The segments are equal in length because a sequence of rotations will preserve the length of segments.

2. Refer to the figure given below.

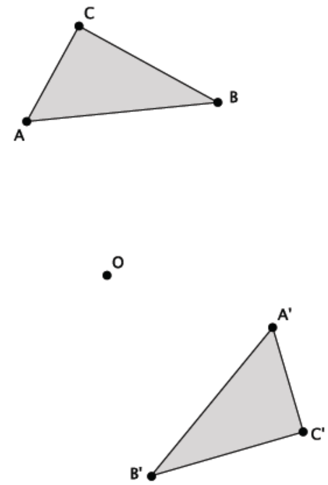


a. Let $Rotation_1$ be a counterclockwise rotation of $90°$ around the center O. Let $Rotation_2$ be a clockwise rotation of $(-45)°$ around the center Q. Determine the approximate location of $Rotation_1(\triangle ABC)$ followed by $Rotation_2$. Label the image of triangle ABC as $A'B'C'$.

b. Describe the sequence of rigid motions that would map $\triangle ABC$ onto $\triangle A'B'C'$.

The image of ABC is shown above. Translate $\triangle ABC$ along vector $\overrightarrow{AA'}$. Rotate $\triangle ABC$ d degrees around center A'. Then, $\triangle ABC$ will map onto $\triangle A'B'C'$.

3. Refer to the figure given below.



Let R be a rotation of $(-90)^\circ$ around the center O. Let $Rotation_2$ be a rotation of $(-45)^\circ$ around the same center O. Determine the approximate location of $Rotation_1(\Delta ABC)$ followed by $Rotation_2(\Delta ABC)$. Label the image of triangle ABC as $A'B'C'$.

The image of ABC is shown above.

Lesson 10: Sequences of Rigid Motions

Student Outcomes

- Students describe a sequence of rigid motions that maps one figure onto another.

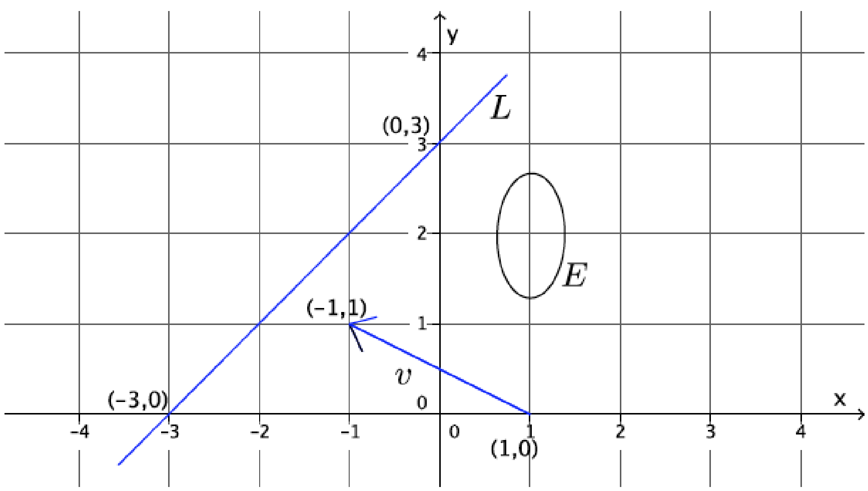
Classwork

Example 1 (8 minutes)

So far, we have seen how to sequence translations, sequence reflections, sequence translations and reflections, and sequence translations and rotations. Now that we know about rotation, we can move geometric figures around the plane by sequencing a combination of translations, reflections, and rotations.

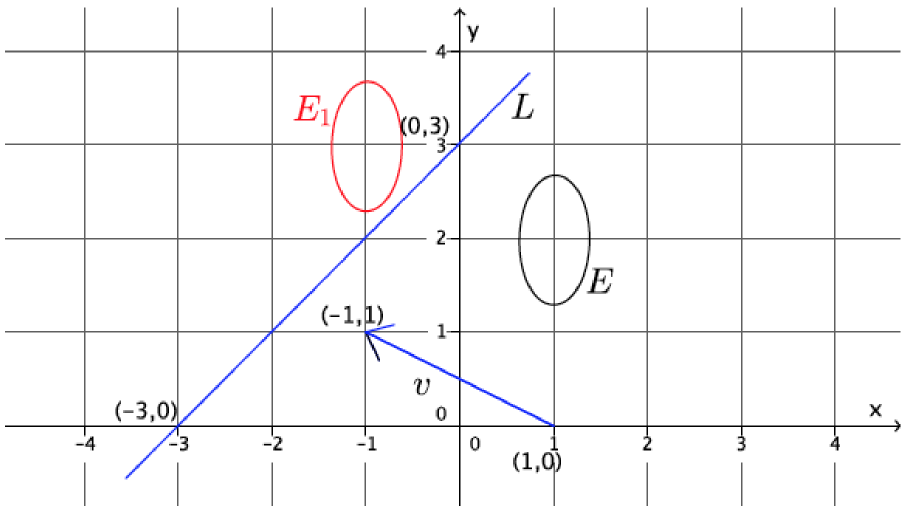
Let's examine the following sequence:

- Let E denote the ellipse in the coordinate plane as shown.

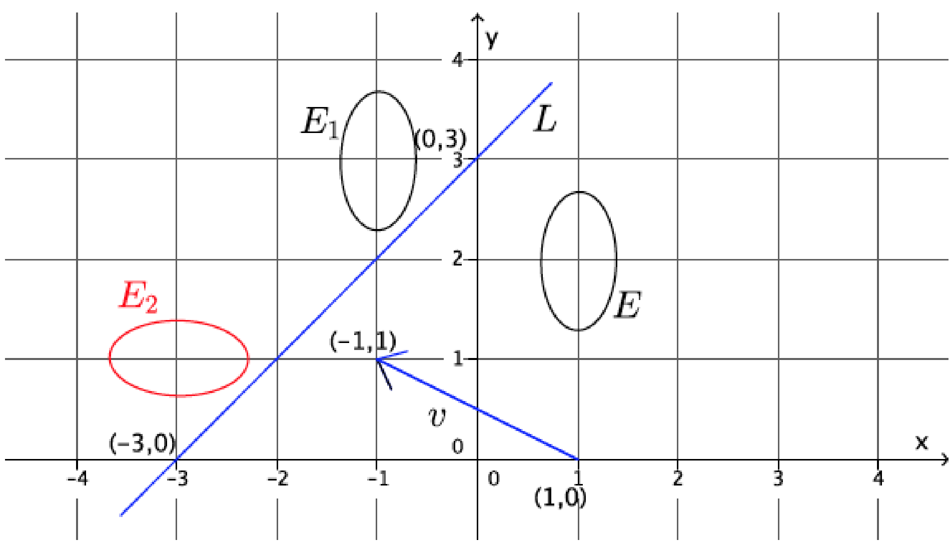


- Let $Translation_1$ be the translation along the vector $\vec{v}$ from $(1,0)$ to $(-1,1)$, let $Rotation_2$ be the 90 degree rotation around $(-1,1)$, and let $Reflection_3$ be the reflection across line L joining $(-3,0)$ and $(0,3)$. What is the $Translation_1(E)$ followed by the $Rotation_2(E)$ followed by the $Reflection_3(E)$?

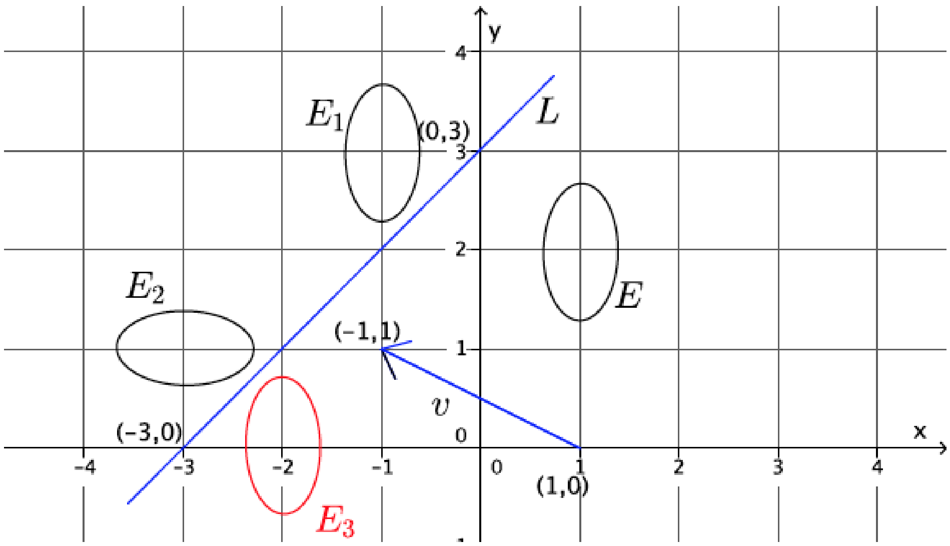
- Which transformation do we perform first, the translation, the reflection, or the rotation? How do you know? Does it make a difference?
  - *The order in which transformations are performed makes a difference. Therefore, we perform the translation first. So now, we let E_1 be $Translation_1(E)$.*



- Which transformation do we perform next?
  - *The rotation. So now, we let E_2 be the image of E after the $Translation_1(E)$ followed by the $Rotation_2(E)$.*



- Now, the only transformation left is $Reflection_3$. So now, we let E_3 be the image of E after the $Translation_1(E)$ followed by the $Rotation_2(E)$ followed by the $Reflection_3(E)$.



Video Presentation (2 minutes)

Students have seen this video[1] in an earlier lesson, but now that they know about rotation, it is worthwhile to watch it again.

www.youtube.com/watch?v=O2XPy3ZLU7Y

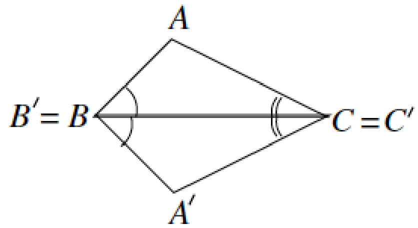
Exercises 1–5 (25 minutes)

Give students one minute or less to work independently on Exercise 1, then have the discussion with them that follows the likely student response. Repeat this process for Exercises 2 and 3. For Exercise 4, have students work in pairs. One student can work on Scenario 1 and the other on Scenario 2, or each student can do both scenarios and then compare with their partner.

[1] The video was developed by Larry Francis.

Exercises

1. **In the following picture, triangle ABC can be traced onto a transparency and mapped onto triangle $A'B'C'$. Which basic rigid motion, or sequence of, would map one triangle onto the other?**



Solution provided below with likely student responses.

Yes, reflection.

Elicit more information from students by asking the following:

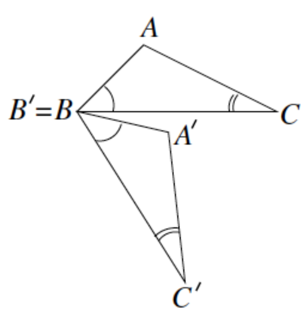
- Reflection requires some information about which line to reflect over; can you provide a clearer answer?
  - *Reflect over line L_{BC}.*

Expand on their answer: Let there be a reflection across the line L_{BC}. We claim that the reflection maps $\triangle ABC$ onto $\triangle A'B'C'$. We can trace $\triangle A'B'C'$ on the transparency and see that when we reflect across line L_{BC}, $\triangle A'B'C'$ maps onto $\triangle ABC$. The reason is because $\angle B$ and $\angle B'$ are equal, and the ray $\overrightarrow{B'A'}$ on the transparency falls on the ray $\overrightarrow{BA}$. Similarly, the ray $\overrightarrow{C'A'}$ falls on the ray $\overrightarrow{CA}$. By the picture, it is obvious that A' on the transparency falls exactly on A, so the reflection of $\triangle A'B'C'$ across L_{BC} is exactly $\triangle ABC$.

Note to Teacher: Here is the precise reasoning without appealing to a transparency. Since a reflection does not move any point on L_{BC}, we already know that $Reflection(B') = B$ and $Reflection(C') = C$. It remains to show the reflection maps A' to A. The hypothesis says $\angle A'BC = \angle ABC$; therefore, the ray $\overrightarrow{BC}$ is the angle bisector [$\angle$ bisector] of $\angle ABA'$. The reflection maps the ray $\overrightarrow{BA'}$ to the ray $\overrightarrow{BA}$. Similarly, the reflection maps the ray $\overrightarrow{CA'}$ to the ray $\overrightarrow{CA}$. Therefore, the reflection maps the intersection of the rays $\overrightarrow{BA'}$ and $\overrightarrow{CA'}$, which is of course just A', to the intersection of rays $\overrightarrow{BA}$ and $\overrightarrow{CA}$, which is, of course, just A. So, $Reflection(A') = A$; therefore, $Reflection(\triangle A'B'C') = \triangle ABC$.

2. **In the following picture, triangle ABC can be traced onto a transparency and mapped onto triangle $A'B'C'$. Which basic rigid motion, or sequence of, would map one triangle onto the other?**

Rotation



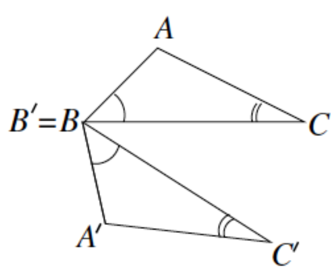
Elicit more information from students by asking:

- Rotation requires some information about what point to rotate around (the center) and how many degrees. If we say we need to rotate d degrees, can you provide a clearer answer?
  - *Rotate around point B as the center, d degrees.*

Expand on their answer: Let there be the (counterclockwise) rotation of d degrees around B, where d is the (positive) degree of the $\angle CBC'$. We claim that the rotation maps $\triangle A'B'C'$ to $\triangle ABC$. We can trace $\triangle A'B'C'$ on the transparency and see that when we pin the transparency at B' (same point as B) and perform a counterclockwise rotation of d degrees, the segment $B'C'$ on the transparency maps onto segment BC (both are equal in length because we can trace one on the transparency and show it is the same length as the other). The point A' on the transparency and A are on the same side (half-plane) of line L_{BC}. Now, we are at the same point we were in the end of Exercise 1; therefore, $\triangle A'B'C'$ and $\triangle ABC$ completely coincide.

Note to Teacher: Here is the precise reasoning without appealing to a transparency. By definition of rotation, rotation maps the ray $\overrightarrow{BC'}$ to the ray $\overrightarrow{BC}$. However, by hypothesis, $BC = BC'$, so $Rotation(C') = C$. Now, the picture implies that after the rotation, A and $Rotation(A)$ lie on the same side of line L_{BC}. If we compare the triangles ABC and $Rotation(A'B'C')$, we are back to the situation at the end of Exercise 1; therefore, the reasoning given here shows that the two triangles coincide.

3. In the following picture, triangle ABC can be traced onto a transparency and mapped onto triangle $A'B'C'$. Which basic rigid motion, or sequence of, would map one triangle onto the other?



Rotation and reflection

Elicit more information from students. Prompt students to think back to what was needed in the last two examples.

- What additional information do we need to provide?
  - *Rotate around point B as the center d degrees; then, reflect across line L_{BC}.*

Expand on their answer: We need a sequence this time. Let there be the (counterclockwise) rotation of d degrees around B, where d is the (positive) degree of the $\angle CBC'$, and let there be the reflection across the line L_{BC}. We claim that the sequence rotation then reflection maps $\triangle A'B'C'$ to $\triangle ABC$. We can trace $\triangle A'B'C'$ on the transparency and see that when we pin the transparency at B' (same point as B) and perform a counterclockwise rotation of d degrees, that the segment $B'C'$ on the transparency maps onto segment BC. Now, $\triangle ABC$ and $\triangle A'B'C'$ are in the exact position as they were in the beginning of Example 2 (Exercise 1); therefore, the reflection across L_{BC} would map $\triangle A'B'C'$ on the transparency to $\triangle ABC$.

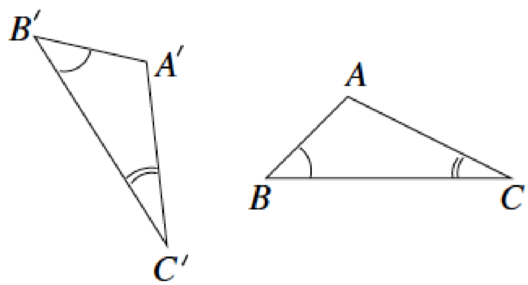
Students may say that they want to reflect first, then rotate. The sequence can be completed in that order, but point out that we need to state which line to reflect across. In that case, we would have to find the appropriate line of reflection. For that reason, it makes more sense to bring a pair of sides together first, i.e., BC and BC', by a rotation, then reflect across the common side of the two triangles. When the rotation is performed first, we can use what we know about Exercise 1.

Note to Teacher: Without appealing to a transparency, the reasoning is as follows. By definition of rotation, rotation maps the ray $\overrightarrow{BC'}$ to the ray $\overrightarrow{BC}$. However, by hypothesis, $BC = BC'$, so $Rotation(C') = C$. Now, when comparing the triangles ABC and $Rotation(A'B'C')$, we see that we are back to the situation in Exercise 1; therefore, the reflection maps the triangle $Rotation\ (\triangle\ A'B'C')$ to triangle $\triangle\ ABC$. This means that rotation then reflection maps $\triangle\ A'B'C'$ to $\triangle\ ABC$.

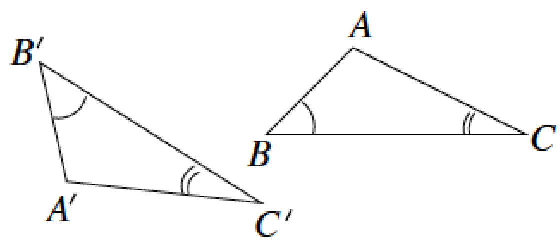
4. **In the following picture, we have two pairs of triangles. In each pair, triangle ABC can be traced onto a transparency and mapped onto triangle $A'B'C'$.**

   Which basic rigid motion, or sequence of, would map one triangle onto the other?

   Scenario 1:



   Scenario 2:



   In Scenario 1, a translation and a rotation; in Scenario 2, a translation, a reflection, then a rotation

Elicit more information from students by asking the following:

- What additional information is needed for a translation?
  - *We need to translate along a vector.*
- Since there is no obvious vector in our picture, which vector should we draw and then use to translate along?

When they do not respond, prompt them to select a vector that would map a point from $\triangle\ A'B'C'$ to a corresponding point in $\triangle\ ABC$. Students will likely respond,

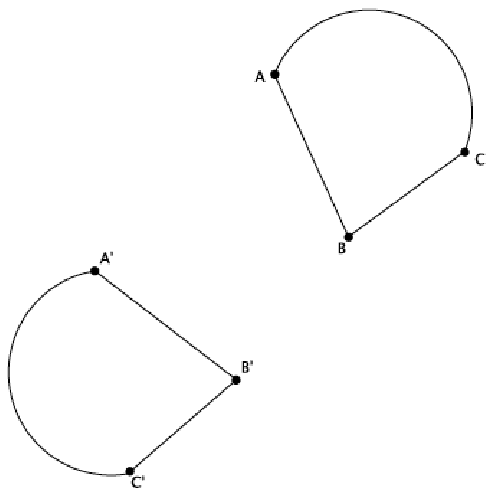
- *Draw vector $\overrightarrow{B'B}$ (or $\overrightarrow{A'A}$ or $\overrightarrow{C'C}$).*

Make it clear to students that we can use any of the vectors they just stated, but using $\overrightarrow{B'B}$ makes the most sense because we can use the reasoning given in the previous exercises rather than constructing the reasoning from the beginning (For example, in Exercises 1–3, $B = B'$).

Expand on their answer: Let there be the translation along vector $\overrightarrow{B'B}$. In Scenario 1, the triangles ABC and $Translation(A'B'C')$ would be similar to the situation of Exercise 2. In Scenario 2, the triangles ABC and $Translation(A'B'C')$ would be similar to the situation of Exercise 3. Based on the work done in Exercises 2 and 3, we can conclude the following: In Scenario 1, the sequence of a translation along $\overrightarrow{B'B}$ followed by a rotation around B would map $\triangle A'B'C'$ to $\triangle ABC$, and in Scenario 2, the sequence of a translation along $\overrightarrow{B'B}$ followed by a rotation around B and finally followed by the reflection across line L_{BC} would map $\triangle A'B'C'$ to $\triangle ABC$.

Students complete Exercise 5 independently or in pairs.

5. **Let two figures ABC and $A'B'C'$ be given so that the length of curved segment AC equals the length of curved segment $A'C'$, $|\angle B| = |\angle B'| = 80°$, and $|AB| = |A'B'| = 5$. With clarity and precision, describe a sequence of rigid motions that would map figure ABC onto figure $A'B'C'$.**



Let there be the translation along vector $\overrightarrow{AA'}$, let there be the rotation around point A d degrees, and let there be the reflection across line L_{AB}. Translate so that $Translation(A') = A$. Rotate so that $Rotation(B') = B$ and $Rotation(A'B')$ coincides with AB (by hypothesis, they are the same length, so we know they will coincide). Reflect across L_{AB} so that $Reflection(C') = C$ and $Reflection(C'B') = CB$ (by hypothesis, $|\angle B| = |\angle B'| = 80°$, so we know that segment $C'B'$ will coincide with CB). By hypothesis, the length of the curved segment $A'C'$ is the same as the length of the curved segment AC, so they will coincide. Therefore, a sequence of translation, then rotation, and then reflection will map figure $A'B'C'$ onto figure ABC.

Closing (5 minutes)

Summarize, or have students summarize, the lesson and what they know of rigid motions to this point:

- We can now describe, using precise language, how to sequence rigid motions so that one figure maps onto another.

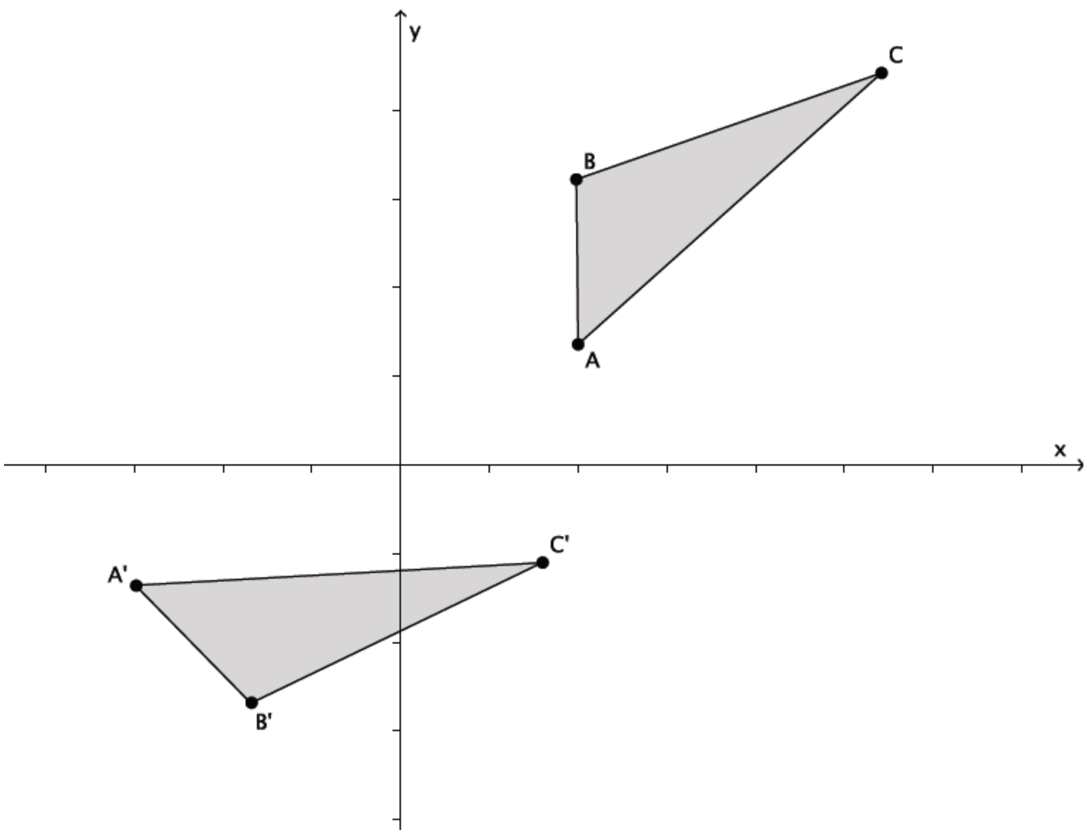
Exit Ticket (5 minutes)

Name ______________________________ Date ________________

Lesson 10: Sequences of Rigid Motions

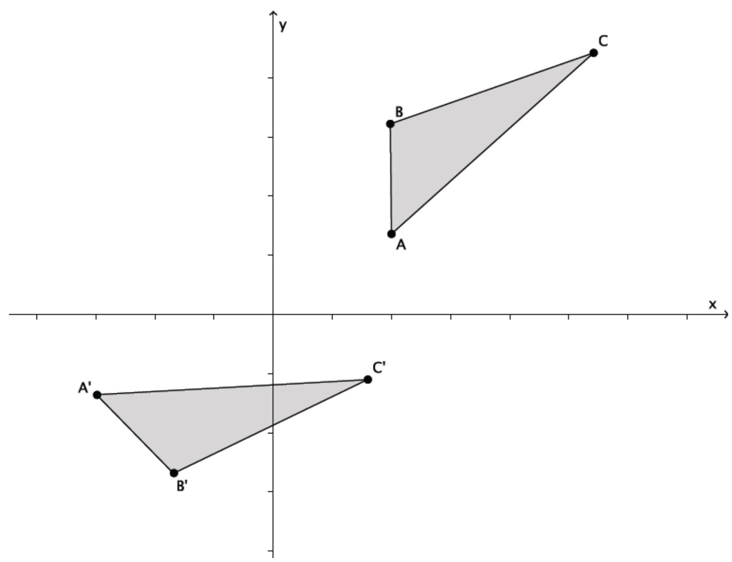
Exit Ticket

Triangle ABC has been moved according to the following sequence: a translation followed by a rotation followed by a reflection. With precision, describe each rigid motion that would map $\triangle ABC$ onto $\triangle A'B'C'$. Use your transparency and add to the diagram if needed.



Exit Ticket Sample Solutions

Triangle ABC has been moved according to the following sequence: a translation followed by a rotation followed by a reflection. With precision, describe each rigid motion that would map $\triangle ABC$ onto $\triangle A'B'C'$. Use your transparency and add to the diagram if needed.

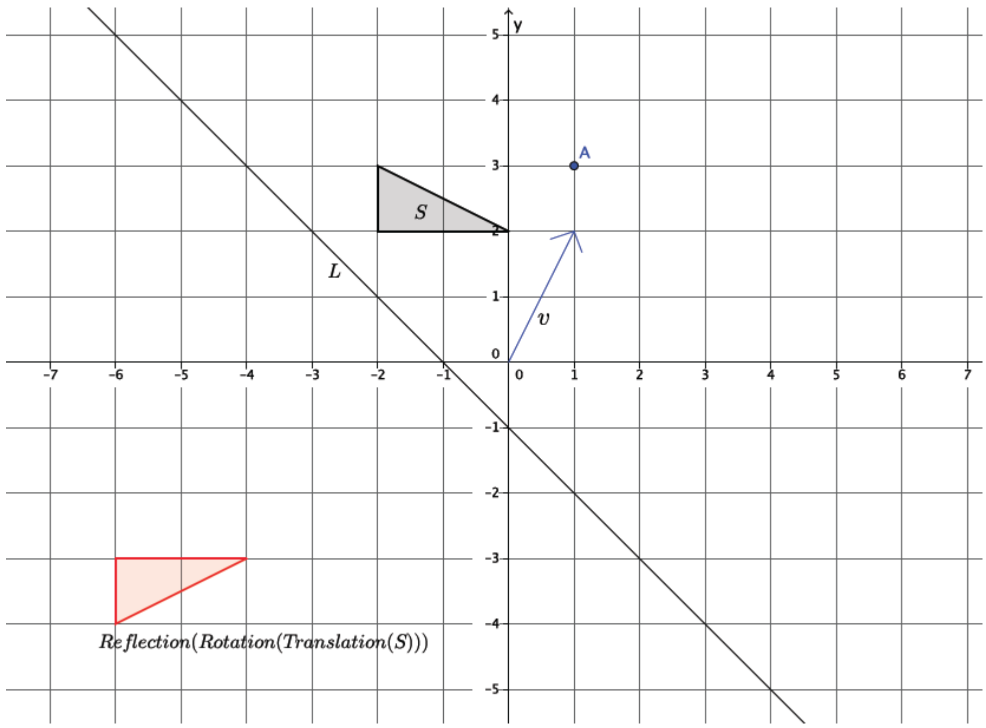


Let there be the translation along vector $\overrightarrow{AA'}$ so that $A = A'$. Let there be the clockwise rotation by d degrees around point A' so that $C = C'$ and $AC = A'C'$. Let there be the reflection across $L_{A'C'}$ so that $B = B'$.

Problem Set Sample Solutions

1. Let there be the translation along vector $\vec{v}$, let there be the rotation around point A, -90 degrees (clockwise), and let there be the reflection across line L. Let S be the figure as shown below. Show the location of S after performing the following sequence: a translation followed by a rotation followed by a reflection.

   Solution is shown in red.

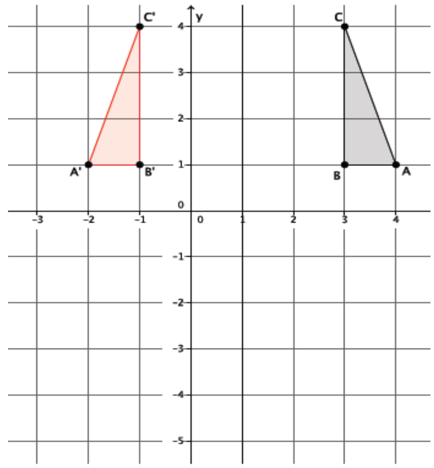


2. Would the location of the image of S in the previous problem be the same if the translation was performed last instead of first, i.e., does the sequence: translation followed by a rotation followed by a reflection equal a rotation followed by a reflection followed by a translation? Explain.

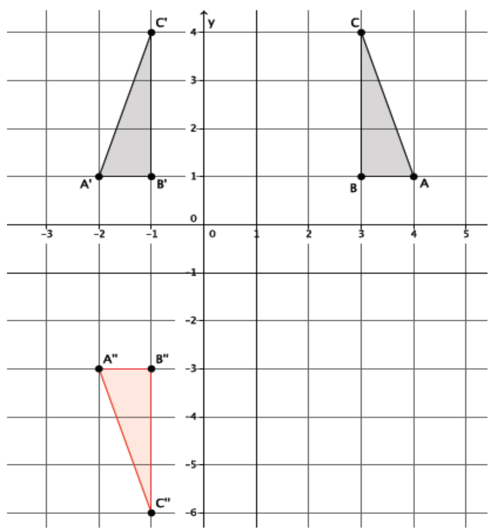
   No, the order of the transformation matters. If the translation was performed last, the location of the image of S, after the sequence, would be in a different location than if the translation was performed first.

3. Use the same coordinate grid to complete parts (a)–(c).

a. Reflect triangle ABC across the vertical line, parallel to the y-axis, going through point $(1, 0)$. Label the transformed points A, B, C as A', B', C', respectively.



b. Reflect triangle $A'B'C'$ across the horizontal line, parallel to the x-axis going through point $(0, -1)$. Label the transformed points of A', B', C' as A'', B'', C'', respectively.



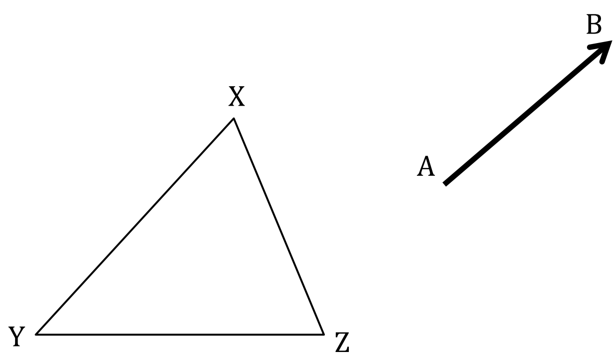
c. Is there a single rigid motion that would map triangle ABC to triangle $A''B''C''$?

Yes, a $180°$ rotation around center $(1, -1)$. The coordinate $(1, -1)$ happens to be the intersection of the two lines of reflection.

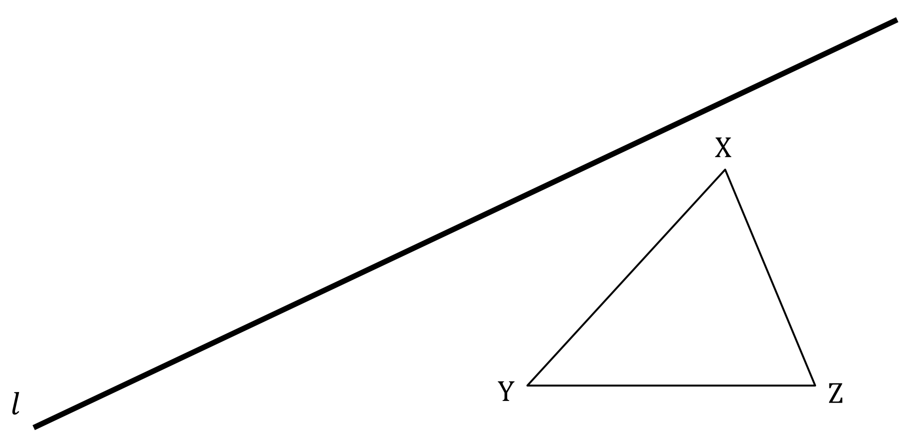
Name ____ Date ____________________

1.

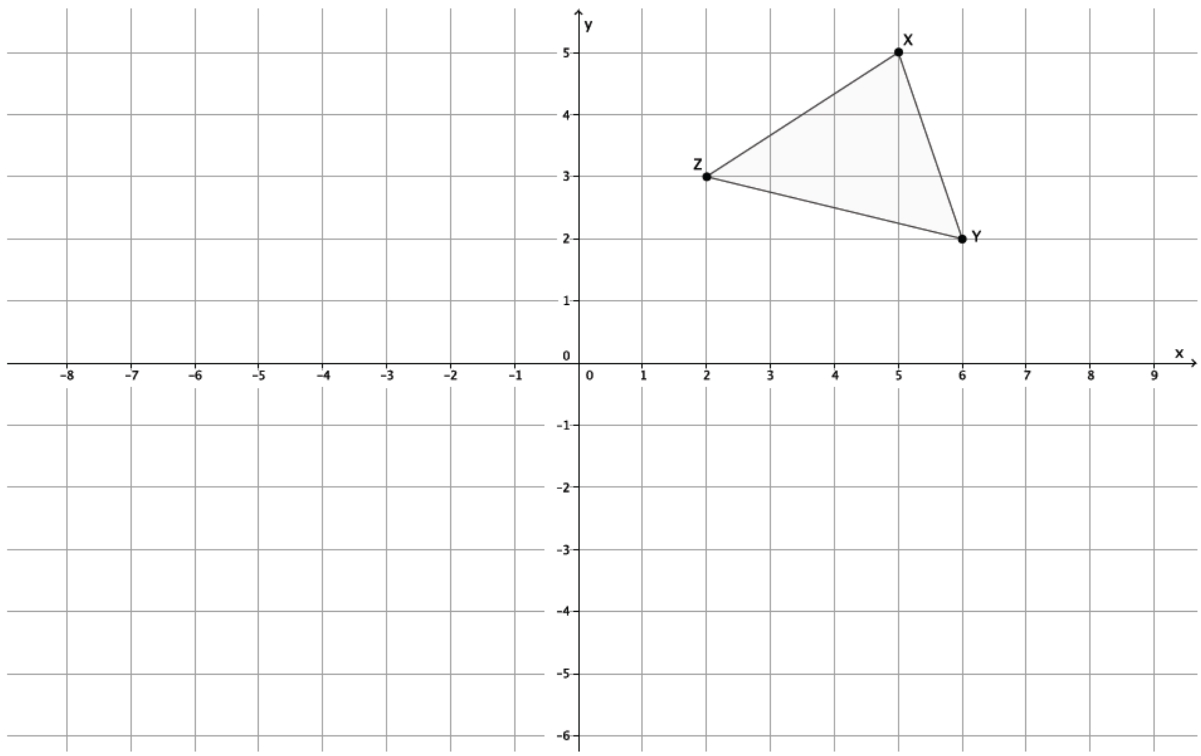
a. Translate ΔXYZ along $\overrightarrow{AB}$. Label the image of the triangle with X', Y', and Z'.



b. Reflect ΔXYZ across the line of reflection, l. Label the image of the triangle with X', Y', and Z'.

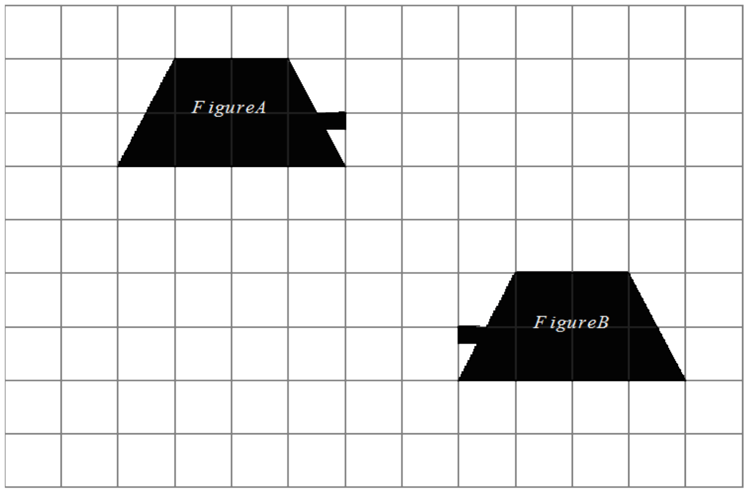


c. Rotate $\triangle XYZ$ around the point $(1,0)$ clockwise $90°$. Label the image of the triangle with X', Y', and Z'.



2. Use the picture below to answer the questions.

Figure A has been transformed to Figure B.

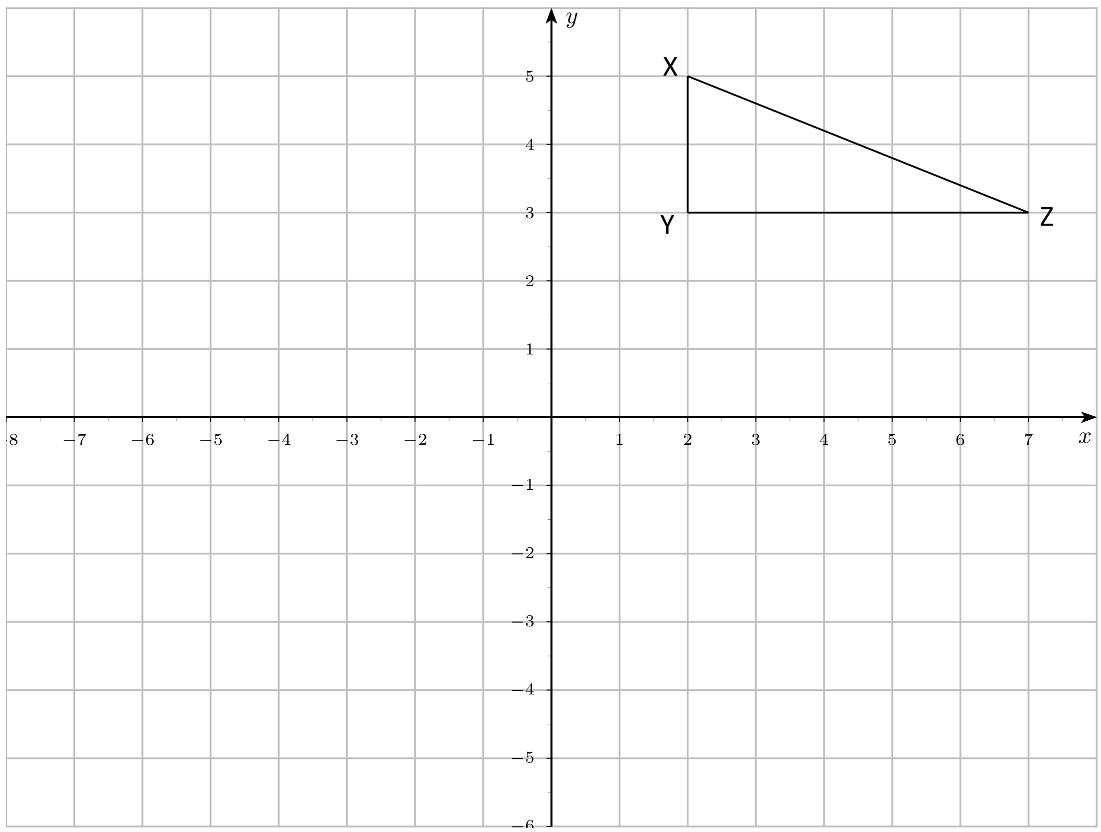


a. Can Figure A be mapped onto Figure B using only translation? Explain. Use drawings as needed in your explanation.

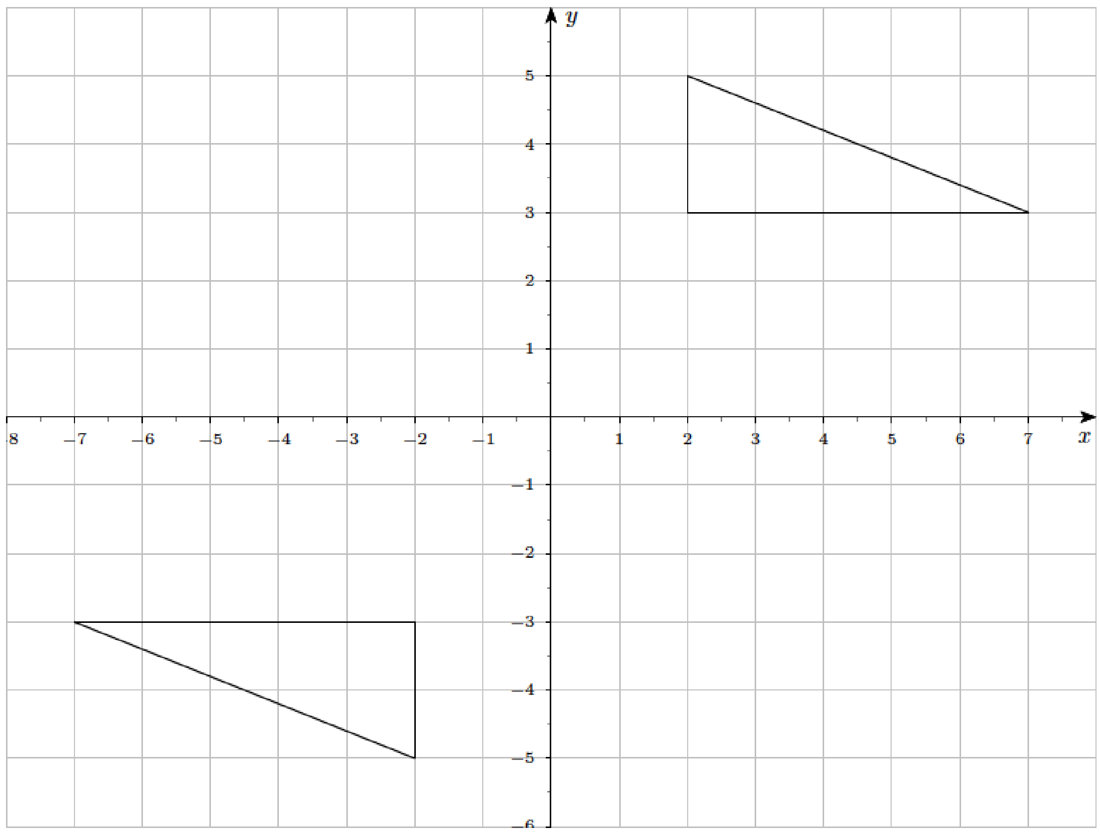
b. Can Figure A be mapped onto Figure B using only reflection? Explain. Use drawings as needed in your explanation.

3. Use the graphs below to answer parts (a) and (b).

   a. Reflect $\triangle XYZ$ over the horizontal line (parallel to the x-axis) through point $(0,1)$. Label the reflected image with $X'Y'Z'$.



b. One triangle in the diagram below can be mapped onto the other using two reflections. Identify the lines of reflection that would map one onto the other. Can you map one triangle onto the other using just one basic rigid motion? If so, explain.



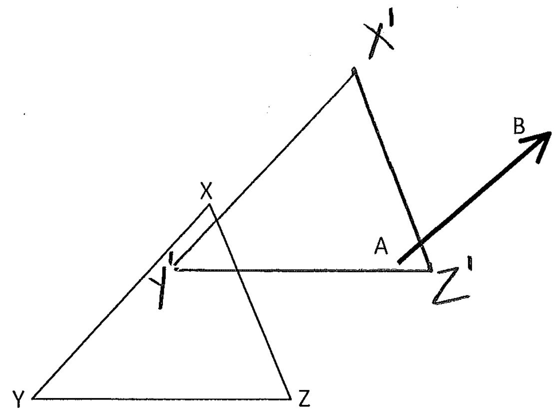
A Progression Toward Mastery					
Assessment Task Item		**STEP 1** **Missing or incorrect answer and little evidence of reasoning or application of mathematics to solve the problem.**	**STEP 2** **Missing or incorrect answer but evidence of some reasoning or application of mathematics to solve the problem.**	**STEP 3** **A correct answer with some evidence of reasoning or application of mathematics to solve the problem, <u>or</u> an incorrect answer with substantial evidence of solid reasoning or application of mathematics to solve the problem.**	**STEP 4** **A correct answer supported by substantial evidence of solid reasoning or application of mathematics to solve the problem.**
1	**a** **8.G.A.1**	Student is unable to respond to the question or leaves item blank. Student enlarges or shrinks image. Student reflects or rotates image.	Student translates along a vector. Student uses a different vector than what was given. Student shortens or lengthens given vector. Student does not label image or labels image incorrectly.	Student translates correctly along vector. Student does not label image or labels image incorrectly.	Student translates correctly along vector and labels image correctly.
	b **8.G.A.1**	Student is unable to respond to the question or leaves item blank. Student enlarges or shrinks image. Student translates or rotates the image.	Student reflects across line, but may reflect across a different line than what is given. Student does not label image or labels image incorrectly. The orientation of the image may be incorrect.	Student reflects correctly across line. Student does not label image or labels image incorrectly.	Student reflects correctly across line and labels image correctly.
	c **8.G.A.1**	Student is unable to respond to the question or leaves item blank. Student translates the triangle to the correct quadrant. Student reflects the triangle to the correct quadrant.	Student rotates about the point $(1,0)$. Student rotates the triangle counter-clockwise $90°$. Student rotates more or less than $90°$. Student does not label image or labels image incorrectly.	Student rotates about the point $(1,0)$ clockwise $90°$. Student does not label image or labels image incorrectly.	Student rotates about the point $(1,0)$ clockwise $90°$ and labels image correctly.
2	**a** **8.G.A.1**	Student answers with yes or no only. Student is unable to give any explanation (pictorially or written).	Student answers with yes or no. Student shows some reasoning (pictorially or written) to solve the problem. Student shows no	Student answers correctly with no. Student uses a pictorial explanation only as evidence of reasoning. Some evidence of	Student answers correctly with no and uses mathematical vocabulary in explanation. Student may use pictorial

			application of mathematics to solve the problem.	mathematical reasoning is evident in explanation. Student does not use mathematical vocabulary in explanation.	explanation to enhance mathematical explanation.
	b **8.G.A.1**	Student answers with yes or no only. Student is unable to give any explanation (pictorially or written).	Student answers with yes or no. Student shows some reasoning (pictorially or written) to solve the problem. Student shows no application of mathematics to solve the problem.	Student answers correctly with no. Student uses a pictorial explanation only as evidence of reasoning. Some evidence of mathematical reasoning is evident in explanation. Student does not use mathematical vocabulary in explanation.	Student answers correctly with no and uses mathematical vocabulary in explanation. Student may use pictorial explanation to enhance mathematical explanation.
3	**a** **8.G.A.1**	Student is unable to respond to the question or leaves item blank. Student shows no reasoning or application of mathematics to solve the problem.	Student reflects triangle across any line other than the line $y = 1$. The orientation of the triangle may or may not be correct. Student may or may not label the triangle correctly.	Student reflects triangle across the line $y = 1$. The orientation of the triangle is correct. Student may or may not label the triangle correctly.	Student reflects triangle across the line $y = 1$, and the orientation of the triangle is correct. Student labels the triangle correctly.
	b **8.G.A.1**	Student is unable to respond to the questions or leaves items blank. Student answers with yes or no only. Student may or may not identify the lines of reflection. No evidence of mathematical reasoning is used in written explanation.	Student answers with yes or no. Student may or may not identify the lines of reflection. Student identifies a rotation as the rigid motion. Student may or may not identify the degree of rotation or the center of rotation. Some evidence of mathematical reasoning is used in written explanation.	Student answers correctly with yes. Student identifies the lines of reflection. Student identifies a rotation as the rigid motion. Student identifies the degree of rotation. Student may or may not identify the center of rotation. Some evidence or mathematical reasoning is used in written explanation.	Student answers correctly with yes. Student correctly identifies the lines of reflection as $y = 0$, then $x = 0$ OR as $x = 0$, then $y = 0$. Student identifies a rotation as the rigid motion. Student identifies the degree of rotation as 180. Student identifies the center of rotation as the origin and substantial evidence of mathematical reasoning is used in written explanation.

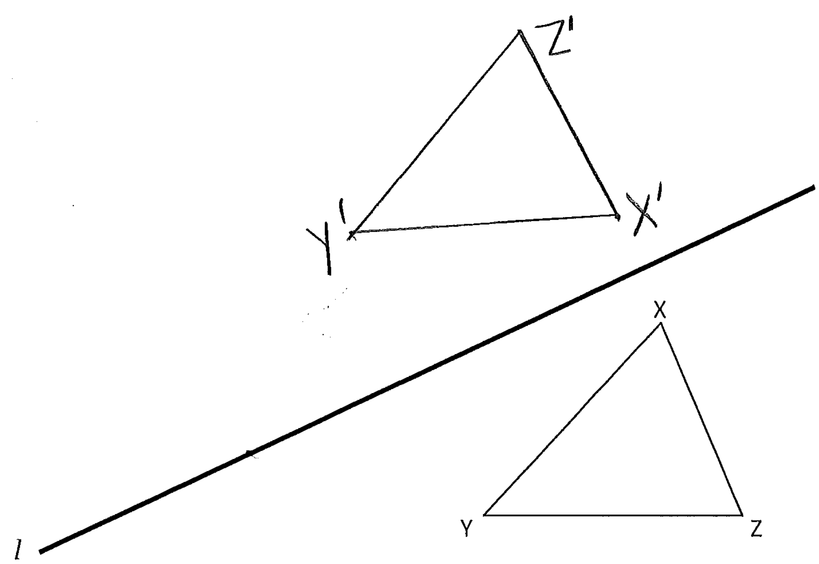
Name ______________________________ Date ________________

1.

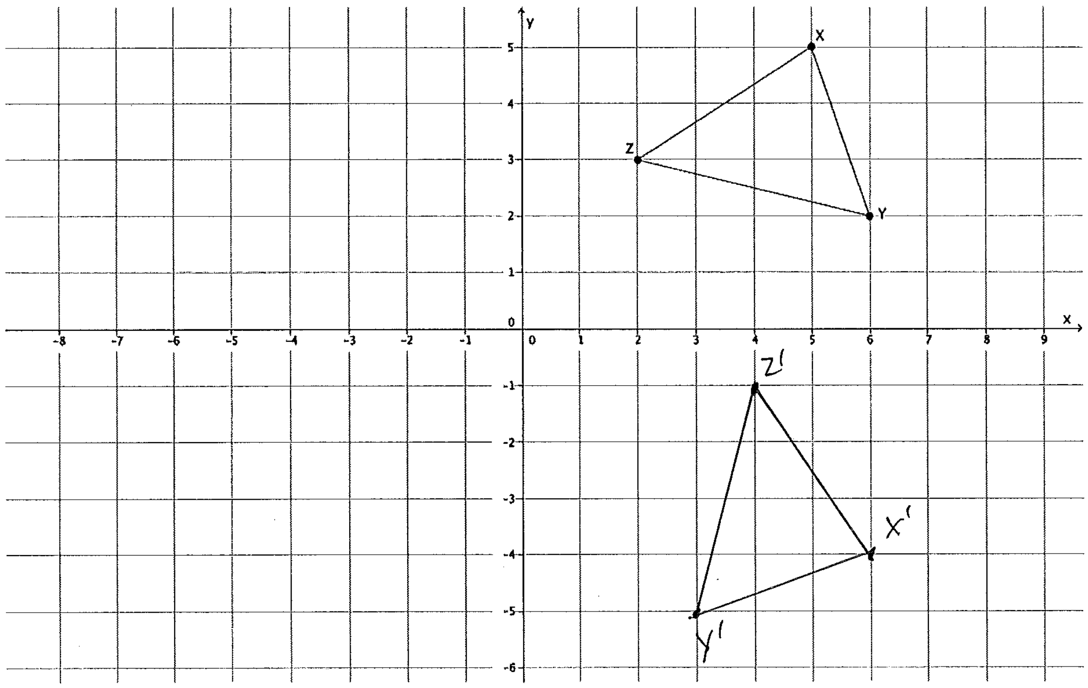
a. Translate ΔXYZ along $\overrightarrow{AB}$. Label the image of the triangle with X', Y', and Z'.



b. Reflect ΔXYZ across the line of reflection, l. Label the image of the triangle with X', Y', and Z'.

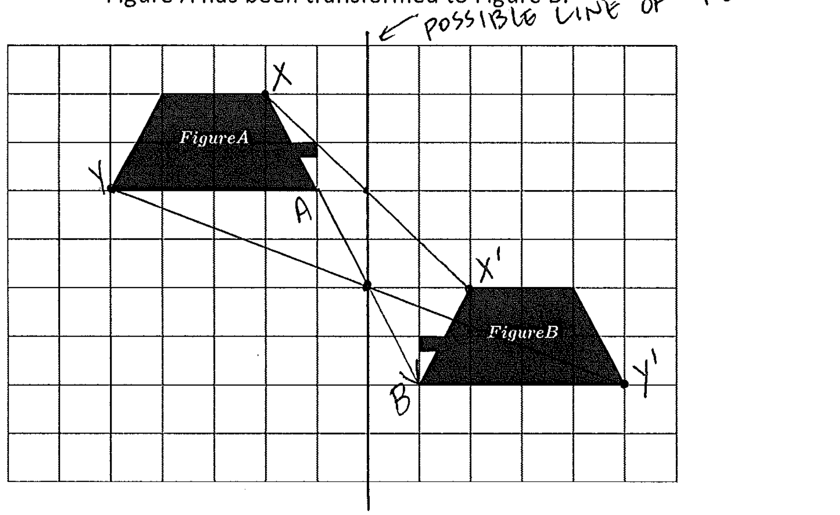


c. Rotate $\triangle XYZ$ around the point $(1,0)$ clockwise $90°$. Label the image of the triangle with X', Y', and Z'.



2. Use the picture below to answer the questions.

Figure A has been transformed to Figure B.



a. Can Figure A be mapped onto Figure B using only translation? Explain. Use drawings as needed in your explanation.

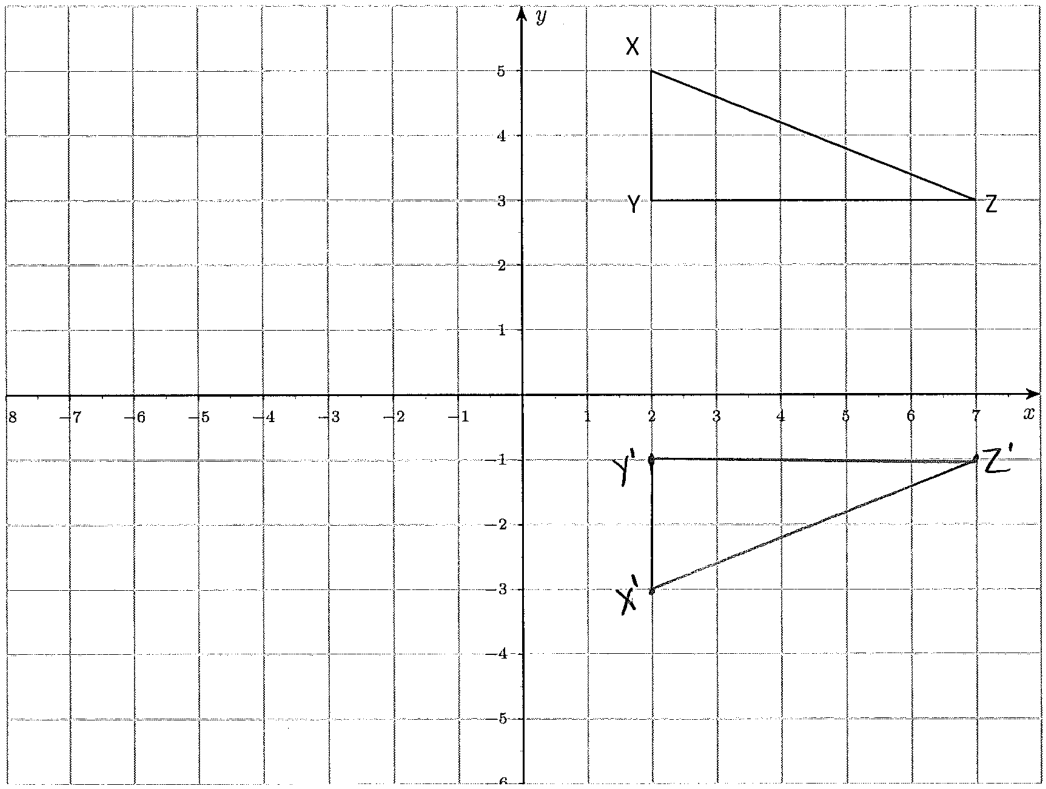
NO, IF I TRANSLATE ALONG VECTOR $\overrightarrow{AB}$ I CAN GET THE LOWER POINT OF FIGURE A TO MAP ONTO THE LOWER LEFT POINT OF FIGURE B (ONE PAIR OF CORRESPONDING POINTS) BUT NO OTHER POINTS OF THE FIGURES COINCIDE.)

b. Can Figure A be mapped onto Figure B using only reflection? Explain. Use drawings as needed in your explanation.

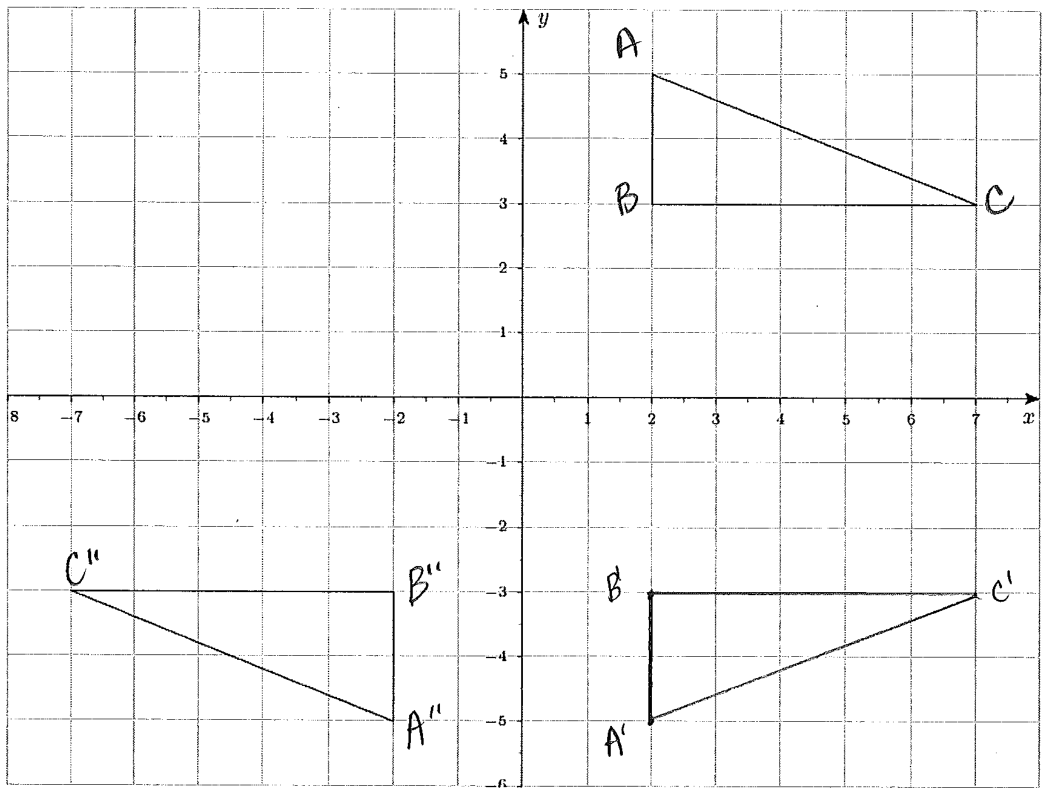
NO, WHEN I CONNECT A POINT OF FIGURE A TO ITS IMAGE ON FIGURE B, THE LINE OF REFLECTION SHOULD BISECT THE SEGMENT. WHEN I CONNECT MIDPOINTS OF $\overline{XX'}$ & $\overline{YY'}$ I GET A POSSIBLE LINE OF REFLECTION, BUT WHEN I CHECK, FIGURE A DOES NOT MAP ONTO FIGURE B.

3. Use the graphs below to answer parts (a) and (b).

 a. Reflect $\triangle XYZ$ over the horizontal line (parallel to the x-axis) through point $(0,1)$. Label the reflected image with $X'Y'Z'$.



b. One triangle in the diagram below can be mapped onto the other using two reflections. Identify the lines of reflection that would map one onto the other. Can you map one triangle onto the other using just one basic rigid motion? If so, explain.



A REFLECTION ACROSS THE x-AXIS MAPS ΔABC TO ΔA'B'C' AND A REFLECTION ACROSS THE Y-AXIS MAPS ΔA'B'C' TO ΔA"B"C".

SINCE AB || A"B", BC || B"C", AND AC || A"C" AND THE LENGTHS AB = A"B", BC = B"C", AC = A"C", THEN A 180° ROTATION ABOUT THE ORIGIN WILL MAP ΔABC TO ΔA"B"C".

Topic C:

Congruence and Angle Relationships

8.G.A.2, 8.G.A.5

Focus Standard:	8.G.A.2	Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.
	8.G.A.5	Use informal arguments to establish facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles. *For example, arrange three copies of the same triangle so that the sum of the three angles appears to form a line, and give an argument in terms of transversals why this is so.*
Instructional Days:	4	
Lesson 11:	Definition of Congruence and Some Basic Properties (S)[1]	
Lesson 12:	Angles Associated with Parallel Lines (E)	
Lesson 13:	Angle Sum of a Triangle (E)	
Lesson 14:	More on the Angles of a Triangle (S)	

Topic C finishes the work of **8.G.A.2** by introducing the concept of congruence as mapping one figure onto another using a sequence of rigid motions. Lesson 11 defines congruence in terms of a sequence of the basic rigid motions, i.e., translations, reflections, and rotations. Students learn the fundamental assumptions that are made about the basic rigid motions that will serve as the basis of all geometric investigations.

The concept of congruence and basic rigid motions are used to determine which angles of parallel lines are equal in measure. In Lesson 12, students show why corresponding angles are congruent using translation and why alternate interior angles are congruent using rotation. In Lessons 13 and 14, the knowledge of rigid motions and angle relationships is put to use to develop informal arguments to show that the sum of the degrees of interior angles of a triangle is $180°$. Students are presented with three such arguments as the importance of the theorem justifies the multiple perspectives. Students also take note of a related fact about the exterior angles of triangles.

[1] Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S-**Socratic Lesson

Lesson 11: Definition of Congruence and Some Basic Properties

Student Outcomes

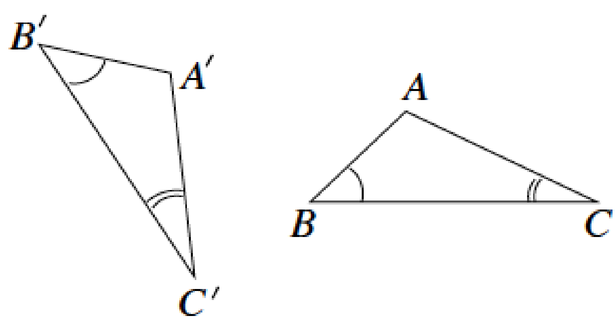
- Students know the definition of congruence and related notation, i.e., $\cong$. Students know that to prove two figures are congruent, there must be a sequence of rigid motions that maps one figure onto the other.
- Students know that the basic properties of congruence are similar to the properties for all three rigid motions (translations, rotations, and reflections).

Classwork

Example 1 (5 minutes)

MP.6

- Sequencing basic rigid motions has been practiced throughout the lessons of Topic B in this module because, in general, the sequence of (a finite number of) basic rigid motions is called a congruence. A geometric figure S is said to be congruent to another geometric figure S' if there is a sequence of rigid motions that maps S to S', i.e., $Congruence(S) = S'$. The notation related to congruence is the symbol $\cong$. When two figures are congruent, like S and S', we can write: $S \cong S'$.
- We want to describe the sequence of rigid motions that demonstrates the two triangles shown below are congruent, i.e., $\triangle ABC \cong \triangle A'B'C'$.



> *Note to Teacher:*
>
> Demonstrate, or have students demonstrate, the rigid motions as they work through the sequence.

- What rigid motion will bring the two triangles together? That is, which motion would bring together at least one pair of corresponding points (vertices)? Be specific.
  - *Translate $\triangle A'B'C'$ along vector $\overrightarrow{A'A}$.*
- What rigid motion would bring together one pair of sides? Be specific.
  - *Rotate d degrees around center A.*
- After these two rigid motions, we have shown that $\triangle ABC \cong \triangle A'B'C'$ through the sequence of a translation followed by a rotation. Notice that only two rigid motions were needed for this sequence. A sequence to demonstrate congruence can be made up of any combination of the basic rigid motions using all three or even just one.

- The concept of *congruence* between two geometric figures is one of the cornerstones of geometry. Congruence is now realized as "a sequence of basic rigid motions that maps one figure onto another."
- Recall the first question raised in this module, "Why move things around?" Now, a complete answer can be given in terms of congruence.

> *Note to Teacher:*
>
> The preceding definition of congruence is meant to replace the existing "same size and same shape" definition.

Example 2 (10 minutes)

- It is said that S is congruent to S' if there is a congruence so that $Congruence(S) = S'$. This leaves open the possibility that, although S is congruent to S', the figure S' may not be congruent to S.

Ask students:

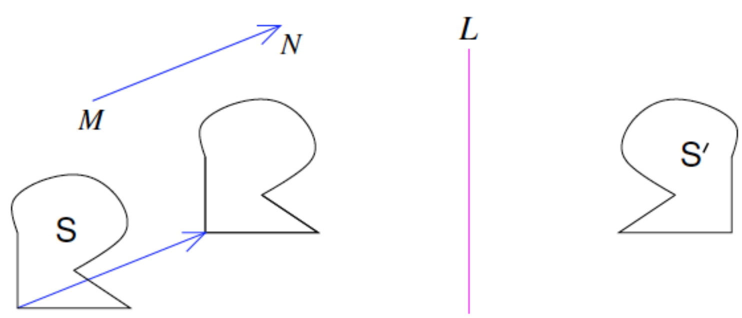
- If there is a $Congruence_1$ so that $Congruence_1(S) = S'$, do we know that there will also be a $Congruence_2$ so that $Congruence_2(S') = S$?

Make sure students understand the question.

- Can you say for certain that if they begin by mapping figure 1 onto figure 2, they can also map figure 2 onto figure 1?
  - *Students will likely say yes, but without proof, further work is necessary.*
- Assume that the congruence is a sequence of a translation followed by a reflection where there is a translation along a given vector $\overrightarrow{MN}$, and there is the reflection across line L. Let S be the figure on the left below, and let S' be the figure on the right below. Then, the equation

$$Congruence(S) = Translation\ (S)\ followed\ by\ the\ Reflection = S' \qquad (7)$$

says that if we trace S in red on a transparency, then translate the transparency along $\overrightarrow{MN}$ and flip it across L, we get the figure to coincide completely with S'.



- Now keeping in mind what we know about how to undo transformations in general, it is obvious how to get a congruence to map S' to S. Namely, tracing the figure S' in red, flip the transparency across L so the red figure arrives at the figure in the middle, and then translate the figure along vector $\overrightarrow{NM}$. (Note the change in direction of the vector from $\overrightarrow{MN}$.) The red figure now coincides completely with S. The sequence of the reflection across L followed by the translation along vector $\overrightarrow{NM}$ achieves the congruence.

- The general argument is that if there is a $Congruence_1$ so that $Congruence_1(S) = S'$, then there will also be a $Congruence_2$ so that $Congruence_2(S') = S$ is similar. The only additional comment to complete the picture is that, in addition to
  1. The sequence required to show that $Congruence_2$ followed by $Congruence_1$ is equal to $Congruence_1$ followed by $Congruence_2$,
  2. A reflection is undone by a reflection across the same line.

  We also have to draw upon the sequence of rotations that maps a figure onto itself to be certain that each of the three basic rigid motions can be undone by another basic rigid motion.
- In summary, if a figure S is congruent to another figure S', then S' is also congruent to S. In symbols $S \cong S'$. It does not matter whether S or S' comes first.

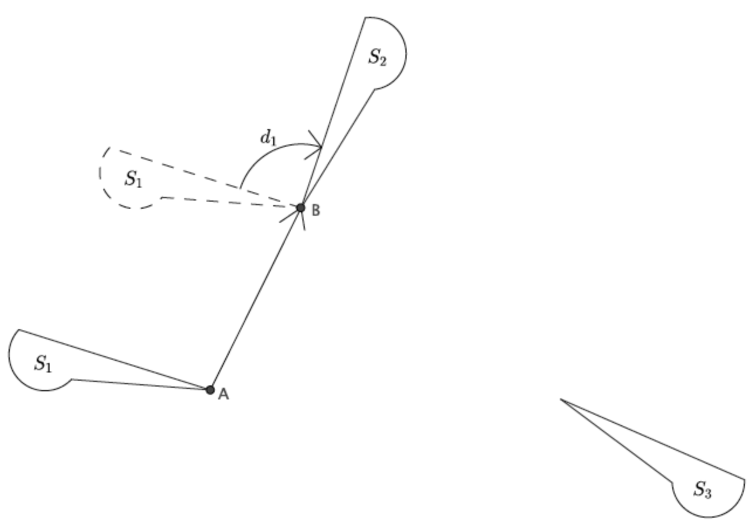
Exercise 1 (10 minutes)

Students work on Exercise 1 in pairs. Students will likely need some guidance with part (a) of Exercise 1. Provide support, and then allow them to work with a partner to complete parts (b) and (c).

Exercise 1

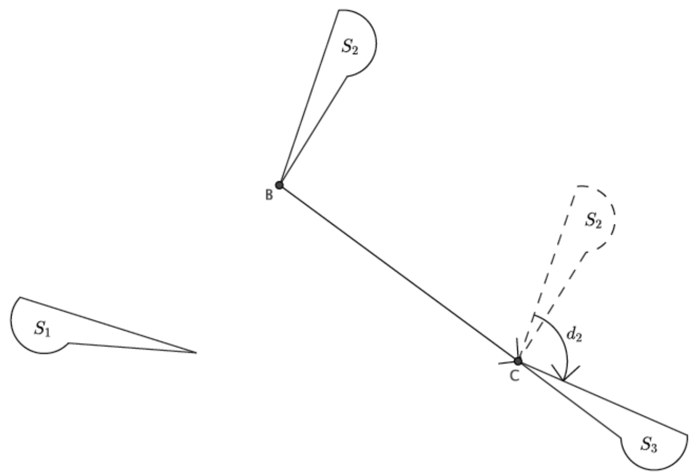
a. **Describe the sequence of basic rigid motions that shows $S_1 \cong S_2$.**

Let there be the translation along vector $\overrightarrow{AB}$. Let there be a rotation around point B, d_1 degrees. Let there be a reflection across the longest side of the figure so that S_1 maps onto S_2. Then, the $Translation(S_1)$ followed by the rotation followed by the reflection $= S_2$.



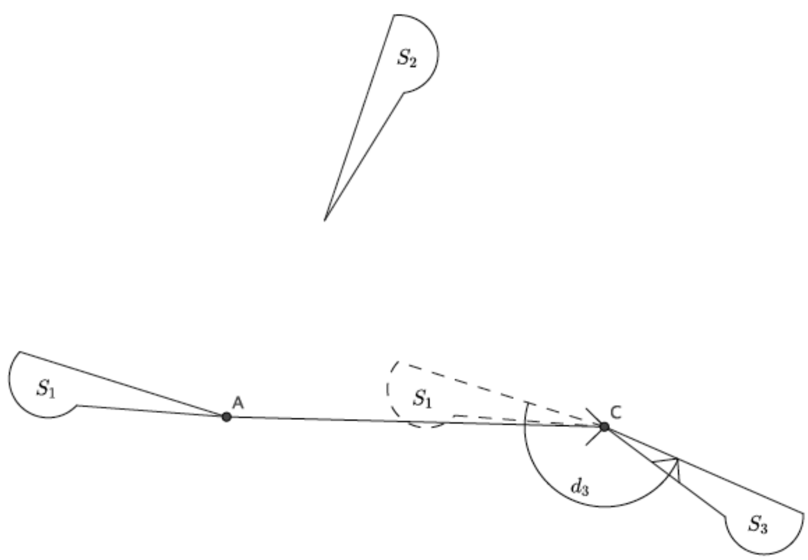
b. **Describe the sequence of basic rigid motions that shows $S_2 \cong S_3$.**

Let there be a translation along vector $\overrightarrow{BC}$. Let there be a rotation around point C, d_2 degrees so that S_2 maps onto S_3. Then, the $Translation(S_2)$ followed by the rotation $= S_3$.



c. **Describe a sequence of basic rigid motions that shows $S_1 \cong S_3$.**

Sample student response: Let there be a translation along vector $\overrightarrow{AC}$. Let there be a rotation around point C, d_3 degrees. Let there be the reflection across the longest side of the figure so that S_1 maps onto S_3. Then, the $Translation\ (S_1)$ followed by the $Rotation$ followed by the $Reflection = S_3$. Because we found a congruence that maps S_1 to S_2; that is, $S_1 \cong S_2$, and another congruence that maps S_2 to S_3; that is, $S_2 \cong S_3$, then we know for certain that $S_1 \cong S_3$.



Discussion and Exercise 2 (10 minutes)

Ask the students if they really need to do all of the work they did in part (c) of Exercise 1.

Students should say no. The reason we do not need to do all of that work is because we already know that translations, rotations, and reflections preserve angle measures and lengths of segments. For that reason, if we know that $S_1 \cong S_2$ and $S_2 \cong S_3$, then $S_1 \cong S_3$.

Ask students to help summarize the basic properties of all three basic rigid motions.

Elicit from students the following three statements:

MP.8

- *A basic rigid motion maps a line to a line, a ray to a ray, a segment to a segment, and an angle to an angle.*
- *A basic rigid motion preserves lengths of segments.*
- *A basic rigid motion preserves measures of angles.*

Ask students if they believe these same facts are true for *sequences* of basic rigid motions.

- Specifically, under a sequence of a translation followed by a rotation: If there is a translation along a vector $\overrightarrow{AB}$, and there is a rotation of d degrees around a center O, will a figure that is sequenced remain rigid? That is, will lengths and angles be preserved? Will lines remain lines, segments remain segments, etc.?
  - *Students should say that yes, sequences of rigid motions also have the same basic properties of rigid motions in general.*

If students are unconvinced, have them complete Exercise 2; then, discuss again.

- Given that sequences enjoy the same basic properties of basic rigid motions, we can state three basic properties of congruences:

MP.8

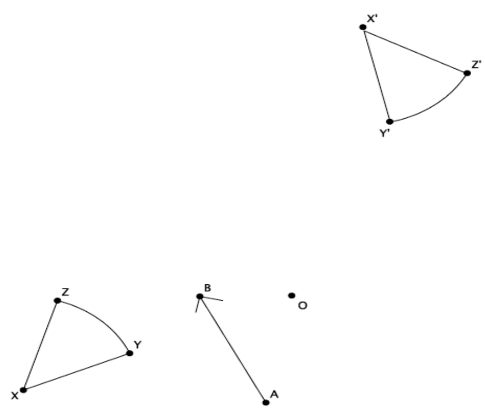
(Congruence 1) A congruence maps a line to a line, a ray to a ray, a segment to a segment, and an angle to an angle.

(Congruence 2) A congruence preserves lengths of segments.

(Congruence 3) A congruence preserves measures of angles.

Exercise 2

Perform the sequence of a translation followed by a rotation of Figure XYZ, where T is a translation along a vector $\overrightarrow{AB}$, and R is a rotation of d degrees (you choose d) around a center O. Label the transformed figure $X'Y'Z'$. Will $XYZ \cong X'Y'Z'$?



After this exercise, students should be convinced that a sequence of rigid motions maintains the basic properties of individual basic rigid motions. They should clearly see that the figure XYZ that they traced in red is exactly the same, i.e., congruent, to the transformed figure $X'Y'Z'$.

Closing (5 minutes)

Summarize, or have students summarize, the lesson.

- We now have a definition for congruence, i.e., a sequence of basic rigid motions.
- We now have new notation for congruence, $\cong$.
- The properties that apply to the individual basic rigid motions also apply to congruences.

Lesson Summary

Given that sequences enjoy the same basic properties of basic rigid motions, we can state three basic properties of congruences:

(Congruence 1) A congruence maps a line to a line, a ray to a ray, a segment to a segment, and an angle to an angle.

(Congruence 2) A congruence preserves lengths of segments.

(Congruence 3) A congruence preserves measures of angles.

The notation used for congruence is $\cong$.

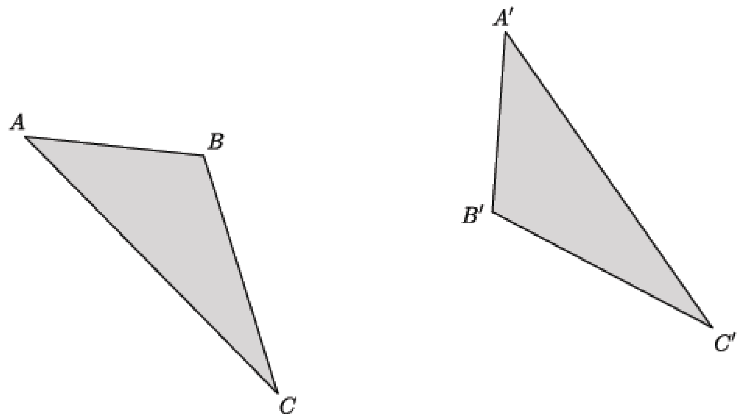
Exit Ticket (5 minutes)

Name ______________________________ Date ______________

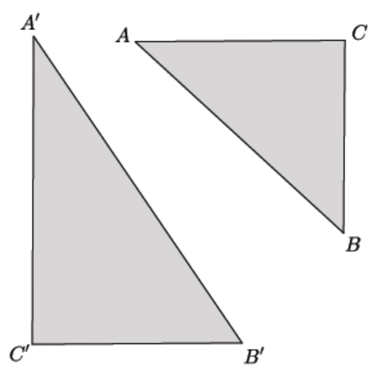
Lesson 11: Definition of Congruence and Some Basic Properties

Exit Ticket

1. Is $\triangle ABC \cong \triangle A'B'C'$? If so, describe a sequence of rigid motions that proves they are congruent. If not, explain how you know.

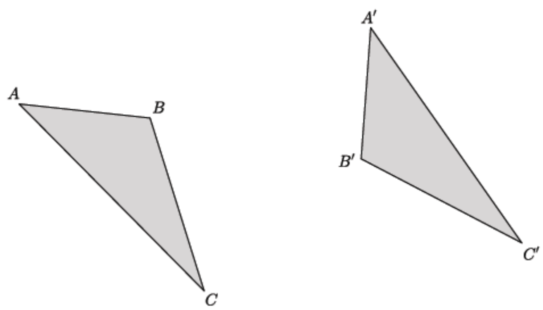


2. Is $\triangle ABC \cong \triangle A'B'C'$? If so, describe a sequence of rigid motions that proves they are congruent. If not, explain how you know.



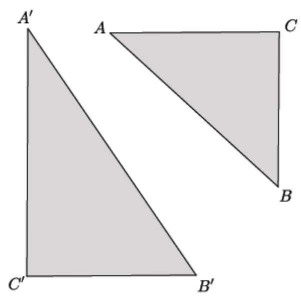
Exit Ticket Sample Solutions

1. **Is $\triangle ABC \cong \triangle A'B'C'$? If so, describe a sequence of rigid motions that proves they are congruent. If not, explain how you know.**



Sample student response: Yes, $\triangle ABC \cong \triangle A'B'C'$. Translate $\triangle A'B'C'$ along vector $\overrightarrow{A'A}$. Rotate $\triangle A'B'C'$ around center A, d degrees until side $A'C'$ coincides with side AC. Then, reflect across line AC.

2. **Is $\triangle ABC \cong \triangle A'B'C'$? If so, describe a sequence of rigid motions that proves they are congruent. If not, explain how you know.**

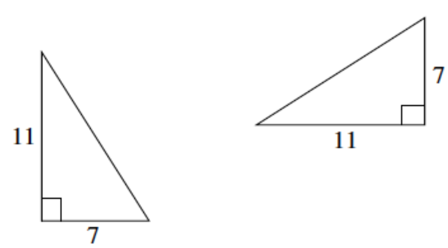


Sample student response: No, $\triangle ABC$ is not congruent to $\triangle A'B'C'$. Though I could translate and rotate to get some of the parts from each triangle to coincide, there is no rigid motion that would map side $A'C'$ to AC or side $A'B'$ to side AB, because they are different lengths. Basic rigid motions preserve length, so no sequence would map $\triangle A'B'C'$ onto $\triangle ABC$.

Problem Set Sample Solutions

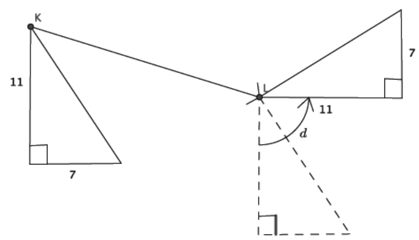
Students practice describing sequences of rigid motions that produce a congruence.

1. **Given two right triangles with lengths shown below, is there one basic rigid motion that maps one to the other? Explain.**



Yes, a rotation of d degrees around some center would map one triangle onto the other. The rotation would map the right angle to the right angle; the sides of length 7 and length 11 would then coincide.

2. **Are the two right triangles shown below congruent? If so, describe a congruence that would map one triangle onto the other.**

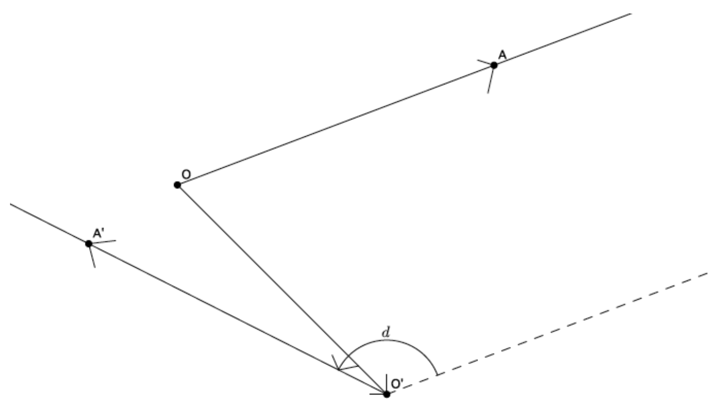


Sample student response: Yes, they are congruent. Let there be the translation along vector $\overrightarrow{KL}$. Let there be the rotation around point L, d degrees. Then, the translation followed by the rotation will map the triangle on the left to the triangle on the right.

3. **Given two rays, $\overrightarrow{OA}$ and $\overrightarrow{O'A'}$:**

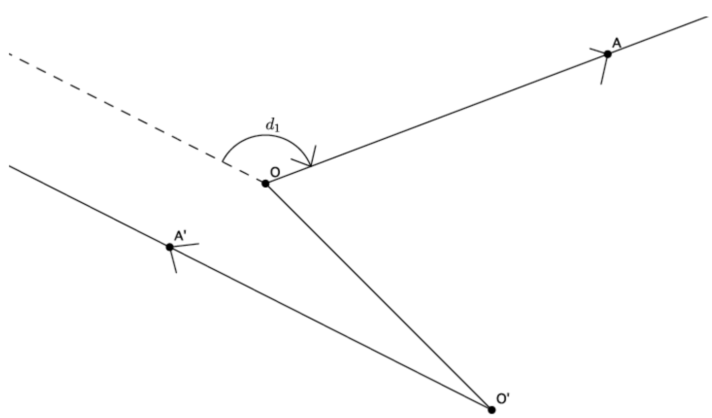
   a. **Describe a congruence that maps $\overrightarrow{OA}$ to $\overrightarrow{O'A'}$.**

   Sample student response: Let there be the translation along vector $\overrightarrow{OO'}$. Let there be the rotation around point O', d degrees. Then, the $Translation(\overrightarrow{OA})$ followed by the $Rotation = \overrightarrow{O'A'}$.



   b. **Describe a congruence that maps $\overrightarrow{O'A'}$ to $\overrightarrow{OA}$.**

   Sample student response: Let there be the translation along vector $\overrightarrow{O'O}$. Let there be the rotation around point O, d_1 degrees. Then, the $Translation\ (\overrightarrow{O'A'})$ followed by the $Rotation = \overrightarrow{OA}$.



Lesson 12: Angles Associated with Parallel Lines

Student Outcomes

- Students know that corresponding angles, alternate interior angles, and alternate exterior angles of parallel lines are equal. Students know that when these pairs of angles are equal, then lines are parallel.
- Students know that corresponding angles of parallel lines are equal because of properties related to translation. Students know that alternate interior angles of parallel lines are equal because of properties related to rotation.
- Students present informal arguments to draw conclusions about angles formed when parallel lines are cut by a transversal.

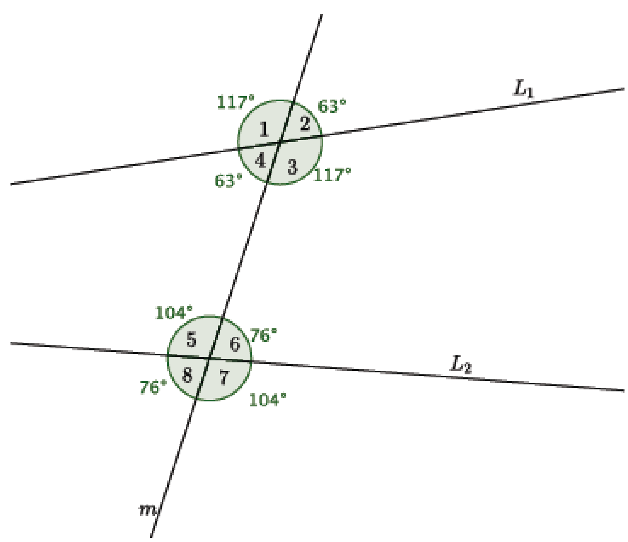
Classwork

Exploratory Challenge 1 (7 minutes)

Students complete the Exploratory Challenge individually or in pairs. Students will investigate the properties of angles formed by two non-parallel lines cut by a transversal.

Exploratory Challenge 1

In the figure below, L_1 is not parallel to L_2, and m is a transversal. Use a protractor to measure angles 1–8. Which, if any, are equal? Explain why. (Use your transparency if needed.)



$\angle 1 = \angle 3$, $\angle 2 = \angle 4$, $\angle 5 = \angle 7$, and $\angle 6 = \angle 8$. The pairs of angles listed are equal because they are vertical angles. Vertical angles are always equal because a rotation of $180°$ around the vertex of the angle will map it to its opposite angle.

Discussion (5 minutes)

Discuss what the students noticed about the angles in the first diagram with non-parallel lines. Ask students the following questions during the discussion.

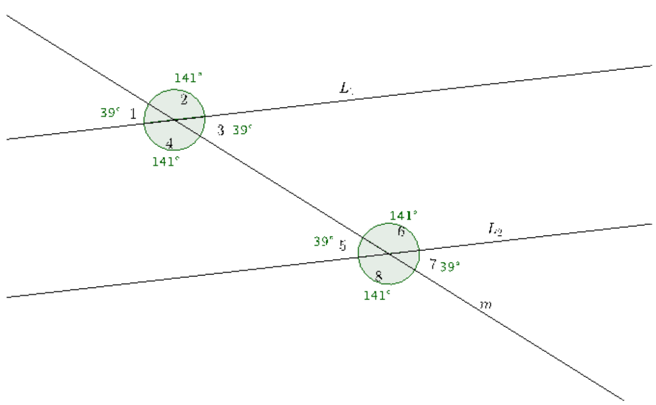
- What did you notice about the pairs of angles in the first diagram when the lines, L_1 and L_2, were not parallel?
  - $\angle 1 = \angle 3$, $\angle 2 = \angle 4$, $\angle 5 = \angle 7$, *and* $\angle 6 = \angle 8$. *Vertical angles were equal in measure.*
- Why are vertical angles equal in measure?
  - *We can rotate the angle around its vertex* $180°$, *and it maps onto its opposite angle. Since rotations are angle-preserving, it means that the angles are equal in measure.*
- Angles that are on the same side of the transversal in corresponding positions (above each of L_1 and L_2 or below each of L_1 and L_2) are called **corresponding angles**. Name a pair of corresponding angles in the diagram. (Note to teacher: Have students name all pairs of corresponding angles from the diagram, one pair at a time.)
  - $\angle 1$ *and* $\angle 5$, $\angle 4$ *and* $\angle 8$, $\angle 2$ *and* $\angle 6$, $\angle 3$ *and* $\angle 7$
- When angles are on opposite sides of the transversal and between (inside) the lines L_1 and L_2, they are called **alternate interior angles**. Name a pair of alternate interior angles. (Note to teacher: Have students name both pairs of alternate interior angles from the diagram, one pair at a time.)
  - $\angle 4$ *and* $\angle 6$, $\angle 3$ *and* $\angle 5$
- When angles are on opposite sides of the transversal and outside of the parallel lines (above L_1 and below L_2), they are called **alternate exterior angles**. Name a pair of alternate exterior angles. (Note to teacher: Have students name both pairs of alternate exterior angles from the diagram, one pair at a time.)
  - $\angle 1$ *and* $\angle 7$, $\angle 2$ *and* $\angle 8$

Exploratory Challenge 2 (9 minutes)

Students complete the Exploratory Challenge individually or in pairs. Students will investigate the properties of angles formed by two parallel lines cut by a transversal.

Exploratory Challenge 2

In the figure below, $L_1 \parallel L_2$, and m is a transversal. Use a protractor to measure angles 1–8. List the angles that are equal in measure.



$\angle 1 = \angle 3 = \angle 5 = \angle 7$ *and* $\angle 2 = \angle 4 = \angle 6 = \angle 8$

a. **What did you notice about the measures of $\angle 1$ and $\angle 5$? Why do you think this is so? (Use your transparency if needed.)**

$\angle 1$ and $\angle 5$ are equal in measure. We can translate $\angle 1$ along a vector on line m so that the vertex of $\angle 1$ maps onto the vertex of $\angle 5$. Translations are angle-preserving, so the two angles will coincide.

b. **What did you notice about the measures of $\angle 3$ and $\angle 7$? Why do you think this is so? (Use your transparency if needed.) Are there any other pairs of angles with this same relationship? If so, list them.**

$\angle 3$ and $\angle 7$ are equal in measure. We can translate $\angle 3$ along a vector on line m so that the vertex of $\angle 3$ maps onto the vertex of $\angle 7$. Translations are angle-preserving, so the two angles will coincide. Other pairs of angles with this same relationship are $\angle 4$ and $\angle 8$, and $\angle 2$ and $\angle 6$.

c. **What did you notice about the measures of $\angle 4$ and $\angle 6$? Why do you think this is so? (Use your transparency if needed.) Is there another pair of angles with this same relationship?**

The measures of $\angle 4$ and $\angle 6$ are equal. A rotation of $180°$ around a center would map $\angle 4$ to $\angle 6$. Rotations are angle-preserving, so we know that $\angle 4$ and $\angle 6$ are equal. $\angle 3$ and $\angle 5$ have the same relationship.

Discussion (15 minutes)

Discuss what the students noticed about the angles in the second diagram with parallel lines. Ask students the following questions during the discussion.

- We are going to discuss what you observed when you measured and compared angles in Exploratory Challenge 2. What we will do is make some informal arguments to prove some of the things you noticed. Each time you answer "why," "how do you know," or "explain," you are making an informal argument.
- Were the vertical angles in Exploratory Challenge 2 equal like they were in Exploratory Challenge 1? Why?
  - *Yes, for the same reason they were equal in the first diagram, rotation.*
- What other angles were equal in the second diagram when the lines L_1 and L_2 were parallel?
  - $\angle 1 = \angle 3 = \angle 5 = \angle 7$ *and* $\angle 2 = \angle 4 = \angle 6 = \angle 8$
- Let's look at just $\angle 1$ and $\angle 5$. What kind of angles are these, and how do you know?
  - *They are corresponding angles because they are on the same side of the transversal and in corresponding positions (i.e., above each of L_1 and L_2 or below each of L_1 and L_2).*
- We have already said that these two angles are equal in measure. Who can explain why this is so?
  - *Translation along a vector in line m will map $\angle 1$ onto $\angle 5$. Translations preserve degrees of angles, so the two angles are equal in measure.*

Note to teacher: Insert and label a point at the vertex of each angle, points A and B, for example. Draw the vector between the two points. Trace one of the angles on a transparency, and demonstrate to students that a translation along a vector of exactly length AB would map one angle onto the other. Further, if there is a translation along vector $\overrightarrow{AB}$, then $Translation(L_1) = L_2$. The vector $\overrightarrow{AB}$ on line m is also translated, but remains on line m. Therefore, all of the angles formed by the intersection of L_1 and m are translated along $\overrightarrow{AB}$ according to the translation. That is why the corresponding angles are equal in measure, i.e., $Translation(\angle 1) = \angle 5$.

- What did you notice about $\angle 3$ and $\angle 7$?
  - *These two angles were also equal in measure for the same reason as $\angle 1$ and $\angle 5$.*
- What other pairs of corresponding angles are in the diagram?
  - *$\angle 4$ and $\angle 8$, and $\angle 2$ and $\angle 6$*

- In Exploratory Challenge 1, the pairs of corresponding angles we named were not equal in measure. Given the information provided about each diagram, can you think of why this is so?
  - *In the first diagram, the lines L_1 and L_2 were not parallel. A translation of one of the angles would not map onto the other angle.*
- Are $\angle 4$ and $\angle 6$ corresponding angles? If not, why not?
  - *No, they are not on the same side of the transversal in corresponding locations.*
- What kind of angles are $\angle 4$ and $\angle 6$? How do you know?
  - *$\angle 4$ and $\angle 6$ are alternate interior angles because they are on opposite sides of the transversal and inside the lines L_1 and L_2.*
- We have already said that $\angle 4$ and $\angle 6$ are equal in measure. Why do you think this is so?
  - *You can use rotation to map one of the angles onto the other. Rotations are angle-preserving.*

Note to teacher: Mark the midpoint, point M for example, of the segment between the vertices of the two angles. Trace one of the angles on a transparency and demonstrate to students that a $180°$ rotation around the point M would map one of the angles onto the other. Further, we know that a $180°$ rotation of a line around a point not on the line maps to a parallel line. If we let there be the rotation of $180°$ around point M, then $Rotation(L_1) = L_2$. The rotation of point M maps to itself under rotation, and the line containing M, line m, will also map to itself under rotation. For that reason, the angles formed at the intersection of L_1 and line m, under rotation, will map to angles at the intersection of L_2 and line m, but on the other side of the transversal, specifically, $Rotation(\angle 4) = \angle 6$.

- Name another pair of alternate interior angles.
  - *$\angle 3$ and $\angle 5$*
- In Exploratory Challenge 1, the pairs of alternate interior angles we named were not equal in measure. Given the information provided about each diagram, can you think of why this is so?
  - *In the first diagram, the lines L_1 and L_2 were not parallel. A rotation around a point would not map one angle onto the other angle.*
- Are $\angle 1$ and $\angle 7$ corresponding angles? If not, why not?
  - *No, they are not on the same side of the transversal, so they cannot be corresponding angles.*
- Are $\angle 1$ and $\angle 7$ alternate interior angles? If not, why not?
  - *No, they are not inside of the lines L_1 and L_2, so they cannot be interior angles.*
- What kind of angles are $\angle 1$ and $\angle 7$?
  - *They are alternate exterior angles because they are on opposite sides of the transversal and above L_1 and below L_2.*
- Name another pair of alternate exterior angles.
  - *$\angle 2$ and $\angle 8$*
- These pairs of alternate exterior angles were not equal in measure in Exploratory Challenge 1. Given the information provided about each diagram, can you think of why this is so?
  - *In the first diagram, the lines L_1 and L_2 were not parallel. A rotation of one of the angles would not map onto the other angle.*
- If you know that pairs of corresponding angles, alternate interior angles, and alternate exterior angles are congruent, what do you think is true about the lines?
  - *The lines are parallel when pairs of corresponding angles, alternate interior angles, and alternate exterior angles are congruent.*

State the following theorem and its converse:

- Theorem: When parallel lines are cut by a transversal, then the pairs of corresponding angles are congruent, the pairs of alternate interior angles are congruent, and the pairs of alternate exterior angles are congruent.
- The converse of the theorem states that if you know that corresponding angles are congruent, then you can be sure that the lines cut by a transversal are parallel. How could you phrase the converse of the theorem with respect to other types of angles we have learned?
    - *When alternate interior angles are congruent, then the lines cut by a transversal are parallel.*
    - *When alternate exterior angles are congruent, then the lines cut by a transversal are parallel.*

Closing (4 minutes)

Provide students two minutes to fill in the example portion of the lesson summary in the student materials.

Summarize, or have students summarize, the lesson.

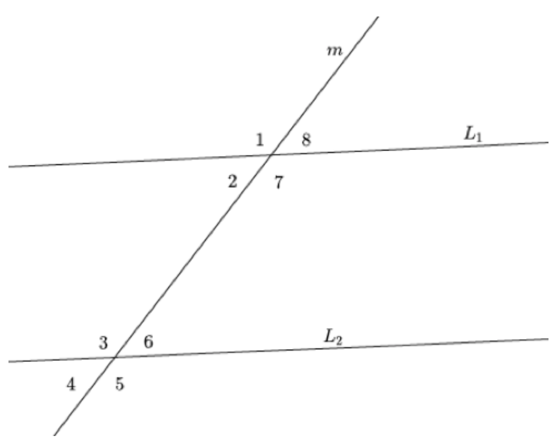
- When a pair of parallel lines are cut by a transversal, then any corresponding angles, any alternate interior angles, and any alternate exterior angles are equal in measure.
- The reason that specific pairs of angles are equal is because of the properties we learned about the basic rigid motions, specifically that they are angle-preserving.
- When a pair of non-parallel lines are cut by a transversal, any corresponding angles, any alternate interior angles, and any alternate exterior angles are not equal.

Lesson Summary

Angles that are on the same side of the transversal in corresponding positions (above each of L_1 and L_2 or below each of L_1 and L_2) are called corresponding angles. For example, $\angle 2$ and $\angle 4$ are corresponding angles.

When angles are on opposite sides of the transversal and between (inside) the lines L_1 and L_2, they are called alternate interior angles. For example, $\angle 3$ and $\angle 7$ are alternate interior angles.

When angles are on opposite sides of the transversal and outside of the parallel lines (above L_1 and below L_2), they are called alternate exterior angles. For example, $\angle 1$ and $\angle 5$ are alternate exterior angles.



When parallel lines are cut by a transversal, any corresponding angles, any alternate interior angles, and any alternate exterior angles are equal in measure. If the lines are not parallel, then the angles are not equal in measure.

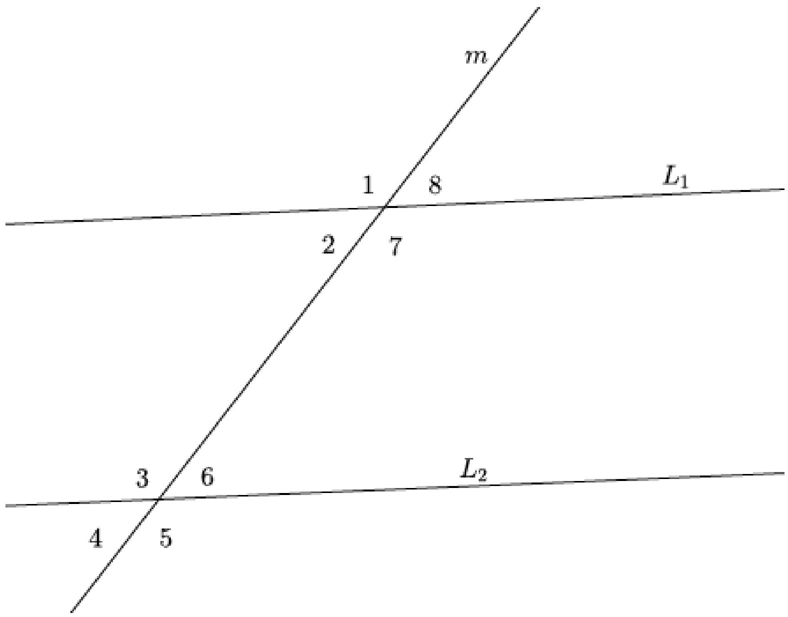
Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 12: Angles Associated with Parallel Lines

Exit Ticket

Use the diagram to answer Questions 1 and 2. In the diagram, lines L_1 and L_2 are intersected by transversal m, forming angles 1–8, as shown.

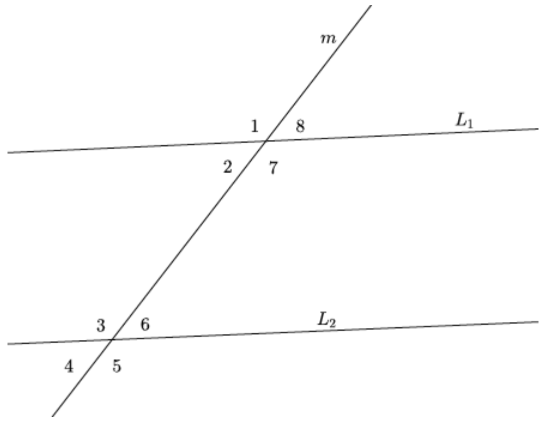


1. If $L_1 \parallel L_2$, what do know about $\angle 2$ and $\angle 6$? Use informal arguments to support your claim.

2. If $L_1 \parallel L_2$, what do know about $\angle 1$ and $\angle 3$? Use informal arguments to support your claim.

Exit Ticket Sample Solutions

Use the diagram to answer Questions 1 and 2. In the diagram, lines L_1 and L_2 are intersected by transversal m, forming angles 1–8, as shown.



1. If $L_1 \parallel L_2$, what do know about $\angle 2$ and $\angle 6$. Use informal arguments to support your claim.

   They are alternate interior angles because they are on opposite sides of the transversal and inside of lines L_1 and L_2. Also, the angles are equal in measure because the lines L_1 and L_2 are parallel. If we rotated angle 2 around the midpoint of the segment between the parallel lines, then it would map onto angle 6.

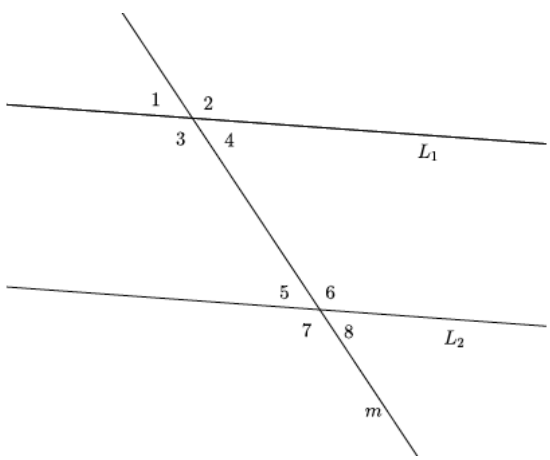
2. If $L_1 \parallel L_2$, what do know about $\angle 1$ and $\angle 3$? Use informal arguments to support your claim.

   They are corresponding angles because they are on the same side of the transversal and above each of lines L_1 and L_2. Also, the angles are equal in measure because the lines L_1 and L_2 are parallel. If we translated angle 1 along a vector (the same length as the segment between the parallel lines), then it would map onto angle 3.

Problem Set Sample Solutions

Students practice identifying corresponding, alternate interior, and alternate exterior angles from a diagram.

Use the diagram below to do Problems 1–6.



1. **Identify all pairs of corresponding angles. Are the pairs of corresponding angles equal in measure? How do you know?**

   $\angle 1$ and $\angle 5$, $\angle 4$ and $\angle 8$, $\angle 2$ and $\angle 6$, $\angle 3$ and $\angle 7$

   There is no information provided about the lines in the diagram being parallel. For that reason, we do not know if the pairs of corresponding angles are equal. If we knew the lines were parallel, we could use translation to map one angle onto another.

2. **Identify all pairs of alternate interior angles. Are the pairs of alternate interior angles equal in measure? How do you know?**

   $\angle 4$ and $\angle 5$, $\angle 3$ and $\angle 6$

   There is no information provided about the lines in the diagram being parallel. For that reason, we do not know if the pairs of alternate interior angles are equal in measure. If the lines were parallel, we could use rotation to show that the pairs of angles would map onto one another proving they are equal in measure.

3. **Use an informal argument to describe why $\angle 1$ and $\angle 8$ are equal in measure if $L_1 \parallel L_2$.**

   The reason that angle 1 and angle 8 are equal in measure when the lines are parallel is because you can rotate around the midpoint of the segment between the parallel lines. A rotation would then map angle 1 onto angle 8, showing that they are congruent and equal in measure.

4. **Assuming $L_1 \parallel L_2$ if the measure of $\angle 4$ is $73°$, what is the measure of $\angle 8$? How do you know?**

   The measure of $\angle 8$ is $73°$. This must be true because they are corresponding angles of parallel lines.

5. **Assuming $L_1 \parallel L_2$, if the measure of $\angle 3$ is $107°$ degrees, what is the measure of $\angle 6$? How do you know?**

   The measure of $\angle 6$ is $107°$. This must be true because they are alternate interior angles of parallel lines.

6. **Assuming $L_1 \parallel L_2$, if the measure of $\angle 2$ is $107°$, what is the measure of $\angle 7$? How do you know?**

   The measure of $\angle 7$ is $107°$. This must be true because they are alternate exterior angles of parallel lines.

7. **Would your answers to Problems 4–6 be the same if you had not been informed that $L_1 \parallel L_2$? Why, or why not?**

   No. The fact that the lines are parallel is the reason we can state that specific pairs of angles are equal. We can use basic rigid motions to prove that angles associated with parallel lines have the property of being equal when they are corresponding, alternate interior, or alternate exterior angles. If the lines are not parallel, then we could still classify the angles, but we would not know anything about their measures.

8. **Use an informal argument to describe why $\angle 1$ and $\angle 5$ are equal in measure if $L_1 \parallel L_2$.**

   The reason that angle 1 and angle 5 are equal in measure when the lines are parallel is because you can translate along a vector equal in length of the segment between the parallel lines; then, angle 1 would map onto angle 5.

9. **Use an informal argument to describe why $\angle 4$ and $\angle 5$ are equal in measure if $L_1 \parallel L_2$.**

   The reason that angle 4 and angle 5 are equal in measure when the lines are parallel is because when you rotate angle 4 around the midpoint of the segment between the parallel lines, angle 4 will map onto angle 5.

10. **Assume that L_1 is not parallel to L_2. Explain why $\angle 3 \neq \angle 7$.**

    If the lines are not parallel, then all we know about angle 3 and angle 7 is that they are corresponding angles. If the lines are parallel, we could use translation to map one angle onto the other to show that they are equal in measure. However, we are to assume that the lines are not parallel, which means that their corresponding angles will not be equal in measure.

Lesson 13: Angle Sum of a Triangle

Student Outcomes

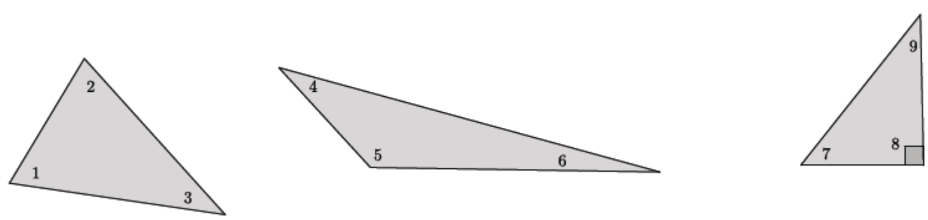
- Students know the angle sum theorem for triangles; the sum of the interior angles of a triangle is always $180°$.
- Students present informal arguments to draw conclusions about the angle sum of a triangle.

Classwork

Concept Development (3 minutes)

- The angle sum theorem for triangles states that the sum of the interior angles of a triangle is always $180°$ ($\angle$ sum of $\triangle$).
- It does not matter what kind of triangle it is (i.e., acute, obtuse, right); when you add the measure of the three angles, you always get a sum of $180°$.

Concept Development



$$\angle 1 + \angle 2 + \angle 3 = \angle 4 + \angle 5 + \angle 6 = \angle 7 + \angle 8 + \angle 9 = 180$$

Note that the sum of angles 7 and 9 must equal $90°$ because of the known right angle in the right triangle.

We want to prove that the angle sum of any triangle is $180°$. To do so, we will use some facts that we already know about geometry:

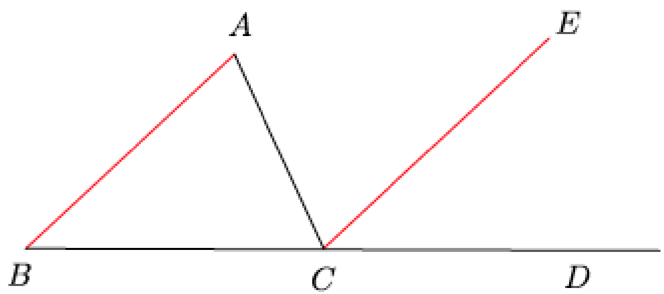
- A straight angle is $180°$ in measure.
- Corresponding angles of parallel lines are equal in measure (corr. $\angle s$, $\overline{AB} \parallel \overline{CD}$).
- Alternate interior angles of parallel lines are equal in measure (alt. $\angle s$, $\overline{AB} \parallel \overline{CD}$).

Exploratory Challenge 1 (13 minutes)

Provide students 10 minutes of work time. Once the 10 minutes have passed, review the solutions with the students before moving on to Exploratory Challenge 2.

Exploratory Challenge 1

Let triangle ABC be given. On the ray from B to C, take a point D so that C is between B and D. Through point C, draw a line parallel to AB, as shown. Extend the parallel lines AB and CE. Line AC is the transversal that intersects the parallel lines.



MP.3

a. **Name the three interior angles of triangle ABC.**

$\angle ABC$, $\angle BAC$, $\angle BCA$

b. **Name the straight angle.**

$\angle BCD$

Our goal is to show that the three interior angles of triangle ABC are equal to the angles that make up the straight angle. We already know that a straight angle is $180°$ in measure. If we can show that the interior angles of the triangle are the same as the angles of the straight angle, then we will have proven that the interior angles of the triangle have a sum of $180°$.

c. **What kinds of angles are $\angle ABC$ and $\angle ECD$? What does that mean about their measures?**

$\angle ABC$ and $\angle ECD$ are corresponding angles. Corresponding angles of parallel lines are equal in measure (corr. $\angle$s, $\overline{AB} \parallel \overline{CE}$).

d. **What kinds of angles are $\angle BAC$ and $\angle ECA$? What does that mean about their measures?**

$\angle BAC$ and $\angle ECA$ are alternate interior angles. Alternate interior angles of parallel lines are equal in measure (alt. $\angle$s, $\overline{AB} \parallel \overline{CE}$).

e. **We know that $\angle BCD = \angle BCA + \angle ECA + \angle ECD = 180°$. Use substitution to show that the three interior angles of the triangle have a sum of $180°$.**

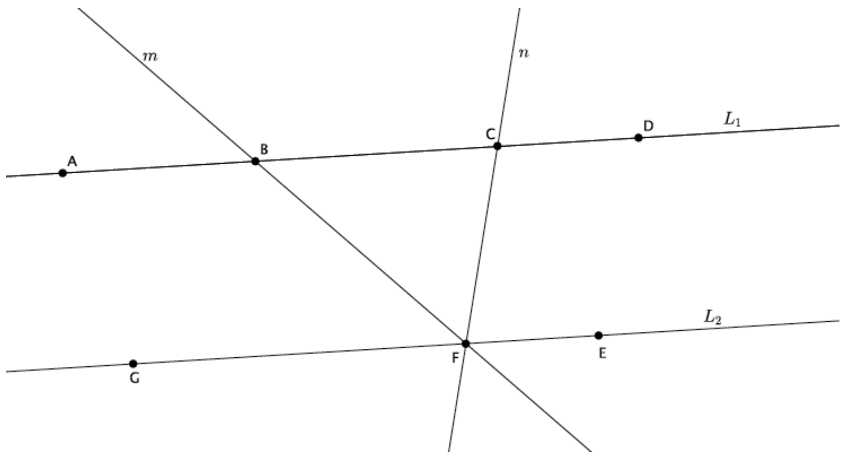
$\angle BCD = \angle BCA + \angle BAC + \angle ABC = 180°$ ($\angle$ sum of $\triangle$).

Exploratory Challenge 2 (20 minutes)

Provide students 15 minutes of work time. Once the 15 minutes have passed, review the solutions with the students.

Exploratory Challenge 2

The figure below shows parallel lines L_1 and L_2. Let m and n be transversals that intersect L_1 at points B and C, respectively, and L_2 at point F, as shown. Let A be a point on L_1 to the left of B, D be a point on L_1 to the right of C, G be a point on L_2 to the left of F, and E be a point on L_2 to the right of F.



a. **Name the triangle in the figure.**

$\triangle BCF$

b. **Name a straight angle that will be useful in proving that the sum of the interior angles of the triangle is $180°$.**

$\angle GFE$

As before, our goal is to show that the interior angles of the triangle are equal to the straight angle. Use what you learned from Exploratory Challenge 1 to show that interior angles of a triangle have a sum of $180°$.

c. **Write your proof below.**

MP.3

The straight angle $\angle GFE$ is comprised of angles $\angle GFB$, $\angle BFC$, and $\angle EFC$. Alternate interior angles of parallel lines are equal in measure (alt. $\angle$s, $\overline{AD} \parallel \overline{CE}$). For that reason, $\angle BCF = \angle EFC$ and $\angle CBF = \angle GFB$. Since $\angle GFE$ is a straight angle, it is equal to $180°$. Then, $\angle GFE = \angle GFB + \angle BFC + \angle EFC = 180°$. By substitution, $\angle GFE = \angle CBF + \angle BFC + \angle BCF = 180°$. Therefore, the sum of the interior angles of a triangle is $180°$ ($\angle$ sum of $\triangle$).

Closing (4 minutes)

Summarize, or have students summarize, the lesson.

- All triangles have a sum of interior angles equal to $180°$.
- We can prove that a triangle has a sum of interior angles equal to that of a straight angle using what we know about alternate interior angles and corresponding angles of parallel lines.

Lesson Summary

All triangles have a sum of interior angles equal to $180°$.

The proof that a triangle has a sum of interior angles equal to $180°$ is dependent upon the knowledge of straight angles and angle relationships of parallel lines cut by a transversal.

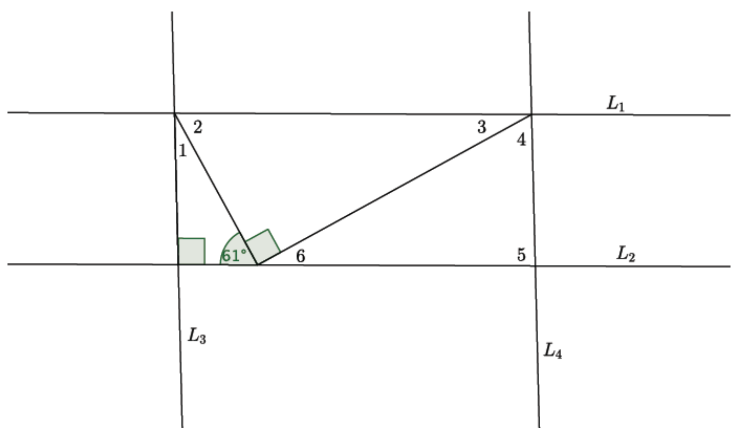
Exit Ticket (5 minutes)

Name ____________________ Date ____________

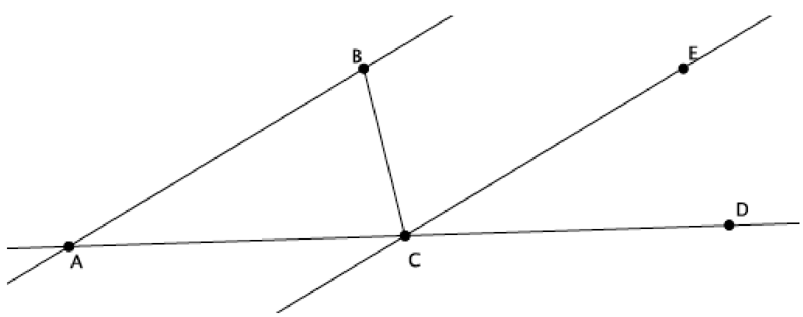
Lesson 13: Angle Sum of a Triangle

Exit Ticket

1. If $L_1 \parallel L_2$, and $L_3 \parallel L_4$, what is the measure of $\angle 1$? Explain how you arrived at your answer.

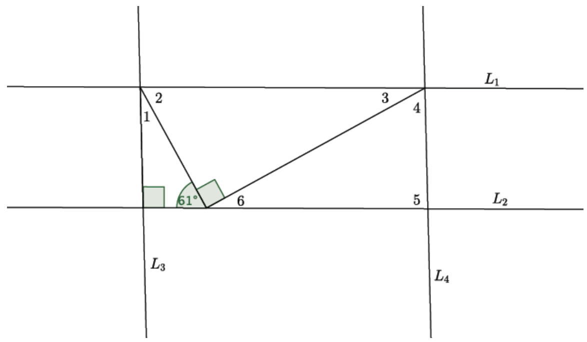


2. Given Line AB is parallel to Line CE, present an informal argument to prove that the interior angles of triangle ABC have a sum of $180°$.



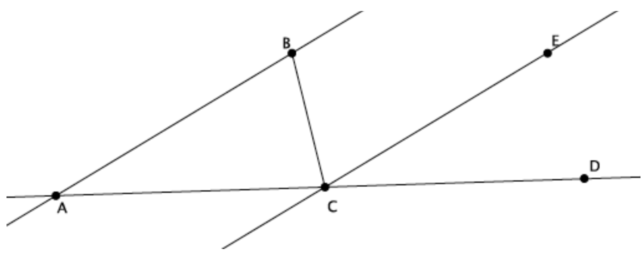
Exit Ticket Sample Solutions

1. If $L_1 \parallel L_2$, and $L_3 \parallel L_4$, what is the measure of $\angle 1$? Explain how you arrived at your answer.



The measure of angle 1 is $29°$. I know that the angle sum of triangles is $180°$. I already know that two of the angles of the triangle are $90°$ and $61°$.

2. Given Line AB is parallel to Line CE, present an informal argument to prove that the interior angles of triangle ABC have a sum of $180°$.

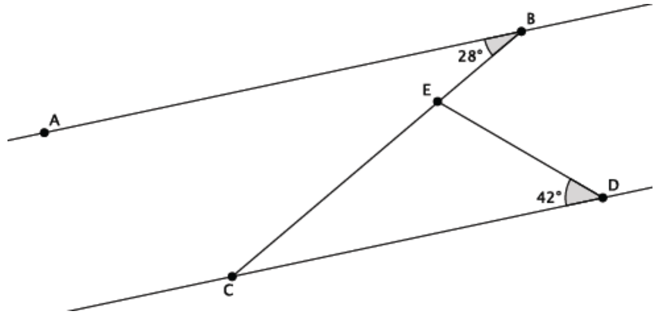


Since AB is parallel to CE, then the corresponding angles $\angle BAC$ and $\angle ECD$ are equal in measure. Similarly, angles $\angle ABC$ and $\angle ECB$ are equal in measure because they are alternate interior angles. Since $\angle ACD$ is a straight angle, i.e., equal to $180°$ in measure, substitution shows that triangle ABC has a sum of $180°$. Specifically, the straight angle is made up of angles $\angle ACB$, $\angle ECB$, and $\angle ECD$. $\angle ACB$ is one of the interior angles of the triangle and one of the angles of the straight angle. We know that angle $\angle ABC$ has the same measure as angle $\angle ECB$ and that angle $\angle BAC$ has the same measure as $\angle ECD$. Therefore, the sum of the interior angles will be the same as the angles of the straight angle, which is $180°$.

Problem Set Sample Solutions

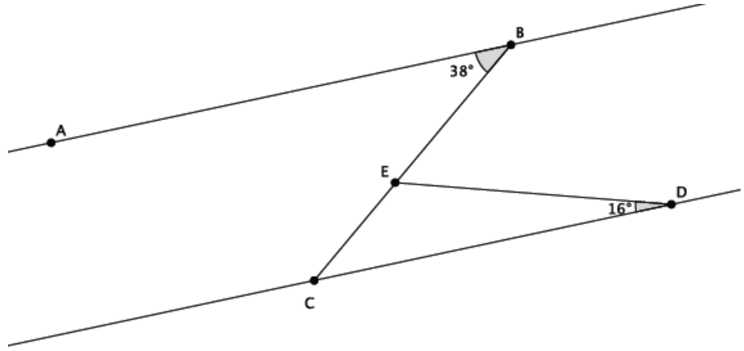
Students practice presenting informal arguments about the sum of the angles of a triangle using the theorem to find the measures of missing angles.

1. In the diagram below, line AB is parallel to line CD, i.e., $L_{AB} \parallel L_{CD}$. The measure of angle $\angle ABC = 28°$, and the measure of angle $\angle EDC = 42°$. Find the measure of angle $\angle CED$. Explain why you are correct by presenting an informal argument that uses the angle sum of a triangle.



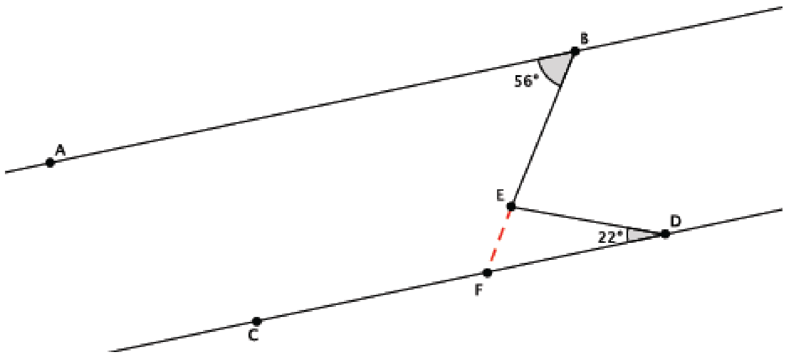
The measure of angle $\angle CED = 110°$. This is the correct measure for the angle because $\angle ABC$ and $\angle DCE$ are alternate interior angles of parallel lines. That means that the angles are congruent and have the same measure. Since the angle sum of a triangle is $180°$, then the measure of $\angle CED = 180° - (28° + 42°) = 110°$.

2. In the diagram below, line AB is parallel to line CD, i.e., $L_{AB} \parallel L_{CD}$. The measure of angle $\angle ABE = 38°$, and the measure of angle $\angle EDC = 16°$. Find the measure of angle $\angle BED$. Explain why you are correct by presenting an informal argument that uses the angle sum of a triangle. (Hint: Find the measure of angle $\angle CED$ first, and then use that measure to find the measure of angle $\angle BED$.)



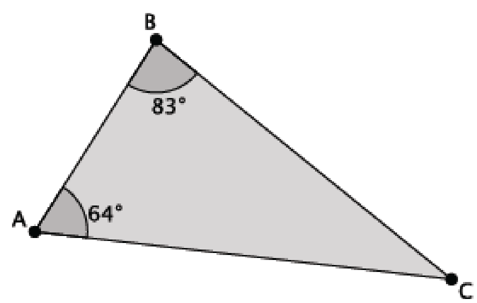
The measure of angle $\angle BED = 54°$. This is the correct measure for the angle because $\angle ABC$ and $\angle DCE$ are alternate interior angles of parallel lines. That means that the angles are congruent and have the same measure. Since the angle sum of a triangle is $180°$, then the measure of $\angle CED = 180° - (38° + 16°) = 126°$. The straight angle $\angle BEC$ is made up of $\angle CED$ and $\angle BED$. Since we know straight angles are $180°$ in measure, and angle $\angle CED = 126°$, then $\angle BED = 54°$.

3. In the diagram below, line AB is parallel to line CD, i.e., $L_{AB} \parallel L_{CD}$. The measure of angle $\angle ABE = 56°$, and the measure of angle $\angle EDC = 22°$. Find the measure of angle $\angle BED$. Explain why you are correct by presenting an informal argument that uses the angle sum of a triangle. (Hint: Extend the segment BE so that it intersects line CD.)



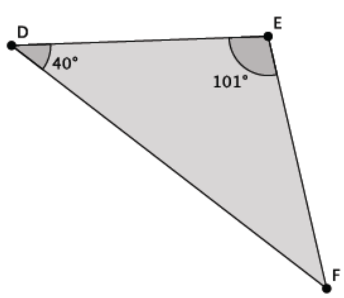
The measure of angle $\angle BED = 78°$. This is the correct measure for the angle because $\angle ABE$ and $\angle DFE$ are alternate interior angles of parallel lines. That means that the angles are congruent and have the same measure. Since the angle sum of a triangle is $180°$, then the measure of $\angle FED = 180° - (56° + 22°) = 102°$. The straight angle $\angle BEF$ is made up of $\angle FED$ and $\angle BED$. Since straight angles are $180°$ in measure, and angle $\angle FED = 102°$, then $\angle BED = 78°$.

4. What is the measure of $\angle ACB$?



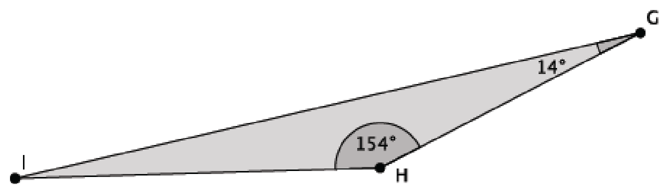
The measure of $\angle ACB$ is $180° - (83° + 64°) = 33°$.

5. What is the measure of $\angle EFD$?



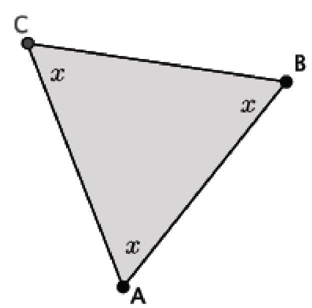
The measure of $\angle EFD$ is $180° - (101° + 40°) = 39°$.

6. What is the measure of $\angle HIG$?



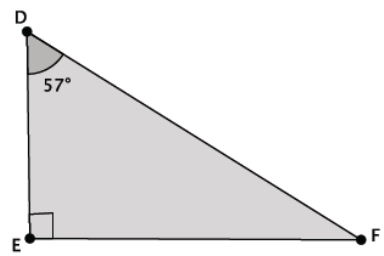
The measure of $\angle HIG$ is $180° - (154° + 14°) = 12°$.

7. What is the measure of $\angle ABC$?



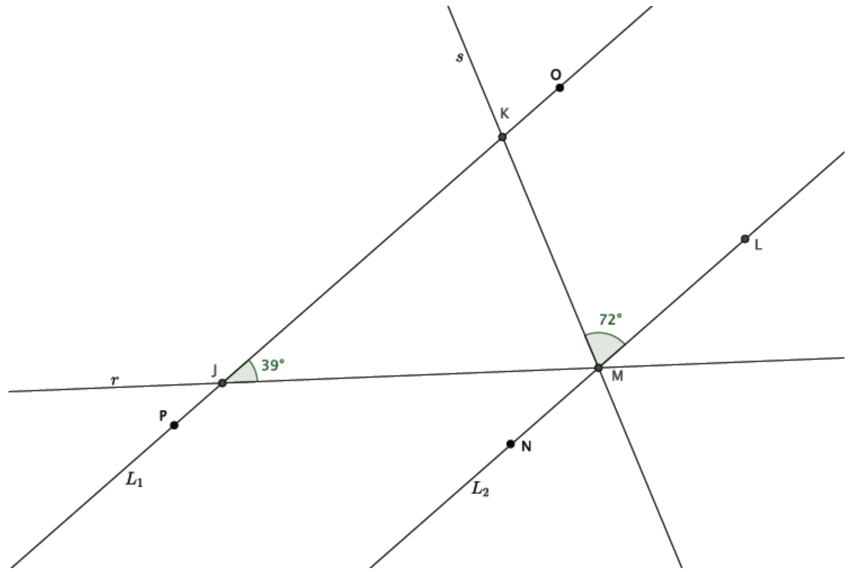
The measure of $\angle ABC$ is $60°$ because $60 + 60 + 60 = 180$.

8. Triangle DEF is a right triangle. What is the measure of $\angle EFD$?



The measure of $\angle EFD$ is $90° - 57° = 33°$.

9. In the diagram below, lines L_1 and L_2 are parallel. Transversals r and s intersect both lines at the points shown below. Determine the measure of $\angle JMK$. Explain how you know you are correct.



The lines L_1 and L_2 are parallel, which means that the alternate interior angles formed by the transversals are equal. Specifically, $\angle LMK = \angle JKM = 72°$. Since triangle ΔJKM has a sum of interior angles equal to $180°$, then $\angle KJM + \angle JMK + \angle JKM = 180°$. By substitution, we have $39 + \angle JMK + 72 = 180$; therefore, $\angle JMK = 69°$.

Lesson 14: More on the Angles of a Triangle

Student Outcomes

- Students know a third informal proof of the angle sum theorem.
- Students know how to find missing interior and exterior angle measures of triangles and present informal arguments to prove their answer is correct.

Lesson Notes

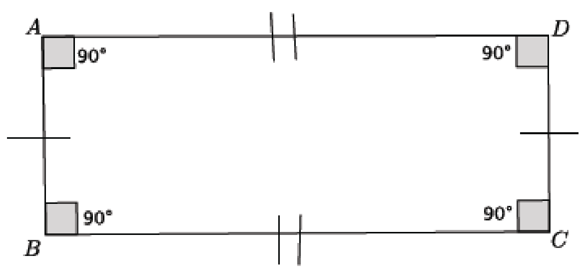
Students will see one final informal proof of the angle sum of a triangle before moving on to working with exterior angles of triangles.

Classwork

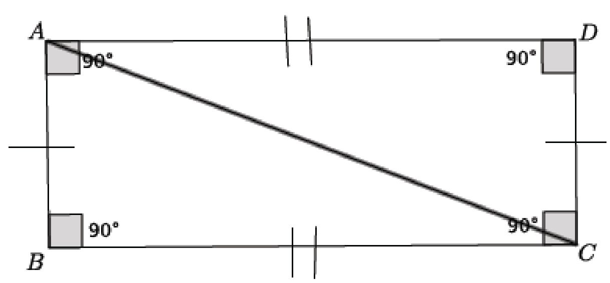
Discussion (7 minutes)

Let's look at one final proof that the sum of the degrees of the interior angles of a triangle is 180.

- Start with a rectangle. What properties do rectangles have?
  - *All four angles are right angles; opposite sides are equal in length.*



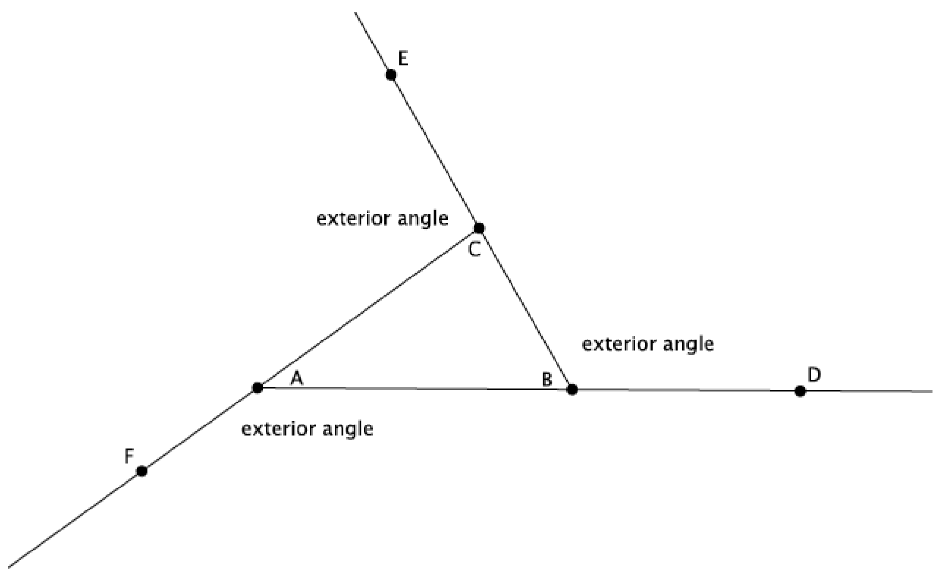
- If we draw a diagonal that connects A to C (or we could choose to connect B to D), what shapes are formed?
  - *We get two triangles.*



- What do we know about these triangles, and how do we know it?
  - *The triangles are congruent. We can trace one of the triangles and, through a sequence of basic rigid motions, map it onto the other triangle.*
- Our goal is to show that the angle sum of a triangle is $180°$. We know that when we draw a diagonal through a rectangle, we get two congruent triangles. How can we put this information together to show that the sum of angles in a triangle is $180°$?
  - *The rectangle has four right angles which means that the sum of the angles of the rectangle is $4(90°) = 360°$. Since the diagonal divides the rectangle into two congruent triangles, each triangle will have exactly half the total degrees of the rectangle. Since $360° \div 2 = 180°$, then each triangle has a sum of angles equal to $180°$.*

Discussion (7 minutes)

Now let's look at what is called the *exterior angle of a triangle*. An exterior angle is formed when one of the sides of the triangle is extended. The interior angles are inside the triangle, so the exterior angle is outside of the triangle along the extended side. In triangle ABC, the exterior angles are $\angle CBD$, $\angle ECA$, and $\angle BAF$.



- What do we know about the sum of interior angles of a triangle? Name the angles.
  - *The sum of the interior angles $\angle ABC$, $\angle BCA$, and $\angle CAB$ of the triangle is $180°$.*
- What do we know about the degree of a straight angle?
  - *A straight angle has a measure of $180°$.*
- Let's look specifically at straight angle $\angle ABD$. Name the angles that make up this straight angle.
  - *$\angle ABC$ and $\angle CBD$*

- Because the triangle and the straight angle both have measures of $180°$, we can write them as equal to one another. That is, since
$$\angle ABC + \angle BCA + \angle CAB = 180,$$
and
$$\angle ABC + \angle CBD = 180,$$
then,
$$\angle ABC + \angle BCA + \angle CAB = \angle ABC + \angle CBD.$$
- Which angle is common to both the triangle and the straight angle?
  - $\angle ABC$
- If we subtract the measure of $\angle ABC$ from both the triangle and the straight angle, we get
$$\angle ABC - \angle ABC + \angle BCA + \angle CAB = \angle ABC - \angle ABC + \angle CBD$$
$$\angle BCA + \angle CAB = \angle CBD.$$
- What kind of angle is $\angle CBD$?
  - *It is the exterior angle of the triangle.*
- We call angles $\angle BCA$ and $\angle CAB$ the remote interior angles because they are the farthest, "remotest" from the exterior angle $\angle CBD$. Each of the remote angles share one side with the angle adjacent to the exterior angle. The equation $\angle BCA + \angle CAB = \angle CBD$ means that the sum of the remote interior angles are equal to the exterior angle of the triangle.

Exercises 1–4 (8 minutes)

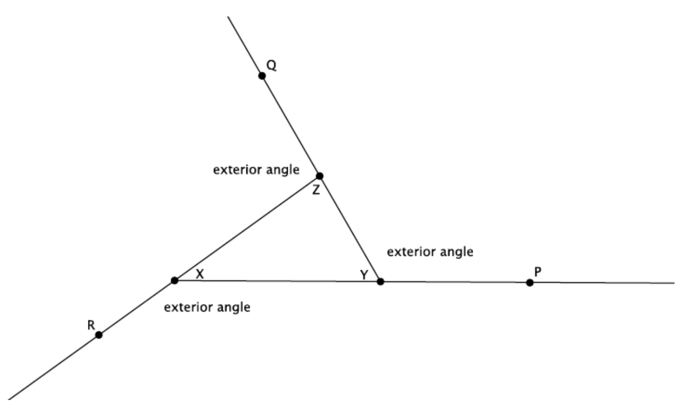
Students work in pairs to identify the remote interior angles and corresponding exterior angle of the triangle in Exercises 1–3. After most of the students have finished Exercises 1–3, provide the correct answers before they move on to the next exercise. In Exercise 4, students recreate the reasoning of Example 1 for another exterior angle of the triangle.

Scaffolding:

Keep the work of Example 1 visible while students work on Exercises 1–4.

Exercises 1–4

Use the diagram below to complete Exercises 1–4.



1. **Name an exterior angle and the related remote interior angles.**

   The exterior angle is $\angle ZYP$, and the related remote interior angles are $\angle YZX$ and $\angle ZXY$.

2. **Name a second exterior angle and the related remote interior angles.**

   The exterior angle is $\angle XZQ$, and the related remote interior angles are $\angle ZYX$ and $\angle ZXY$.

3. **Name a third exterior angle and the related remote interior angles.**

   The exterior angle is $\angle RXY$, and the related remote interior angles are $\angle ZYX$ and $\angle XZY$.

4. **Show that the measure of an exterior angle is equal to the sum of the related remote interior angles.**

   Triangle XYZ has interior angles $\angle XYZ$, $\angle YZX$, and $\angle ZXY$. The sum of those angles is $180°$. The straight angle $\angle XYP$ also has a measure of $180°$ and is made up of angles $\angle XYZ$ and $\angle ZYP$. Since the triangle and the straight angle have the same number of degrees, we can write the sum of their respective angles as an equality:

   $$\angle XYZ + \angle YZX + \angle ZXY = \angle XYZ + ZYP.$$

   Both the triangle and the straight angle share $\angle XYZ$. We can subtract the measure of that angle from the triangle and the straight angle. Then, we have

   $$\angle YZX + \angle ZXY = \angle ZYP,$$

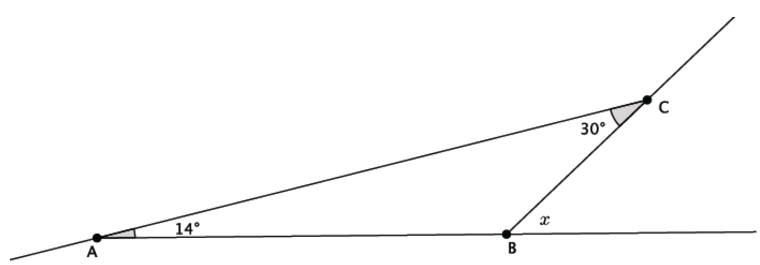
   where the angle $\angle ZYP$ is the exterior angle, and the angles $\angle YZX$ and $\angle ZXY$ are the related remote interior angles of the triangle. Therefore, the sum of the remote interior angles of a triangle are equal to the exterior angle.

Example 1 (2 minutes)

- Ask students what we need to do to find the measure of angle x. Then, have them work on white boards and show you their answer.

Example 1

Find the measure of angle x.



We need to find the sum of the remote interior angles to find the measure of the exterior angle x:

$14 + 30 = 44$. Therefore, the measure of angle $x = 44°$.

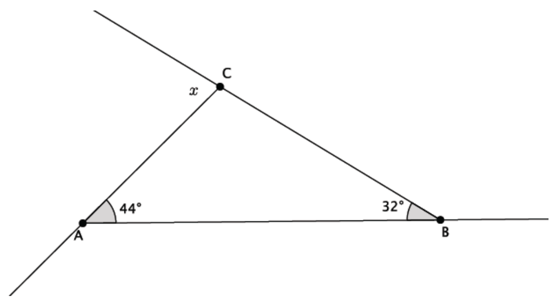
- Present an informal argument that proves you are correct.
  - *We know that triangles have a sum of interior angles that is equal to $180°$. We also know that straight angles are $180°$. Angle $\angle ABC$ must be $136°$, which means that $\angle x = 44°$.*

Example 2 (2 minutes)

Ask students what we need to do to find the measure of angle x. Then, have them work on white boards and show you their answer.

Example 2

Find the measure of angle x.



We need to find the sum of the remote interior angles to find the measure of the exterior angle x: $44+32=76$. Therefore, the measure of angle $x=76°$.

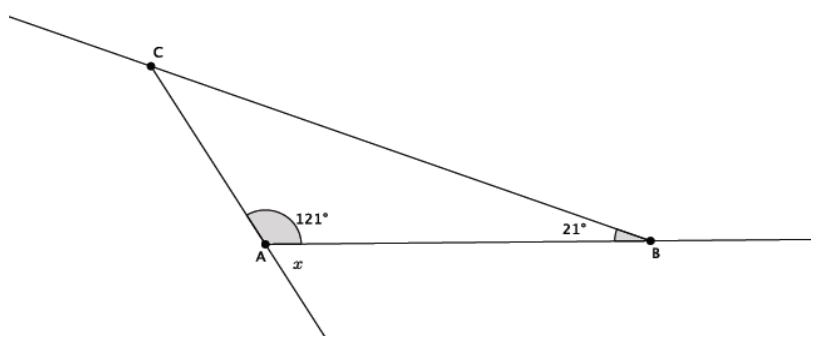
- Present an informal argument that proves you are correct.
  - *We know that triangles have a sum of interior angles that is equal to $180°$. We also know that straight angles are $180°$. Angle $\angle ACB$ must be $104°$, which means that $\angle x=76°$.*

Example 3 (2 minutes)

Ask students what we need to do to find the measure of angle x. Then, have them work on white boards and show you their answers. Make sure students see that this is not like the last two examples. They must pay attention to the information that is provided and not expect to always do the same procedure.

Example 3

Find the measure of angle x.



$180-121=59$. *Therefore, the measure of angle* $=59°$.

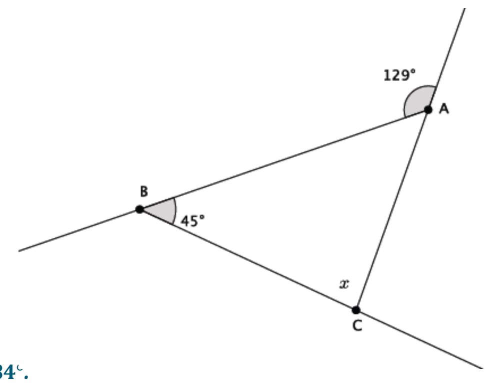
Students should notice that we are not given the two remote interior angles associated with the exterior angle x. For that reason, we must use what we know about straight angles (or supplementary angles) to find the measure of angle x.

Example 4 (2 minutes)

Ask students what we need to do to find the measure of angle x. Then, have them work on white boards and show you their answers. Make sure students see that this is not like the last three examples. They must pay attention to the information that is provided and not expect to always do the same procedure.

Example 4

Find the measure of angle x.



$129 - 45 = 84$. Therefore, the measure of angle $x = 84^\circ$.

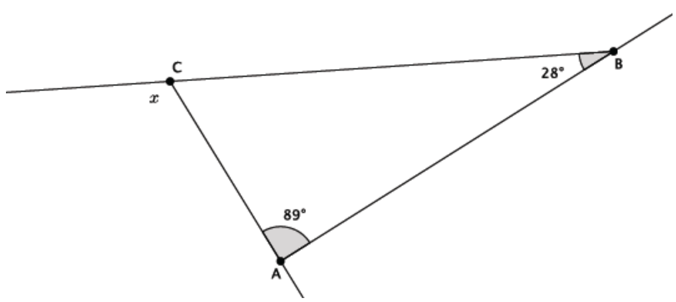
Students should notice that we are given just one of the remote interior angle measures and the exterior angle measure. For that reason, we will need to subtract 45 from the exterior angle to find the measure of angle x.

Exercises 5–10 (6 minutes)

Students complete Exercises 5–10 independently. Check solutions once most students have finished.

Exercise 5–10

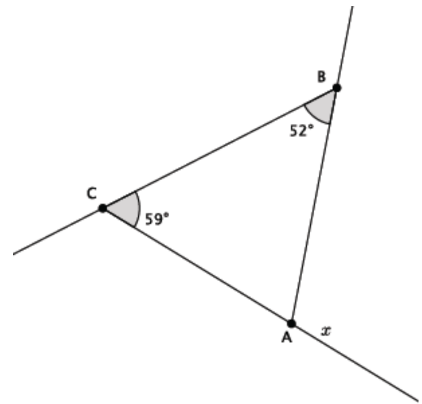
5. **Find the measure of angle x. Present an informal argument showing that your answer is correct.**



Since $89 + 28 = 117$, the measure of angle x is 117°. We know that triangles have a sum of interior angles that is equal to 180°. We also know that straight angles are 180°. Angle $\angle ACB$ must be 63°, which means that $\angle x = 117^\circ$.

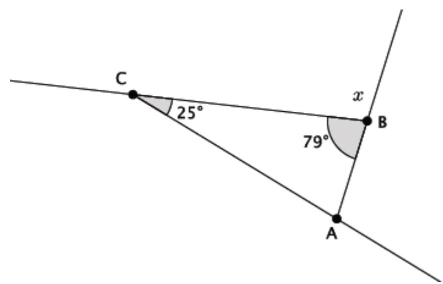
6. Find the measure of angle x. Present an informal argument showing that your answer is correct.

Since $59 + 52 = 111$, the measure of angle x is $111°$. We know that triangles have a sum of interior angles that is equal to $180°$. We also know that straight angles are $180°$. Angle $\angle CAB$ must be $69°$, which means that $\angle x = 111°$.



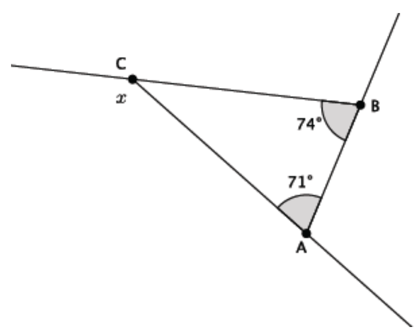
7. Find the measure of angle x. Present an informal argument showing that your answer is correct.

Since $180 - 79 = 101$, the measure of angle x is $101°$. We know that straight angles are $180°$, and the straight angle in the diagram is made up of angle $\angle ABC$ and angle $\angle x$. Angle $\angle ABC$ is $79°$, which means that $\angle x = 101°$.



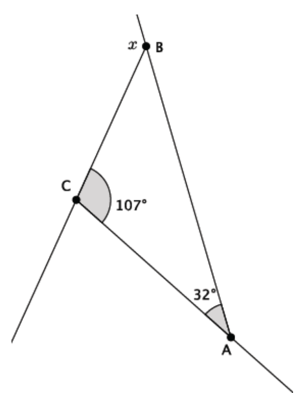
8. Find the measure of angle x. Present an informal argument showing that your answer is correct.

Since $71 + 74 = 145$, the measure of angle x is $145°$. We know that triangles have a sum of interior angles that is equal to $180°$. We also know that straight angles are $180°$. Angle $\angle ACB$ must be $35°$, which means that $\angle x = 145°$.

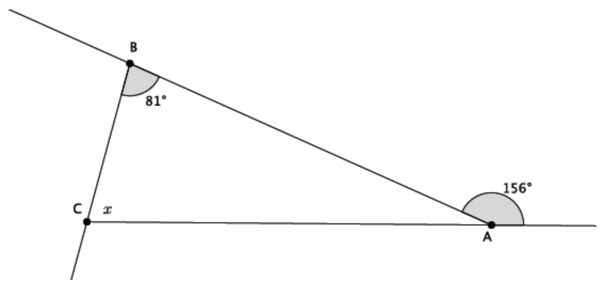


9. Find the measure of angle x. Present an informal argument showing that your answer is correct.

Since $107 + 32 = 139$, the measure of angle x is $139°$. We know that triangles have a sum of interior angles that is equal to $180°$. We also know that straight angles are $180°$. Angle $\angle CBA$ must be $41°$, which means that $\angle 139°$.



10. Find the measure of angle x. Present an informal argument showing that your answer is correct.



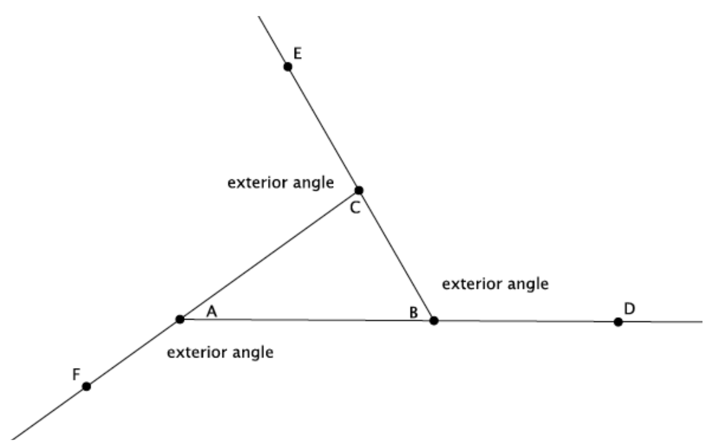
Since $156 - 81 = 75$, the measure of angle x is 75°. We know that triangles have a sum of interior angles that is equal to 180°. We also know that straight angles are 180°. Angle $\angle BAC$ must be 24° because it is part of the straight angle. Then, $\angle x = 180^\circ - (81^\circ + 24^\circ) = 75^\circ$.

Closing (4 minutes)

Summarize, or have students summarize, the lesson.

- We learned another proof as to why the interior angles of a triangle are equal to 180° with respect to a triangle being exactly half of a rectangle.
- We learned the definitions of exterior angles and remote interior angles.
- The sum of the remote interior angles of a triangle is equal to the measure of the related exterior angle.

Lesson Summary



The sum of the remote interior angles of a triangle is equal to the measure of the related exterior angle. For example, $\angle CAB + \angle ABC = \angle ACE$.

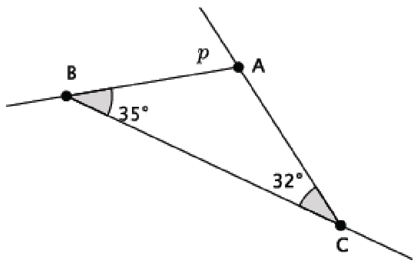
Exit Ticket (5 minutes)

Name ____ Date ________________

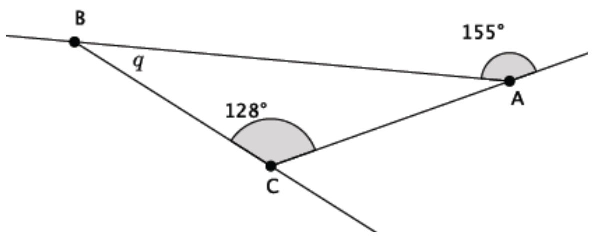
Lesson 14: More on the Angles of a Triangle

Exit Ticket

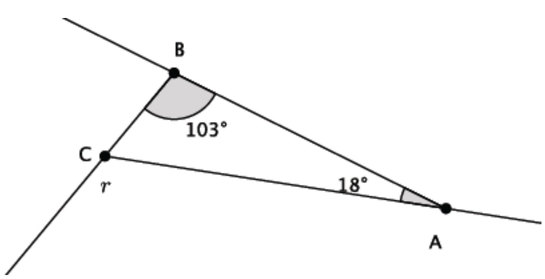
1. Find the measure of angle p. Present an informal argument showing that your answer is correct.



2. Find the measure of angle q. Present an informal argument showing that your answer is correct.

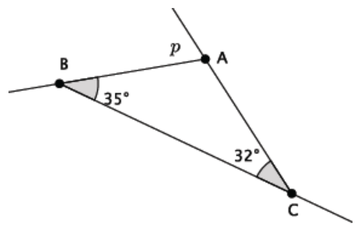


3. Find the measure of angle r. Present an informal argument showing that your answer is correct.



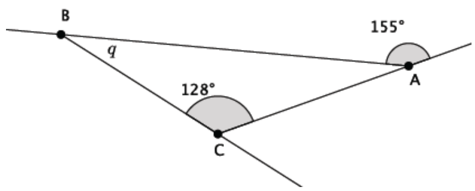
Exit Ticket Sample Solutions

1. **Find the measure of angle p. Present an informal argument showing that your answer is correct.**



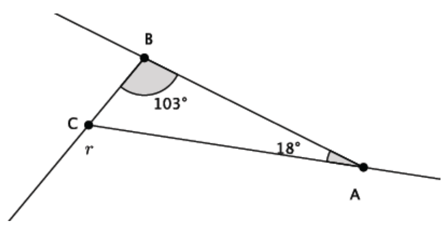
The measure of angle p is $67°$. We know that triangles have a sum of interior angles that is equal to $180°$. We also know that straight angles are $180°$. Angle $\angle BAC$ must be $113°$, which means that $\angle p = 67°$.

2. **Find the measure of angle q. Present an informal argument showing that your answer is correct.**



The measure of angle q is $27°$. We know that triangles have a sum of interior angles that is equal to $180°$. We also know that straight angles are $180°$. Angle $\angle CAB$ must be $25°$, which means that $\angle q = 27°$.

3. **Find the measure of angle r. Present an informal argument showing that your answer is correct.**



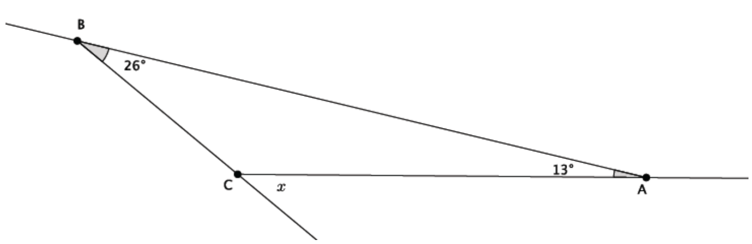
The measure of angle r is $121°$. We know that triangles have a sum of interior angles that is equal to $180°$. We also know that straight angles are $180°$. Angle $\angle BCA$ must be $59°$, which means that $\angle r = 121°$.

Problem Set Sample Solutions

Students practice finding missing angle measures of triangles.

For each of the problems below, use the diagram to find the missing angle measure. Show your work.

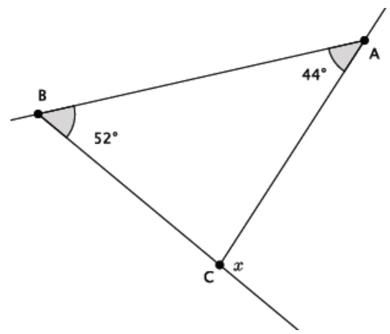
1. **Find the measure of angle x. Present an informal argument showing that your answer is correct.**



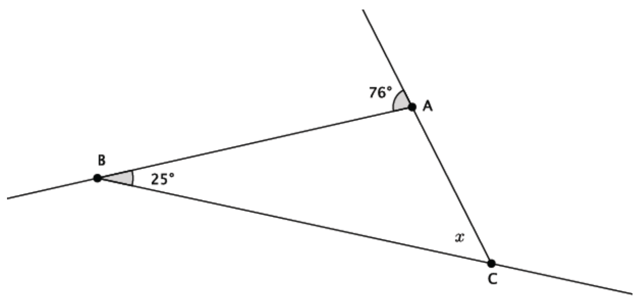
Since $26 + 13 = 39$, the measure of angle x is $39°$. We know that triangles have a sum of interior angles that is equal to $180°$. We also know that straight angles are $180°$. Angle $\angle BCA$ must be $141°$, which means that $\angle x = 39°$.

2. Find the measure of angle x.

Since $52 + 44 = 96$, the measure of angle x is $96°$.



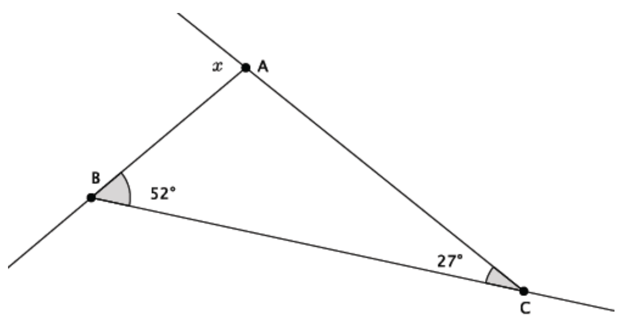
3. Find the measure of angle x. Present an informal argument showing that your answer is correct.



Since $76 - 25 = 51$, the measure of angle x is $51°$. We know that triangles have a sum of interior angles that is equal to $180°$. We also know that straight angles are $180°$. Angle $\angle BAC$ must be $104°$ because it is part of the straight angle. Then, $x = 180° - (104° + 25°) = 51°$.

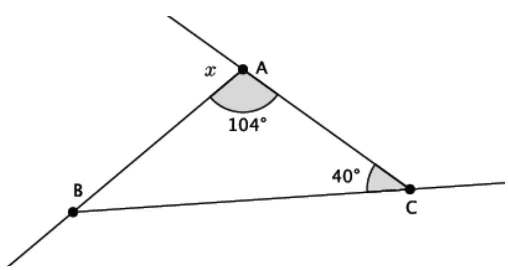
4. Find the measure of angle x.

Since $27 + 52 = 79$, the measure of angle x is $79°$.

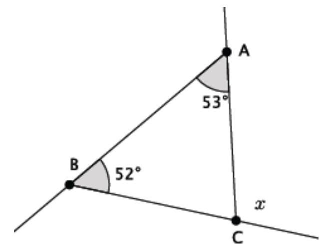


5. Find the measure of angle x.

Since $180 - 104 = 76$, the measure of angle x is $76°$.

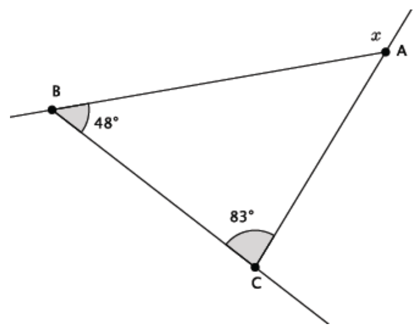


6. Find the measure of angle x.



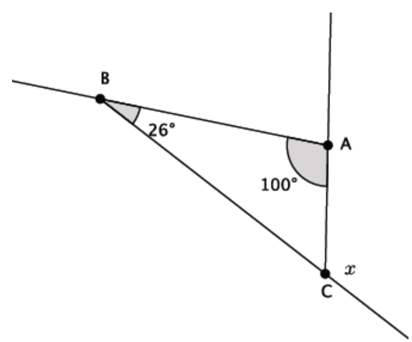
Since $52 + 53 = 105$*, the measure of angle* x *is* $105°$.

7. Find the measure of angle x.



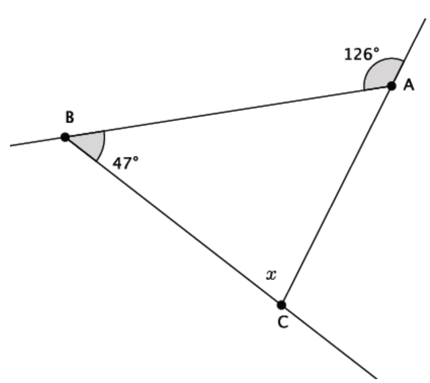
Since $48 + 83 = 131$*, the measure of angle* x *is* $131°$.

8. Find the measure of angle x.



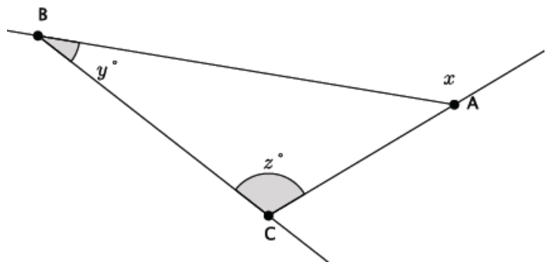
Since $100 + 26 = 126$*, the measure of angle* x *is* $126°$.

9. Find the measure of angle x.



Since $126 - 47 = 79$*, the measure of angle* x *is* $79°$.

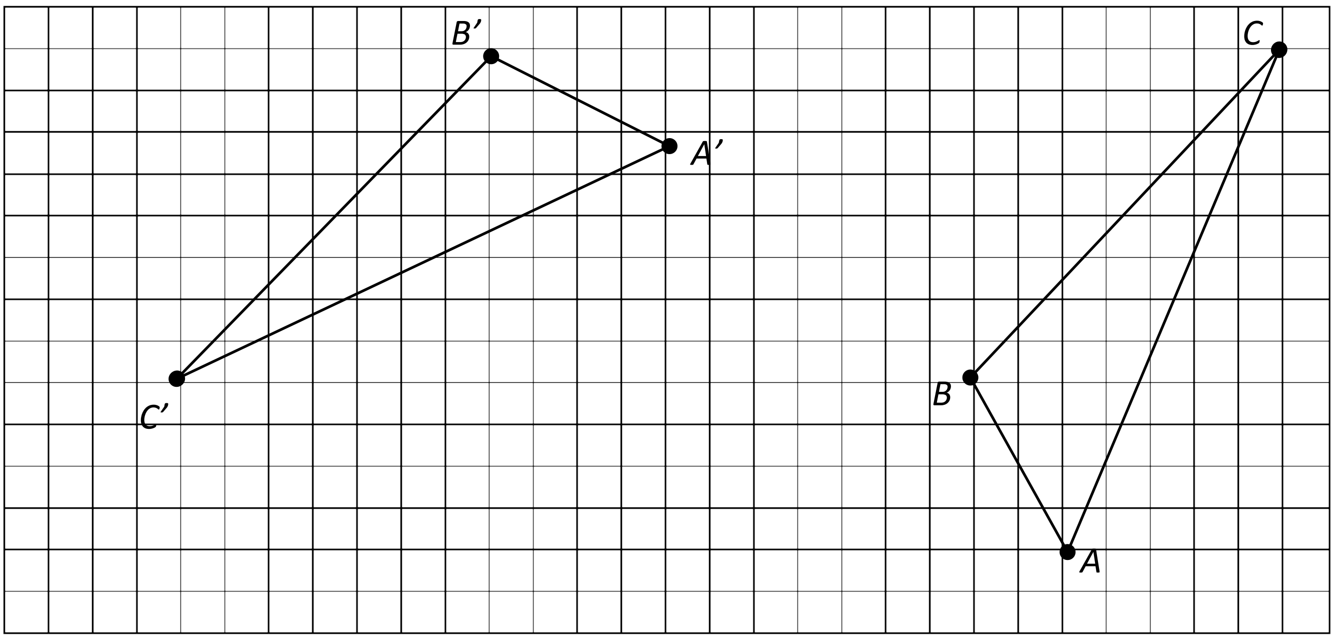
10. **Write an equation that would allow you to find the measure of angle x. Present an informal argument showing that your answer is correct.**



Since $y + z = x$, the measure of angle x is $(y + z)°$. We know that triangles have a sum of interior angles that is equal to $180°$. We also know that straight angles are $180°$. Then, $\angle y + \angle z + \angle BAC = 180°$, and $\angle x + \angle BAC = 180°$. Since both equations are equal to $180°$, then $\angle y + \angle z + \angle BAC = \angle x + \angle BAC$. Subtract $\angle BAC$ from each side of the equation, and you get $\angle y + \angle z = \angle x$.

Name ____________________________________ Date ____________________

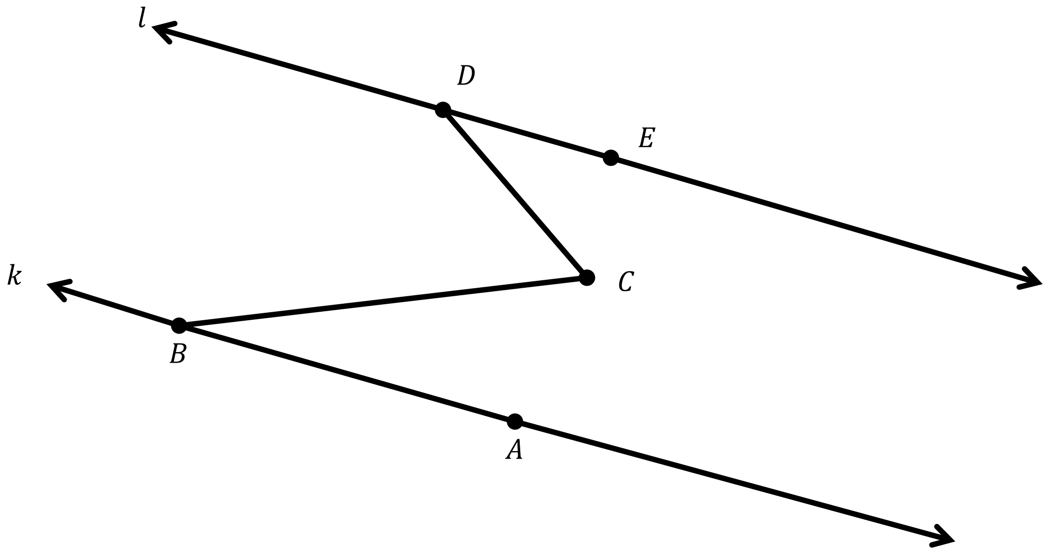
1. $\triangle ABC \cong \triangle A'B'C'$. Use the picture to answer the question below.



Describe a sequence of rigid motions that would prove a congruence between $\triangle ABC$ and $\triangle A'B'C'$.

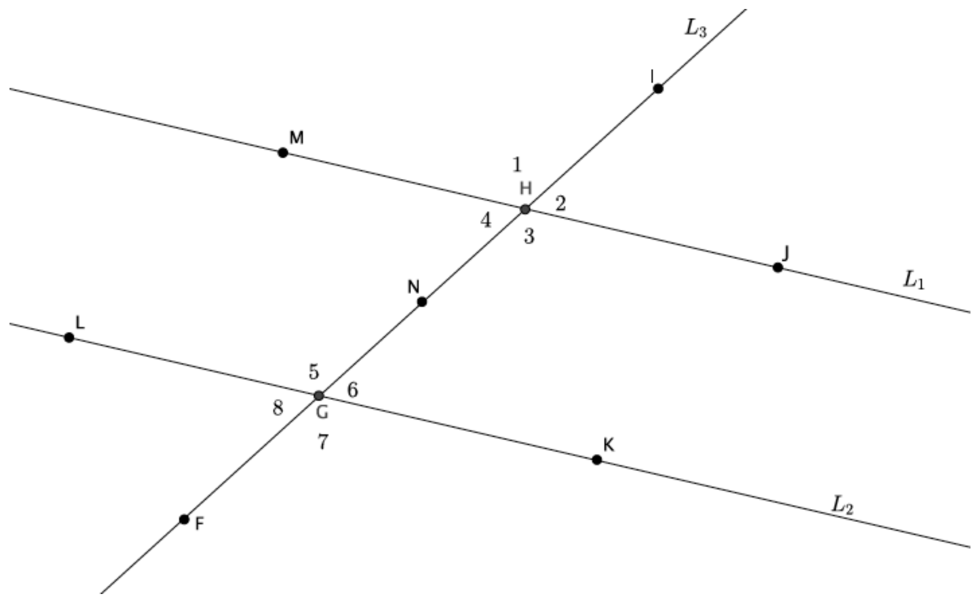
2. Use the diagram to answer the question below.

$k || l$



Line k is parallel to line l. $m\angle EDC = 41°$ and $m\angle ABC = 32°$. Find the $m\angle BCD$. Explain in detail how you know you are correct. Add additional lines and points as needed for your explanation.

3. Use the diagram below to answer the questions that follow. Lines L_1 and L_2 are parallel, $L_1 \parallel L_2$. Point N is the midpoint of segment GH.



a. If $\angle IHM = 125°$, what is the measure of $\angle IHJ$? $\angle JHN$? $\angle NHM$?

b. What can you say about the relationship between $\angle 4$ and $\angle 6$? Explain using a basic rigid motion. Name another pair of angles with this same relationship.

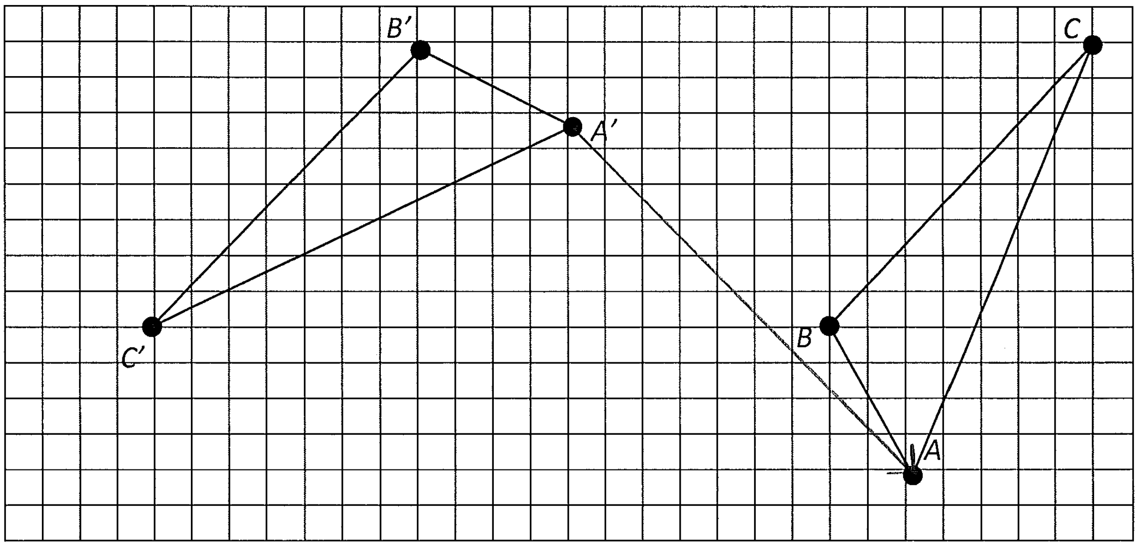
c. What can you say about the relationship between $\angle 1$ and $\angle 5$? Explain using a basic rigid motion. Name another pair of angles with this same relationship.

A Progression Toward Mastery					
Assessment Task Item		**STEP 1** **Missing or incorrect answer and little evidence of reasoning or application of mathematics to solve the problem.**	**STEP 2** **Missing or incorrect answer but evidence of some reasoning or application of mathematics to solve the problem.**	**STEP 3** **A correct answer with some evidence of reasoning or application of mathematics to solve the problem, or an incorrect answer with substantial evidence of solid reasoning or application of mathematics to solve the problem.**	**STEP 4** **A correct answer supported by substantial evidence of solid reasoning or application of mathematics to solve the problem.**
1	**8.G.A.2**	Student is s unable to respond to the question or left item blank. Student does not describe a sequence. Student shows no reasoning or application of mathematics to solve the problem.	Student identifies an incorrect sequence of rigid motions. Student uses little or no mathematical vocabulary or notation in sequence. Some evidence of mathematical reasoning is used in sequence.	Student identifies a correct sequence of rigid motions but lacks precision. Student may or may not use mathematical vocabulary or notation sequence. Some evidence of mathematical reasoning is used in sequence.	Student identifies a correct sequence of rigid motions with precision. Student uses mathematical vocabulary and notation in sequence. Substantial evidence of mathematical reasoning is used in sequence.
2	**8.G.A.5**	Student is unable to respond to the questions or leaves items blank. Student shows no reasoning or application of mathematics to solve the problem.	Student calculates the measurement of the angle but makes calculation errors. Student attempts to use auxiliary lines to solve the problem. Student shows little or no reasoning in written explanation. Student does not use any theorem in written explanation.	Student calculates the measurement of the angle but makes calculation errors. Student uses auxiliary lines to solve the problem. Student shows some reasoning in written explanation. Student may or may not use the correct theorem in the written explanation.	Student calculates the measurement of the angle correctly as $73°$. Student uses auxiliary lines to solve the problem. Student shows substantial reasoning in written explanation including information about congruent angles being equal, straight angles having $180°$, triangle sum being $180°$, sum of remote interior angles equal to exterior angle of a triangle, etc.

3	**a** **8.G.A.5**	Student is unable to respond to the questions or leaves items blank. Student shows no reasoning or application of mathematics to solve the problem.	Student makes calculation errors. Student answers part of the question correctly, i.e., $\angle IHM = \angle JHN = 125°$ but omits $\angle IHJ = \angle NHM = 55°$ or answers with all four angles as the same measure.	Student shows some application of mathematics to solve the problem. Student makes calculation errors. Student reverses the answers, i.e., $\angle IHM = \angle JHN = 55°$ or $\angle IHJ = \angle NHM = 125°$.	Student answers correctly with $\angle IHM = \angle JHN = 125°$ and $\angle IHJ = \angle NHM = 55°$ for measures of ALL four angles.
	b **8.G.A.5**	Student is unable to respond to the questions or leaves items blank. Student shows no reasoning or application of mathematics to solve the problem. Student does not include a written explanation.	Student answers the name of the angles incorrectly. Student incorrectly identifies the other angles with the same relationship. Student includes a written explanation. Student references a rigid motion, translation, rotation, reflection. Written explanation is not mathematically based, e.g., "they look the same."	Student may answer the name of the angles incorrectly but correctly identifies the other angles with the same relationship. Student uses some mathematical vocabulary in written explanation. Student references rotation but may not reference all of the key points in written explanation.	Student answers correctly by calling the angles Alternate Interior Angles. Student names $\angle 3$ and $\angle 5$ as angles with the same relationship. Student uses mathematical vocabulary in written explanation. Student references ALL of the following key points: N is the midpoint of HG, rotation of $180°$ around N, and rotation is angle-preserving in the written explanation. Written explanation is thorough and complete.
	c **8.G.A.5**	Student is unable to respond to the questions or leaves items blank. Student shows no reasoning or application of mathematics to solve the problem. Student does not include a written explanation.	Student ma answers the name of the angles incorrectly. Student incorrectly identifies the other angles with the same relationship. Student includes a written explanation. Student references a rigid motion, translation, rotation, reflection. Written explanation is not mathematically based, e.g., "they look the same."	Student identifies the name of the angles incorrectly but does correctly identify the other angles with the same relationship. Student uses some mathematical vocabulary in written explanation. Student references translation but may not reference all of the key points in written explanation.	Student answers correctly by calling the angles corresponding angles. Student names $\angle 2$ and $\angle 6$ (or $\angle 3$ and $\angle 7$ or $\angle 4$ and $\angle 8$) as angles with the same relationship. Student uses mathematical vocabulary in written explanation. Student references ALL of the following key points: translation along vector HG, translation maps parallel lines to parallel lines, and translation is angle-preserving in written explanation. Written explanation is thorough and complete.

Name ______________________________ Date ______________

1. $\triangle ABC \cong \triangle A'B'C'$. Use the picture to answer the question below.

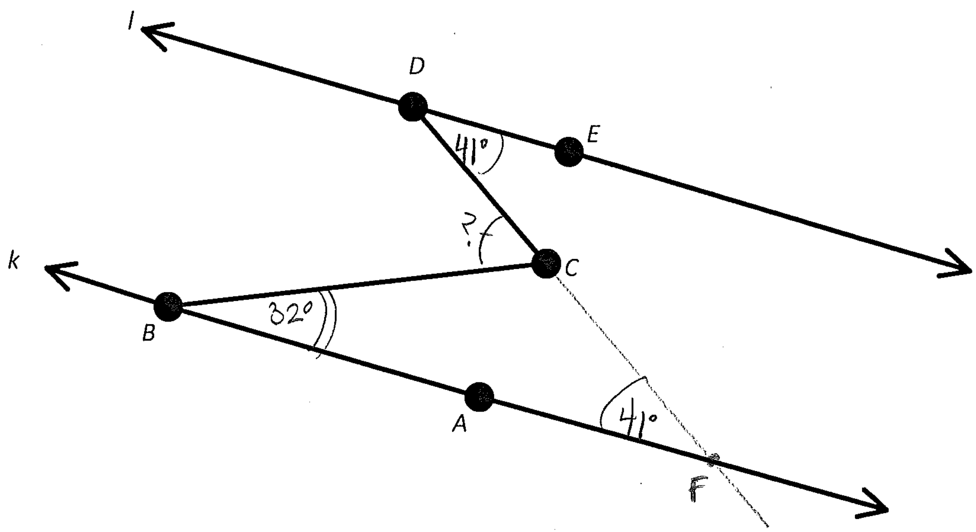


Describe a sequence of rigid motions that would prove a congruence between $\triangle ABC$ and $\triangle A'B'C'$.

LET T BE THE TRANSLATION ALONG $\overrightarrow{A'A}$ SO THAT $T(A') = A$.
LET R BE THE ROTATION AROUND A, d DEGREES SO THAT $R(A'B') = AB$. BY HYPOTHESIS $|AB| = |A'B'|$.
LET Λ BE THE REFLECTION ACROSS L_{AB}. AGAIN BY HYPOTHESIS $|\angle A| = |\angle A|$, $|\angle B| = |\angle B|$, SO THE COMPOSITION $\Lambda \circ R \circ T$ WILL MAP $\Delta A'B'C'$ TO ΔABC, ie., $\Lambda(R(T(\Delta A'B'C'))) = \Delta ABC$.

2. Use the diagram to answer the question below.

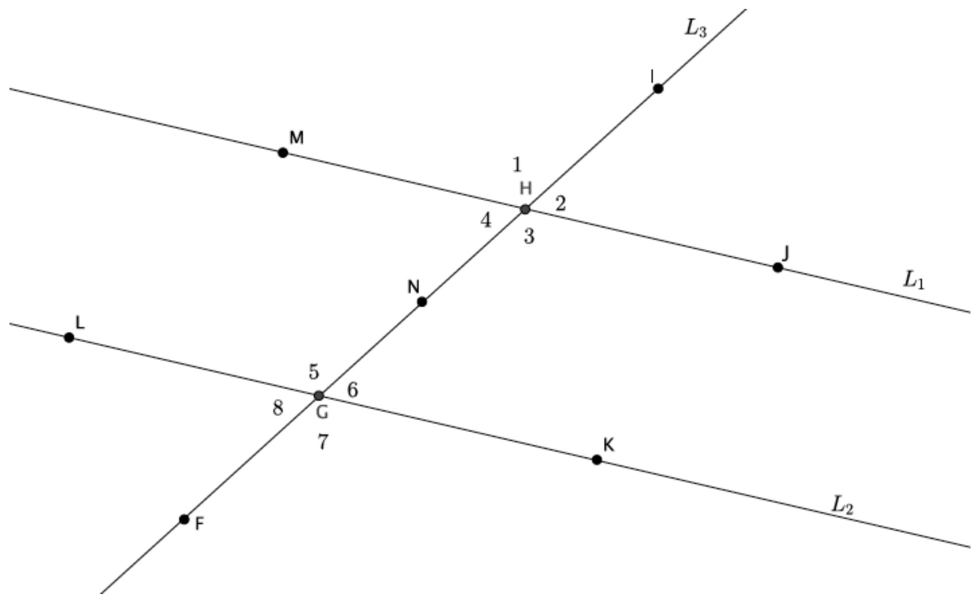
$k||l$



Line k is parallel to line l. $m\angle EDC = 41°$ and $m\angle ABC = 32°$. Find the $m\angle BCD$. Explain in detail how you know you are correct. Add additional lines and points as needed for your explanation.

LET F BE A POINT ON LINE k SO THAT $\angle DCF$ IS A STRAIGHT ANGLE. THEN BECAUSE $k \parallel l$, $\angle EDC \cong \angle CFA$ AND HAVE EQUAL MEASURE. $\angle ABC$ AND $\angle CFA$ ARE THE REMOTE INTERIOR ANGLES OF ΔBCF WHICH MEANS $\angle BCD = \angle ABC + CFA$. THEREFORE $\angle BCD = 32 + 41 = 73°$.

3. Use the diagram below to answer the questions that follow. Lines L_1 and L_2 are parallel, $L_1 \parallel L_2$. Point N is the midpoint of segment GH.



a. If $\angle IHM = 125°$, what is the measure of $\angle IHJ$? $\angle JHN$? $\angle NHM$?

$\angle IHJ = 55°$ $\angle JHN = 125°$ $\angle NHM = 55°$

b. What can you say about the relationship between $\angle 4$ and $\angle 6$? Explain using a basic rigid motion. Name another pair of angles with this same relationship.

$\angle 4$ & $\angle 6$ ARE ALTERNATE INTERIOR ANGLES THAT ARE EQUAL BECAUSE $L_1 // L_2$. LET R BE A ROTATION OF 180° AROUND POINT N. THEN $R(N) = N$; $R(L_3) = L_3$; AND $R(L_1) = L_2$. ROTATIONS ARE DEGREE PRESERVING SO $R(\angle 4) = \angle 6$.

$\angle 3$ & $\angle 5$ ARE ALSO ALTERNATE INTERIOR ANGLES THAT ARE EQUAL.

c. What can you say about the relationship between $\angle 1$ and $\angle 5$? Explain using a basic rigid motion. Name another pair of angles with this same relationship.

$\angle 1$ & $\angle 5$ ARE CORRESPONDING ANGLES THAT ARE EQUAL BECAUSE $L_1 // L_2$. LET T BE THE TRANSLATION ALONG VECTOR $\overrightarrow{GH}$. THEN $T(L_2) = L_1$ AND $T(\angle 5) = \angle 1$.

$\angle 3$ & $\angle 7$ ARE ALSO CORRESPONDING ANGLES THAT ARE EQUAL.

(Optional) Topic D:

The Pythagorean Theorem

8.G.B.6, 8.G.B.7

Focus Standard:	8.G.B.6	Explain a proof of the Pythagorean Theorem and its converse.
	8.G.B.7	Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.
Instructional Days:	2	
Lesson 15:	Informal Proof of the Pythagorean Theorem (S)[1]	
Lesson 16:	Applications of the Pythagorean Theorem (P)	

In Topic D, students are guided through the square within a square proof of the Pythagorean theorem, which requires students to know that congruent figures also have congruent areas. Once proved, students will practice using the Pythagorean theorem and its converse in Lesson 16 to find unknown side lengths in right triangles. Students apply their knowledge of the Pythagorean theorem to real-world problems that involve two-and three-dimensional figures.

[1] Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson

Lesson 15: Informal Proof of the Pythagorean Theorem

Student Outcomes

- Students are introduced to the Pythagorean theorem and will be shown an informal proof of the theorem.
- Students will use the Pythagorean theorem to find the length of the hypotenuse of a right triangle.

Lesson Notes

Since **8.G.B.6** and **8.G.B.7** are post-test standards, this lesson is designated as an extension lesson for this module. However, the content within this lesson is prerequisite knowledge for Module 7. If this lesson is not used with students as part of the work within Module 2, it must be used with students prior to beginning work on Module 7. Please realize that many mathematicians agree that the Pythagorean theorem is the most important theorem in geometry and has immense implications in much of high school mathematics in general (e.g., especially when studying quadratics and trigonometry). It is crucial that students see the teacher explain several proofs of the Pythagorean theorem and practice using it before being expected to produce a proof on their own.

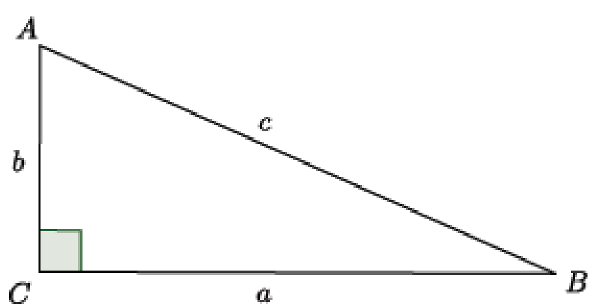
Classwork

Concept Development (5 minutes)

The Pythagorean theorem is a famous theorem that will be used throughout much of high school mathematics. For that reason, you will see several proofs of the theorem throughout the year and have plenty of practice using it. The first thing you need to know about the Pythagorean theorem is what it states.

Pythagorean Theorem: If the lengths of the legs of a right triangle are a and b, and the length of the hypotenuse is c, then $a^2 + b^2 = c^2$.

Given a right triangle ABC with C being the vertex of the right angle, then the sides AC and BC are called the *legs* of ΔABC, and AB is called the *hypotenuse* of ΔABC.



> *Scaffolding:*
>
> Draw arrows, one at a time, to show that each side is the opposite of the given angle.

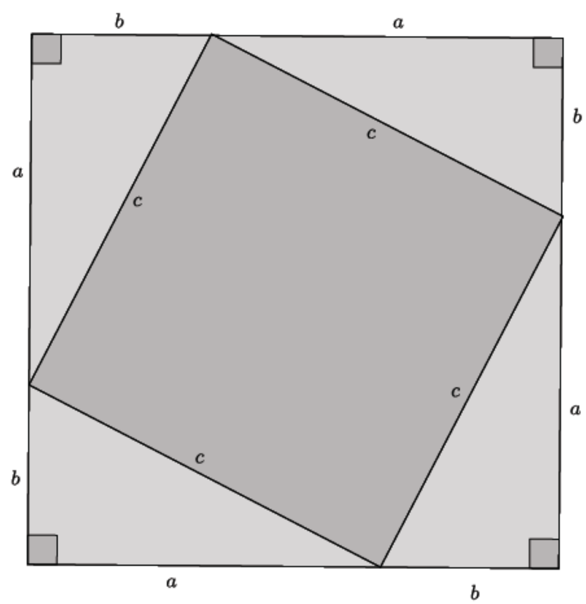
Take note of the fact that side a is opposite the angle A, side b is opposite the angle B, and side c is opposite the angle C.

Discussion (15 minutes)

The first proof of the Pythagorean theorem that you will see requires you to know some basic facts about geometry.

1. Congruent triangles have equal areas.
2. All corresponding parts of congruent triangles are congruent.
3. The triangle sum theorem. (∠ sum of △)
   a. For right triangles, the two angles that are not the right angle have a sum of $90°$. (∠ sum of rt. △)

What we will look at next is what is called a square within a square. The outside square has side lengths $(a+b)$, and the inside square has side lengths c. Our goal is to show that $a^2+b^2=c^2$. To accomplish this goal, we will compare the total area of the outside square with the parts it is comprised of, i.e., the four triangles and the smaller inside square.



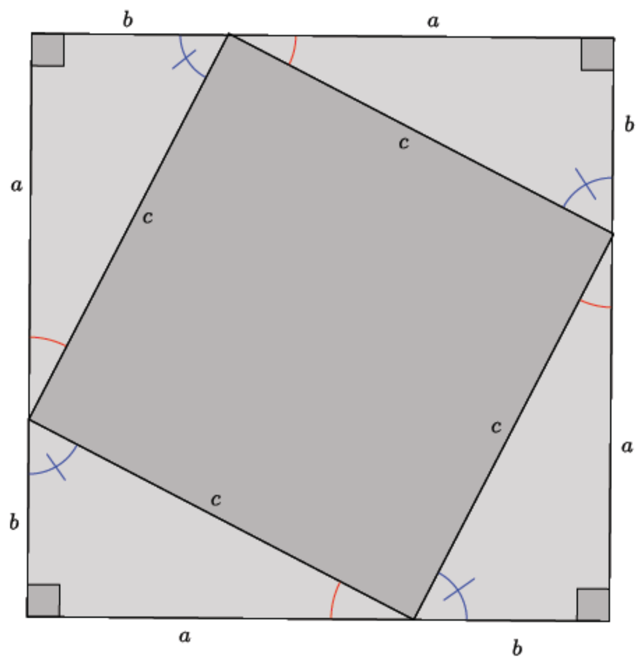
Note to Teacher:

Remind students to use the distributive law to determine the area of the outside square. Also remind them to use what they know about exponential notation to simplify the expression.

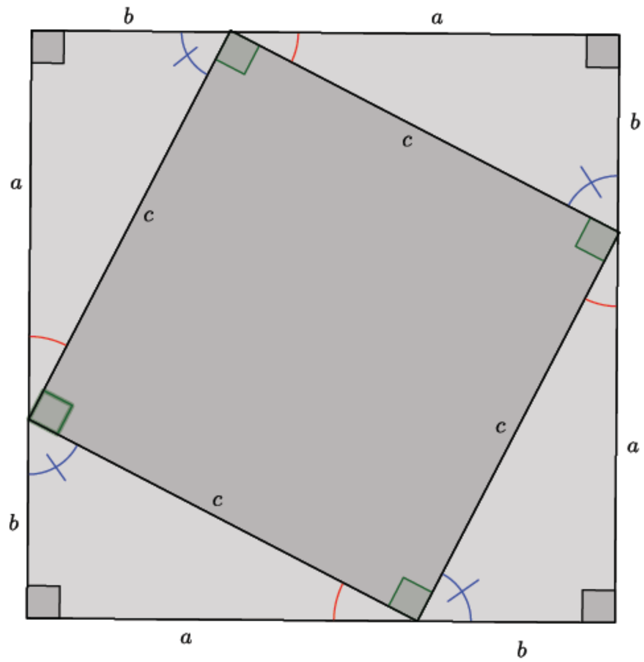
Ask students the following questions during the discussion:

- Looking at the outside square only, the square with side lengths $(a+b)$; what is its area?
  - *The area of the outside square is* $(a+b)^2=a^2+2ab+b^2$.
- Are the four triangles with sides lengths a and b congruent? If so, how do you know?
  - *Yes, the triangles are congruent. From the diagram, we can see that each triangle has a right angle, and each triangle has side lengths of a and b. Our rigid motions will preserve those measures, and we can trace one triangle and use rigid motions to prove that they are congruent.*
- What is the area of just one triangle?
  - $\frac{1}{2}ab$
- Does each triangle have the same area? If so, what is the sum of all four of those areas?
  - *Yes, each triangle has the same area because they are congruent. The sum of all four triangles is* $4\left(\frac{1}{2}ab\right)=2ab$.

- We called this entire figure a square within a square, but we want to make sure that the figure in the center is indeed a square. To do so, we need to look at the angles of the triangles. First of all, what do we know about corresponding angles of congruent triangles?
  - *Corresponding angles of congruent triangles are also congruent and equal in measure.*



- So we know that the angles marked by the red arcs are equal in measure, and the angles marked with the blue arcs and line are equal in measure. What do we know about the sum of the interior angles of a triangle (∠ sum of △)?
  - *The sum of the interior angles of a triangle is* $180°$.
- What is the sum of the two interior angles of a right triangle, not including the right angle (∠ sum of rt. △)? How do you know?
  - *For right triangles, we know that one angle has a measure of* $90°$. *Since the sum of all three angles must be* $180°$, *then we know that the other two angles must have a sum of* $90°$.
- Now look at just one side of the figure. We have an angle with a red arc and an angle with a blue arc. In between them is another angle that we do not know the measure of. All three angles added together make up the straight side of the outside figure. What must be the measure of the unknown angle (the measure of the angle between the red and blue arcs)? How do you know?
  - *Since the angle with the red arc and the angle with the blue arc must have a sum of* $90°$, *and all three angles together must make a straight angle measuring* $180°$ *(∠s on a line), then the unknown angle must equal* $90°$.
- That means that the figure with side lengths c must be a square. It is a figure with four equal sides and four right angles. What is the area of this square?
  - *The area of the square must be* c^2.



- Recall our goal: To show that $a^2 + b^2 = c^2$. To accomplish this goal, we will compare the total area of the outside square with the parts it is comprised of, i.e., the four triangles and the smaller, inside square. Do we have everything we need to accomplish our goal?
  - *Yes, we know the area of the outside square, $(a+b)^2 = a^2 + 2ab + b^2$, the sum of the areas of the four triangles, $4\left(\frac{1}{2}ab\right) = 2ab$, and the area of the inside square, c^2.*

Show students the end of the square within a square proof:

Total area of the outside square = area of four triangles + area of inside square

$$a^2 + 2ab + b^2 = 2ab + c^2$$
$$a^2 + 2ab - 2ab + b^2 = 2ab - 2ab + c^2$$
$$a^2 + b^2 = c^2.$$

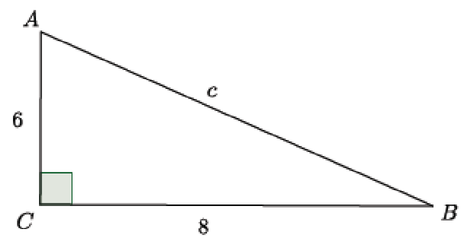
Thus, we have shown the Pythagorean theorem to be true using a square within a square.

Example 1 (2 minutes)

Example 1

Now that we know what the Pythagorean theorem is, let's practice using it to find the length of a hypotenuse of a right triangle.

Determine the length of the hypotenuse of the right triangle.



The Pythagorean theorem states that for right triangles $a^2 + b^2 = c^2$, where a and b are the legs and c is the hypotenuse. Then,

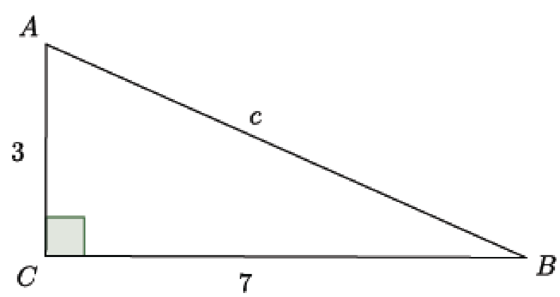
$$a^2 + b^2 = c^2$$
$$6^2 + 8^2 = c^2$$
$$36 + 64 = c^2$$
$$100 = c^2.$$

Since we know that $100 = 10^2$, we can say that the hypotenuse $c = 10$.

Example 2 (3 minutes)

Example 2

Determine the length of the hypotenuse of the right triangle.



- Based on our work in the last example, what should we do to find the length of the hypotenuse?
  - *Use the Pythagorean theorem and replace a and b with 3 and 7. Then,*

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 3^2 + 7^2 &= c^2 \\ 9 + 49 &= c^2 \\ 58 &= c^2. \end{aligned}$$

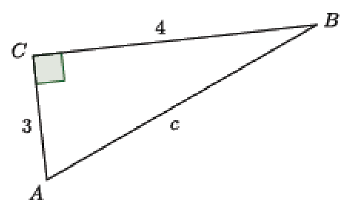
- Since we do not know what number times itself produces 58, for now we can leave our answer as $58 = c^2$. Later this year, we will learn how to determine the actual value for c for problems like this one.

Exercises 1–5 (10 minutes)

Exercises 1–5

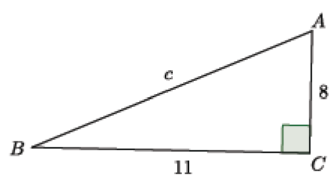
For each of the exercises, determine the length of the hypotenuse of the right triangle shown. Note: Figures not drawn to scale.

1.



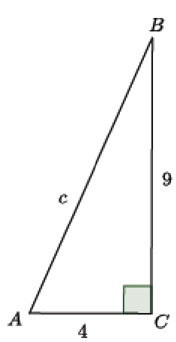
$$\begin{aligned} \boldsymbol{a^2 + b^2} &= \boldsymbol{c^2} \\ \boldsymbol{3^2 + 4^2} &= \boldsymbol{c^2} \\ \boldsymbol{9 + 16} &= \boldsymbol{c^2} \\ \boldsymbol{25} &= \boldsymbol{c^2} \\ \boldsymbol{5} &= \boldsymbol{c} \end{aligned}$$

2.



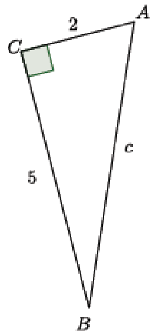
$$\begin{aligned} \boldsymbol{a^2 + b^2} &= \boldsymbol{c^2} \\ \boldsymbol{8^2 + 11^2} &= \boldsymbol{c^2} \\ \boldsymbol{64 + 121} &= \boldsymbol{c^2} \\ \boldsymbol{185} &= \boldsymbol{c^2} \end{aligned}$$

3.



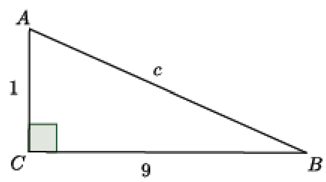
$$\begin{aligned} \boldsymbol{a^2 + b^2} &= \boldsymbol{c^2} \\ \boldsymbol{4^2 + 9^2} &= \boldsymbol{c^2} \\ \boldsymbol{16 + 81} &= \boldsymbol{c^2} \\ \boldsymbol{97} &= \boldsymbol{c^2} \end{aligned}$$

4.



$$a^2 + b^2 = c^2$$
$$2^2 + 5^2 = c^2$$
$$4 + 25 = c^2$$
$$29 = c^2$$

5.



$$a^2 + b^2 = c^2$$
$$1^2 + 9^2 = c^2$$
$$1 + 81 = c^2$$
$$82 = c^2$$

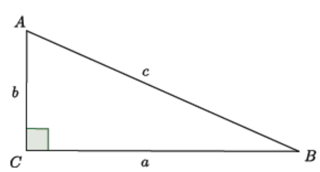
Closing (5 minutes)

Summarize, or have students summarize, the lesson.

- We were shown a proof for the Pythagorean theorem that required us to find the area of four congruent triangles and two squares.
- We learned that right triangles have sides a and b known as legs and a side c known as the hypotenuse.
- We know that for right triangles, $a^2 + b^2 = c^2$.
- We learned how to use the Pythagorean theorem in order to find the length of the hypotenuse of a right triangle.

Lesson Summary

Given a right triangle ABC with C being the vertex of the right angle, then the sides AC and BC are called the *legs* of ΔABC, and AB is called the *hypotenuse* of ΔABC.



Take note of the fact that side a is opposite the angle A, side b is opposite the angle B, and side c is opposite the angle C.

The Pythagorean theorem states that for any right triangle, $a^2 + b^2 = c^2$.

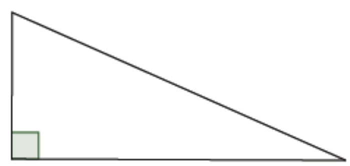
Exit Ticket (5 minutes)

Name ______________________________ Date ____________

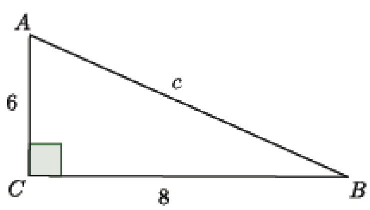
Lesson 15: Informal Proof of the Pythagorean Theorem

Exit Ticket

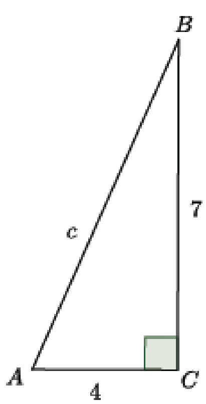
1. Label the sides of the right triangle with leg, leg, and hypotenuse.



2. Determine the length of c in the triangle shown.

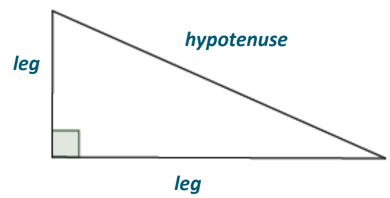


3. Determine the length of c in the triangle shown.

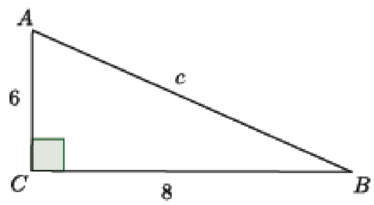


Exit Ticket Sample Solutions

1. **Label the sides of the right triangle with leg, leg, and hypotenuse.**

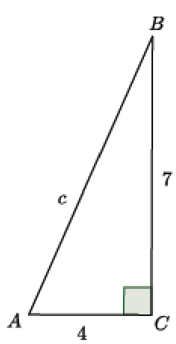


2. **Determine the length of c in the triangle shown.**



$$a^2 + b^2 = c^2$$
$$6^2 + 8^2 = c^2$$
$$36 + 64 = c^2$$
$$100 = c^2$$
$$10 = c$$

3. **Determine the length of c in the triangle shown.**



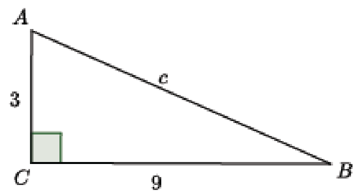
$$a^2 + b^2 = c^2$$
$$4^2 + 7^2 = c^2$$
$$16 + 49 = c^2$$
$$65 = c^2$$

Problem Set Sample Solutions

Students practice using the Pythagorean theorem to find the length of the hypotenuse of a right triangle.

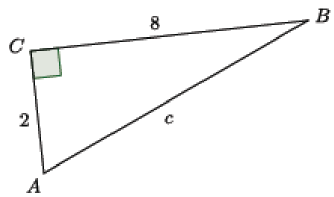
For each of the problems below, determine the length of the hypotenuse of the right triangle shown. Note: Figures not drawn to scale.

1.



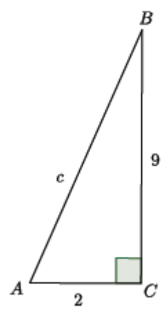
$$a^2 + b^2 = c^2$$
$$3^2 + 9^2 = c^2$$
$$9 + 81 = c^2$$
$$90 = c^2$$

2.



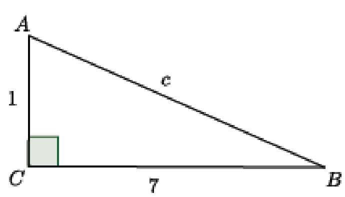
$$a^2 + b^2 = c^2$$
$$8^2 + 2^2 = c^2$$
$$64 + 4 = c^2$$
$$68 = c^2$$

3.



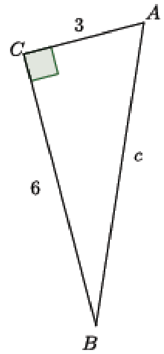
$$a^2 + b^2 = c^2$$
$$9^2 + 2^2 = c^2$$
$$81 + 4 = c^2$$
$$85 = c^2$$

4.



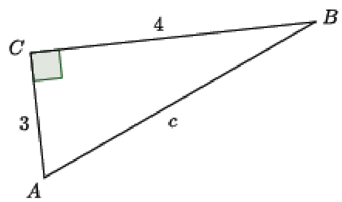
$$a^2 + b^2 = c^2$$
$$7^2 + 1^2 = c^2$$
$$49 + 1 = c^2$$
$$50 = c^2$$

5.



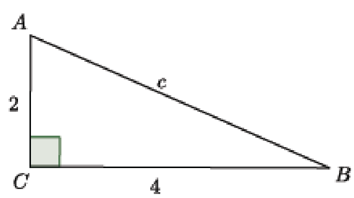
$$\begin{aligned} a^2 + b^2 &= c^2 \\ 6^2 + 3^2 &= c^2 \\ 36 + 9 &= c^2 \\ 45 &= c^2 \end{aligned}$$

6.



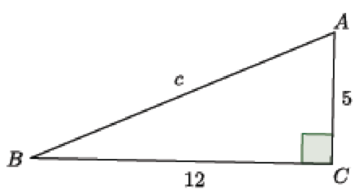
$$\begin{aligned} a^2 + b^2 &= c^2 \\ 4^2 + 3^2 &= c^2 \\ 16 + 9 &= c^2 \\ 25 &= c^2 \\ 5 &= c \end{aligned}$$

7.



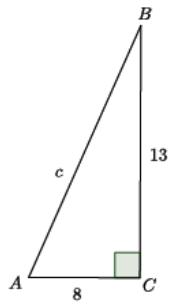
$$\begin{aligned} a^2 + b^2 &= c^2 \\ 4^2 + 2^2 &= c^2 \\ 16 + 4 &= c^2 \\ 20 &= c^2 \end{aligned}$$

8.



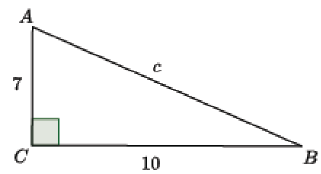
$$\begin{aligned} a^2 + b^2 &= c^2 \\ 12^2 + 5^2 &= c^2 \\ 144 + 25 &= c^2 \\ 169 &= c^2 \\ 13 &= c \end{aligned}$$

9.



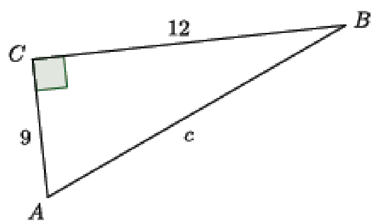
$$a^2 + b^2 = c^2$$
$$13^2 + 8^2 = c^2$$
$$169 + 64 = c^2$$
$$233 = c^2$$

10.



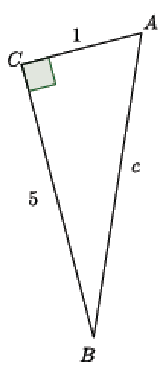
$$a^2 + b^2 = c^2$$
$$10^2 + 7^2 = c^2$$
$$100 + 49 = c^2$$
$$149 = c^2$$

11.



$$a^2 + b^2 = c^2$$
$$12^2 + 9^2 = c^2$$
$$144 + 81 = c^2$$
$$225 = c^2$$
$$15 = c$$

12.



$$a^2 + b^2 = c^2$$
$$5^2 + 1^2 = c^2$$
$$25 + 1 = c^2$$
$$26 = c^2$$

Lesson 16: Applications of the Pythagorean Theorem

Student Outcomes

- Students use the Pythagorean theorem to determine missing side lengths of right triangles.

Lesson Notes

Since **8.G.B.6** and **8.G.B.7** are post-test standards, this lesson is designated as an extension lesson for this module. However, the content within this lesson is prerequisite knowledge for Module 7. If this lesson is not used with students as part of the work within Module 2, it must be used with students prior to beginning work on Module 7. Please realize that many mathematicians agree that the Pythagorean theorem is the most important theorem in geometry and has immense implications in much of high school mathematics in general (e.g., especially when studying quadratics and trigonometry). It is crucial that students see the teacher explain several proofs of the Pythagorean Theorem and practice using it before being expected to produce a proof on their own.

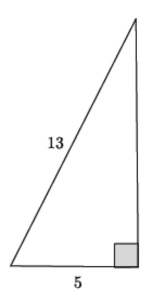
Classwork

Example 1 (4 minutes)

Pythagorean theorem as it applies to missing side lengths of triangles:

Example 1

Given a right triangle with a hypotenuse with length 13 units and a leg with length 5 units, as shown, determine the length of the other leg.



$$\begin{aligned} 5^2 + b^2 &= 13^2 \\ 5^2 - 5^2 + b^2 &= 13^2 - 5^2 \\ b^2 &= 13^2 - 5^2 \\ b^2 &= 169 - 25 \\ b^2 &= 144 \\ b &= 12 \end{aligned}$$

The length of the leg is 12 units.

- Let b represent the missing leg of the right triangle; then, by the Pythagorean theorem:
$$5^2 + b^2 = 13^2.$$
- If we let a represent the missing leg of the right triangle, then by the Pythagorean theorem:
$$a^2 + 5^2 = 13^2.$$

- Which of these two equations is correct: $5^2 + b^2 = 13^2$ or $a^2 + 5^2 = 13^2$?
- It does not matter which equation we use as long as we are showing the sum of the squares of the legs as equal to the square of the hypotenuse.
- Using the first of our two equations, $5^2 + b^2 = 13^2$, what can we do to solve for b in the equation?
  - *We need to subtract* 5^2 *from both sides of the equation.*

$$5^2 + b^2 = 13^2$$
$$5^2 - 5^2 + b^2 = 13^2 - 5^2$$
$$b^2 = 13^2 - 5^2$$

Scaffolding:

If students do not believe that we could use either equation, solve each of them and show that the answer is the same.

- Point out to students that we are looking at the Pythagorean theorem in a form that allows us to find the length of one of the legs of the right triangle. That is, $b^2 = c^2 - a^2$.

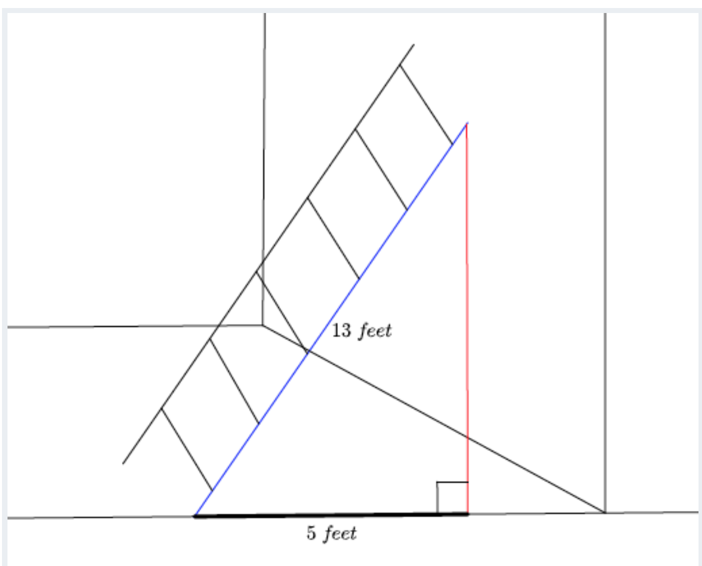
$$b^2 = 13^2 - 5^2$$
$$b^2 = 169 - 25$$
$$b^2 = 144$$
$$b = 12$$

  - *The length of the leg of the right triangle is* 12 *units.*

Example 2 (4 minutes)

Pythagorean theorem as it applies to missing side lengths of triangles in a real-world problem:

- Suppose you have a ladder of length 13 feet. Suppose that to make it sturdy enough to climb, you must place the ladder exactly 5 feet from the wall of a building. You need to post a banner on the building 10 feet above the ground. Is the ladder long enough for you to reach the location you need to post the banner?



The ladder against the wall forms a right angle. For that reason, we can use the Pythagorean theorem to find out how far up the wall the ladder will reach. If we let h represent the height the ladder can reach, what equation will represent this problem?

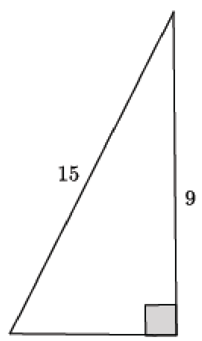
  - $5^2 + h^2 = 13^2$ *or* $h^2 = 13^2 - 5^2$

- Using either equation, we see that this is just like Example 1. We know that the missing side of the triangle is 12 feet. Is the ladder long enough for you to reach the 10-foot banner location?
  - *Yes, the ladder allows us to reach* 12 *feet up the wall.*

Example 3 (3 minutes)

Pythagorean theorem as it applies to missing side lengths of a right triangle:

- Given a right triangle with a hypotenuse of length 15 units and a leg of length 9, what is the length of the other leg?



- If we let the length of the missing leg be represented by a, what equation will allow us to determine its value?
  - $a^2 + 9^2 = 15^2$ *or* $a^2 = 15^2 - 9^2$.
- Finish the computation:

$$a^2 = 225 - 81$$
$$a^2 = 144$$
$$a = 12$$

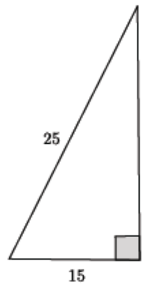
  - *The length of the missing leg of this triangle is* 12 *units.*

Exercises 1–2 (5 minutes)

Students work on Exercises 1 and 2 independently.

Exercises 1–2

1. **Use the Pythagorean theorem to find the missing length of the leg in the right triangle.**

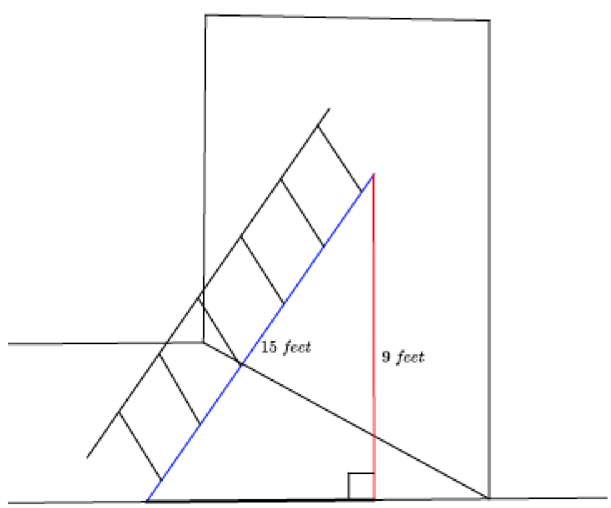


Let b represent the missing leg length; then,

$$15^2 + b^2 = 25^2$$
$$15^2 - 15^2 + b^2 = 25^2 - 15^2$$
$$b^2 = 625 - 225$$
$$b^2 = 400$$
$$b = 20.$$

The length of the leg is 20 units.

2. **You have a 15-foot ladder and need to reach exactly 9 feet up the wall. How far away from the wall should you place the ladder so that you can reach your desired location?**



Let a represent the distance the ladder must be placed from the wall; then,

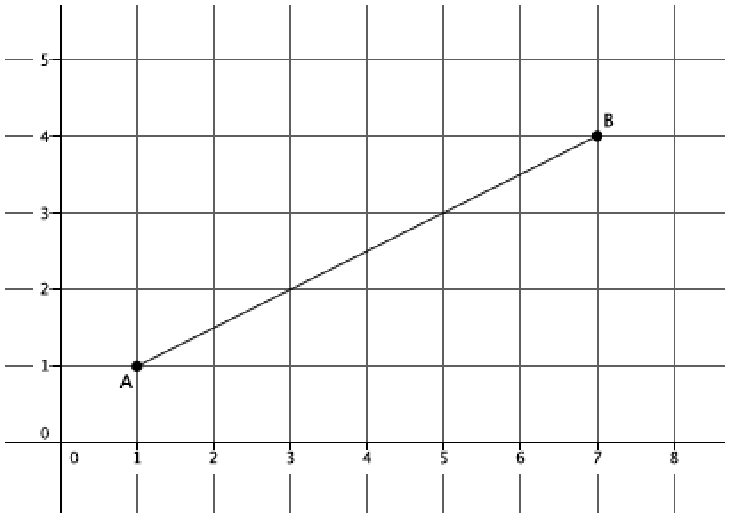
$$a^2 + 9^2 = 15^2$$
$$a^2 + 9^2 - 9^2 = 15^2 - 9^2$$
$$a^2 = 225 - 81$$
$$a^2 = 144$$
$$a = 12.$$

The ladder must be placed exactly 12 feet from the wall.

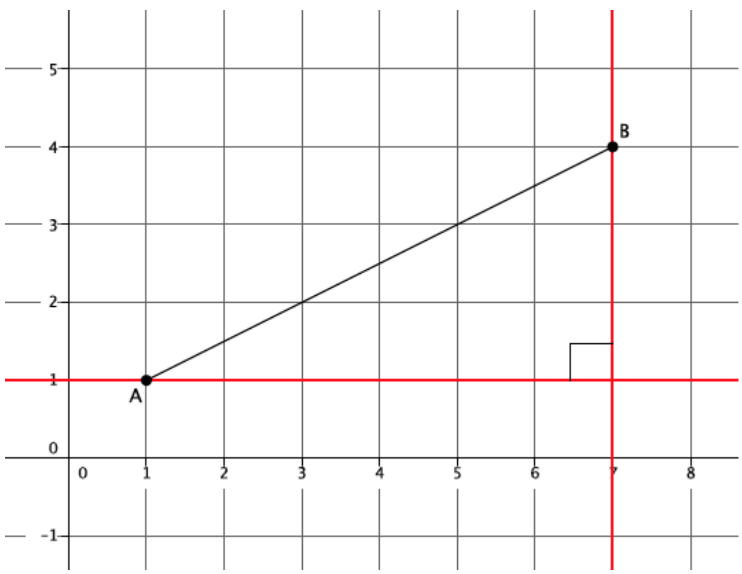
Example 4 (5 minutes)

Pythagorean theorem as it applies to distances on a coordinate plane:

- We want to find the length of the segment AB on the coordinate plane, as shown.

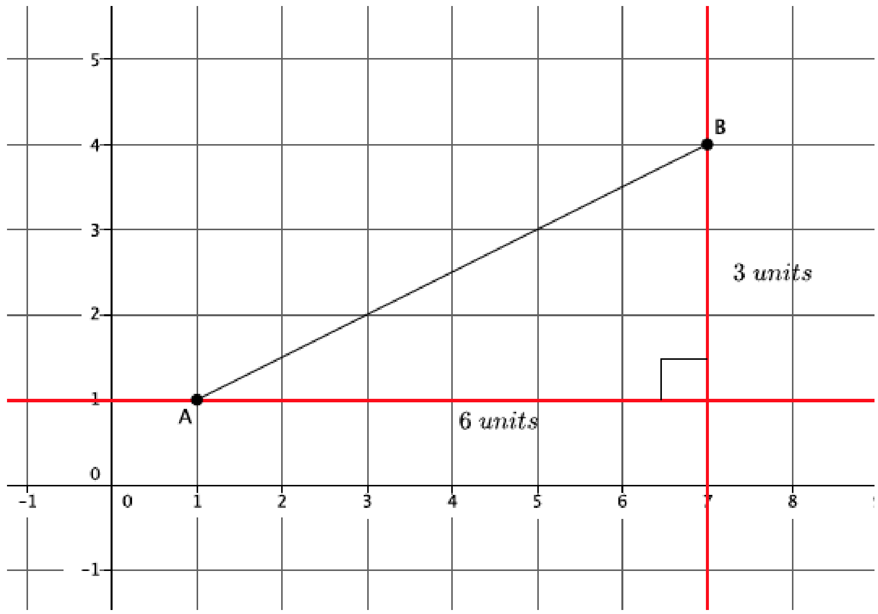


- If we had a right triangle, then we could use the Pythagorean theorem to determine the length of the segment. Let's draw a line parallel to the y-axis through point B. We will also draw a line parallel to the x-axis through point A.



- How can we be sure we have a right triangle?
  - *The coordinate plane is set up so that the intersection of the x-axis and y-axis are perpendicular. The line parallel to the y-axis through B is just a translation of the y-axis. Similarly, the line parallel to the x-axis through A is a translation of the x-axis. Since translations preserve angle measure, the intersection of the two red lines are also perpendicular, meaning we have a $90°$ angle and a right triangle.*

- Now that we are sure we can use the Pythagorean theorem, we need to know the lengths of the legs. Count the units from point A to the right angle and point B to the right angle. What are those lengths?
  - *The base of the triangle is* 6 *units, and the height of the triangle is* 3 *units.*



- What equation can we use to find the length of the segment AB? Let's represent that length by c.
  - $3^2 + 6^2 = c^2$
  - *The length of* c *is*

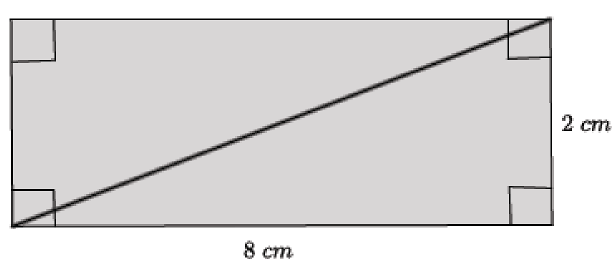
$$\begin{aligned} 3^2 + 6^2 &= c^2 \\ 9 + 36 &= c^2 \\ 45 &= c^2. \end{aligned}$$

- We cannot get a precise answer, so we will leave the length of c as $c^2 = 45$.

Example 5 (3 minutes)

Pythagorean Theorem as it applies to the length of a diagonal in a rectangle:

- Given a rectangle with side lengths of 8 cm and 2 cm, as shown, what is the length of the diagonal?



- If we let the length of the diagonal be represented by c, what equation can we use to find its length?
  - $2^2 + 8^2 = c^2$
  - *The length of c is*

$$\begin{aligned} 2^2 + 8^2 &= c^2 \\ 4 + 64 &= c^2 \\ 68 &= c^2. \end{aligned}$$

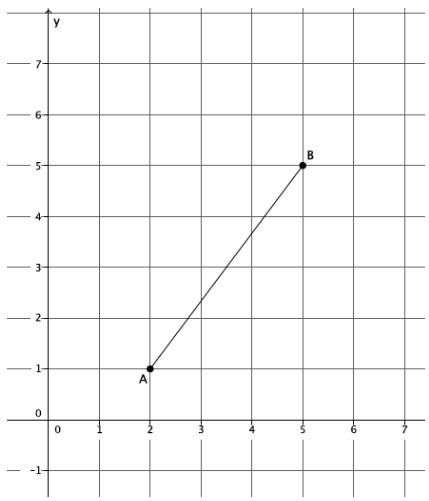
- We cannot get a precise answer, so we will leave the length of c as $c^2 = 68$.

Exercises 3–6 (11 minutes)

Students work independently on Exercises 3–6.

Exercises 3–6

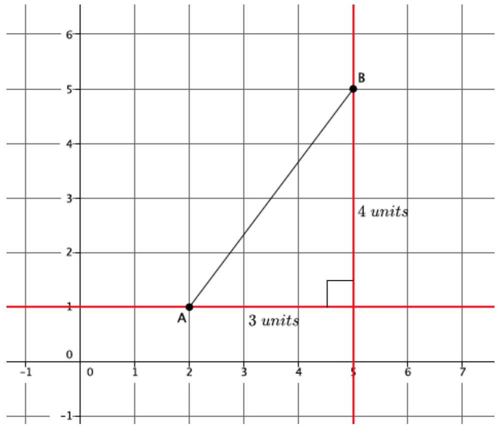
3. Find the length of the segment AB, if possible.



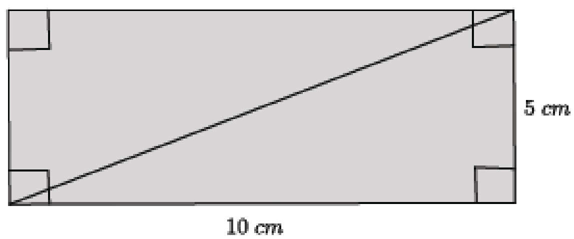
If we let the length of AB be represented by c, then

$$\begin{aligned} 3^2 + 4^2 &= c^2 \\ 9 + 16 &= c^2 \\ 25 &= c^2 \\ 5 &= c \end{aligned}$$

The length of segment AB is 5 units.



4. Given a rectangle with dimensions 5 cm and 10 cm, as shown, find the length of the diagonal, if possible.



Let c represent the length of the diagonal; then,

$$c^2 = 5^2 + 10^2$$
$$c^2 = 25 + 100$$
$$c^2 = 125.$$

We cannot find a precise answer for c, so the length is $(c^2 = 125)$ *cm.*

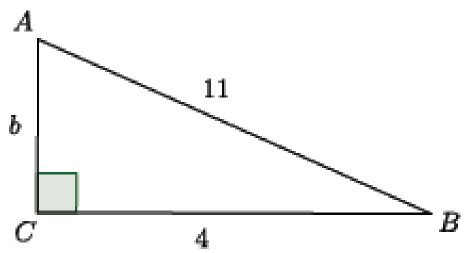
5. A right triangle has a hypotenuse of length 13 in. and a leg with length 4 in. What is the length of the other leg?

If we let a represent the length of the other leg, then

$$a^2 + 4^2 = 13^2$$
$$a^2 + 4^2 - 4^2 = 13^2 - 4^2$$
$$a^2 = 13^2 - 4^2$$
$$a^2 = 169 - 16$$
$$a^2 = 153.$$

We cannot find a precise length for a, so the leg is $(a^2 = 153)$ *in.*

6. **Find the length of b in the right triangle below, if possible.**



By the Pythagorean theorem,

$$\begin{aligned} 4^2 + b^2 &= 11^2 \\ 4^2 - 4^2 + b^2 &= 11^2 - 4^2 \\ b^2 &= 121 - 16 \\ b^2 &= 105. \end{aligned}$$

A precise length for side b cannot be found, so $b^2 = 105$.

Closing (5 minutes)

Summarize, or have students summarize, the lesson.

- We know how to use the Pythagorean theorem to find the length of a missing side of a right triangle whether it be one of the legs or the hypotenuse.
- We know how to apply the Pythagorean theorem to a real life problem like how high a ladder will reach along a wall.
- We know how to find the length of a diagonal of a rectangle.
- We know how to determine the length of a segment that is on the coordinate plane.

Lesson Summary

The Pythagorean theorem can be used to find the unknown length of a leg of a right triangle.

An application of the Pythagorean theorem allows you to calculate the length of a diagonal of a rectangle, the distance between two points on the coordinate plane, and the height that a ladder can reach as it leans against a wall.

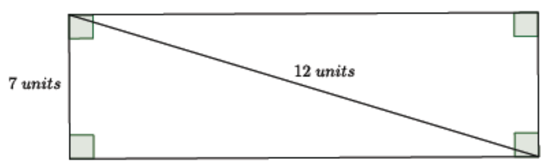
Exit Ticket (5 minutes)

Name ________________________________ Date ______________

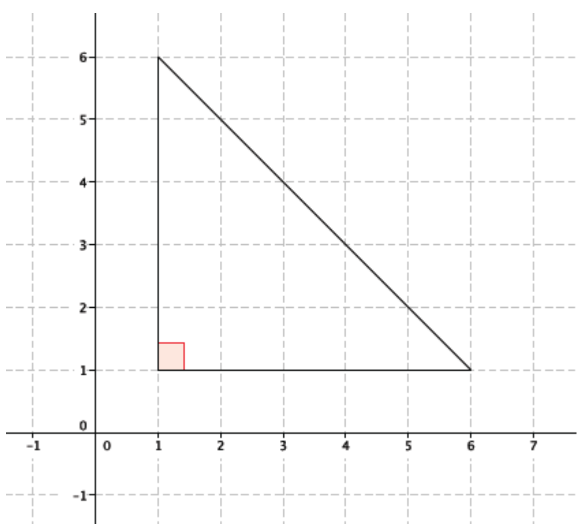
Lesson 16: Applications of the Pythagorean Theorem

Exit Ticket

1. Find the length of the missing side of the rectangle shown below, if possible.

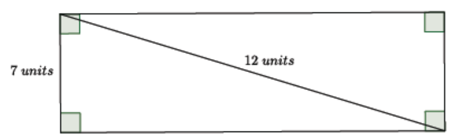


2. Find the length of all three sides of the right triangle shown below, if possible.



Exit Ticket Sample Solutions

1. **Find the length of the missing side of the rectangle shown below, if possible.**

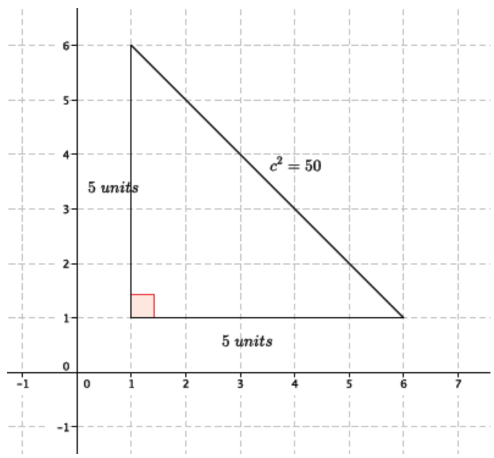


Let a represent the length of the unknown leg. Then,

$$\begin{aligned} a^2 + 7^2 &= 12^2 \\ a^2 + 7^2 - 7^2 &= 12^2 - 7^2 \\ a^2 &= 12^2 - 7^2 \\ a^2 &= 144 - 49 \\ a^2 &= 95. \end{aligned}$$

The precise length of the side cannot be found, but $a^2 = 95$ units.

2. **Find the length of all three sides of the right triangle shown below, if possible.**



The two legs are each 5 units in length. The hypotenuse is

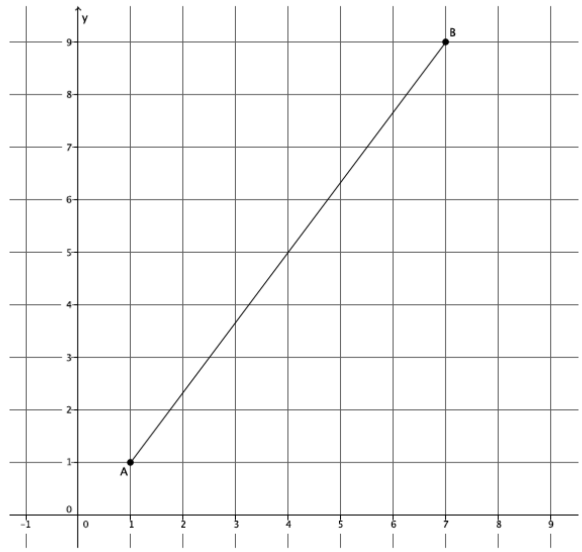
$$\begin{aligned} 5^2 + 5^2 &= c^2 \\ 25 + 25 &= c^2 \\ 50 &= c^2. \end{aligned}$$

The precise length of the hypotenuse cannot be found, but $c^2 = 50$ units.

Problem Set Sample Solutions

Students practice using the Pythagorean theorem to find missing lengths in right triangles.

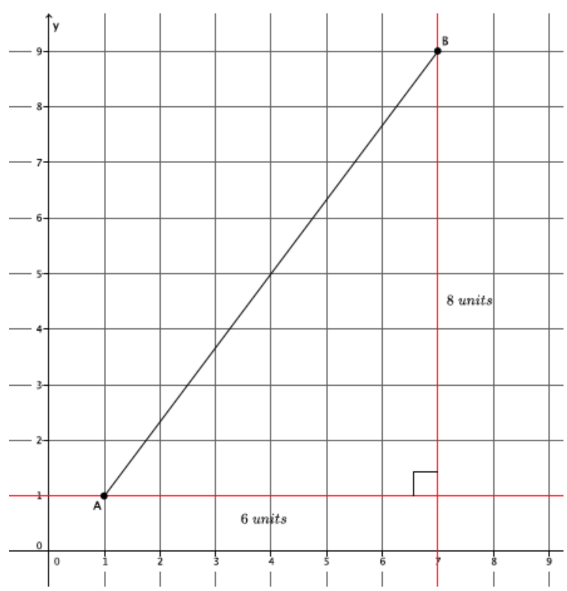
1. **Find the length of the segment AB shown below, if possible.**



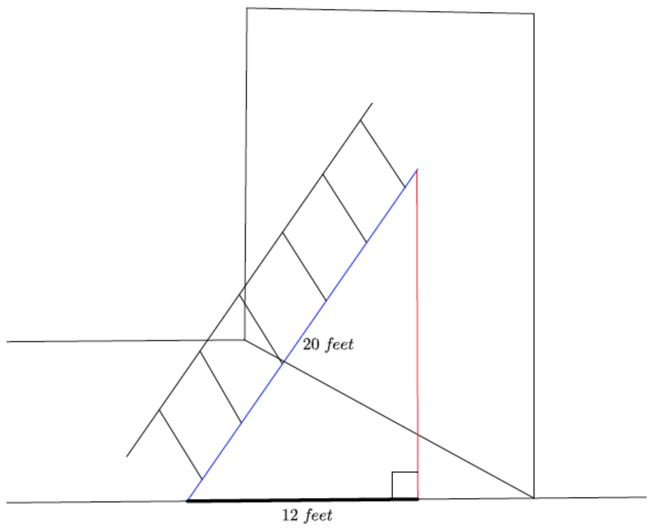
If we let the length of AB be represented by c, then by the Pythagorean theorem

$$6^2 + 8^2 = c^2$$
$$36 + 64 = 100$$
$$100 = c^2$$
$$10 = c.$$

The length of the segment AB is 10 units.



2. A 20-foot ladder is placed 12 feet from the wall, as shown. How high up the wall will the ladder reach?



Let a represent the height up the wall that the ladder will reach. Then,

$$\begin{aligned} a^2 + 12^2 &= 20^2 \\ a^2 + 12^2 - 12^2 &= 20^2 - 12^2 \\ a^2 &= 20^2 - 12^2 \\ a^2 &= 400 - 144 \\ a^2 &= 256 \\ a &= 16. \end{aligned}$$

The ladder will reach 16 feet up the wall.

3. A rectangle has dimensions 6 in. by 12 in. What is the length of the diagonal of the rectangle?

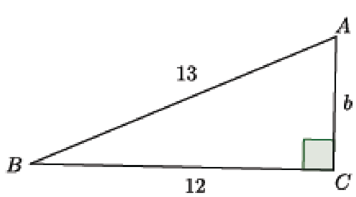
If we let c represent the length of the diagonal, then

$$\begin{aligned} 6^2 + 12^2 &= c^2 \\ 36 + 144 &= c^2 \\ 180 &= c^2. \end{aligned}$$

A precise answer cannot be determined for the length of the diagonal, so we say that ($c^2 = 180$) in.

Use the Pythagorean theorem to find the missing side lengths for the triangles shown in Problems 4–8.

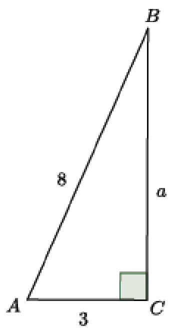
4. Determine the length of the missing side, if possible.



$$\begin{aligned} 12^2 + b^2 &= 13^2 \\ 12^2 - 12^2 + b^2 &= 13^2 - 12^2 \\ b^2 &= 13^2 - 12^2 \\ b^2 &= 169 - 144 \\ b^2 &= 25 \\ b &= 5 \end{aligned}$$

The length of the missing side is 5 units.

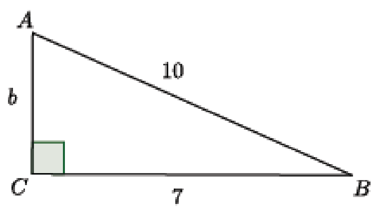
5. **Determine the length of the missing side, if possible.**



$$a^2 + 3^2 = 8^2$$
$$a^2 + 3^2 - 3^2 = 8^2 - 3^2$$
$$a^2 = 8^2 - 3^2$$
$$a^2 = 64 - 9$$
$$a^2 = 55$$

We cannot get a precise answer, but $a^2 = 55$.

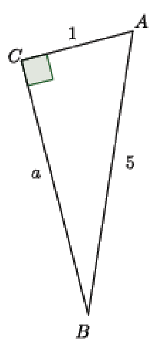
6. **Determine the length of the missing side, if possible.**



$$7^2 + b^2 = 10^2$$
$$7^2 - 7^2 + b^2 = 10^2 - 7^2$$
$$b^2 = 10^2 - 7^2$$
$$b^2 = 100 - 49$$
$$b^2 = 51$$

We cannot get a precise answer, but $b^2 = 51$.

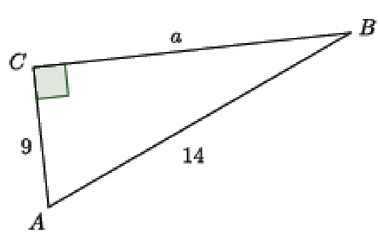
7. **Determine the length of the missing side, if possible.**



$$a^2 + 1^2 = 5^2$$
$$a^2 + 1^2 - 1^2 = 5^2 - 1^2$$
$$a^2 = 5^2 - 1^2$$
$$a^2 = 25 - 1$$
$$a^2 = 24$$

We cannot get a precise answer, but $a^2 = 24$.

8. **Determine the length of the missing side, if possible.**



$$a^2 + 9^2 = 14^2$$
$$a^2 + 9^2 - 9^2 = 14^2 - 9^2$$
$$a^2 = 14^2 - 9^2$$
$$a^2 = 196 - 81$$
$$a^2 = 115$$

We cannot get a precise answer, but $a^2 = 115$.