Lesson 1: Wishful Thinking—Does Linearity Hold?

Classwork

Exercises 1–2

Look at these common mistakes that students make, and answer the questions that follow.

1. If $f(x) = \sqrt{x}$, does $f(a + b) = f(a) + f(b)$, when a and b are not negative?
   a. Can we find a counterexample to refute the claim that $f(a + b) = f(a) + f(b)$ for all nonnegative values of a and b?

   b. Find some nonnegative values for a and b for which the statement, by coincidence, happens to be true.

   c. Find all values of a and b for which the statement is true. Explain your work and the results.

d. Why was it necessary for us to consider only nonnegative values of a and b?

e. Does $f(x) = \sqrt{x}$ display ideal linear properties? Explain.

2. If $f(x) = x^3$, does $f(a+b) = f(a) + f(b)$?

   a. Substitute in some values of a and b to show this statement is not true in general.

   b. Find some values for a and b for which the statement, by coincidence, happens to work.

   c. Find all values of a and b for which the statement is true. Explain your work and the results.

d. Is this true for all positive and negative values of a and b? Explain and prove by choosing positive and negative values for the variables.

e. Does $f(x) = x^3$ display ideal linear properties? Explain.

Problem Set

Study the statements given in Problems 1–3. Prove that each statement is false, and then find all values of a and b for which the statement is true. Explain your work and the results.

1. If $f(x) = x^2$, does $f(a+b) = f(a) + f(b)$?

2. If $f(x) = x^{\frac{1}{3}}$, does $f(a+b) = f(a) + f(b)$?

3. If $f(x) = \sqrt{4x}$, does $f(a+b) = f(a) + f(b)$?

4. Think back to some mistakes that you have made in the past simplifying or expanding functions. Write the statement that you assumed was correct that was not, and find numbers that prove your assumption was false.

Lesson 2: Wishful Thinking—Does Linearity Hold?

Classwork

Exercises 1–5

1. Let $f(x) = \sin x$. Does $f(2x) = 2f(x)$ for all values of x? Is it true for any values of x? Show work to justify your answer.

2. Let $f(x) = \log(x)$. Find a value for a such that $f(2a) = 2f(a)$. Is there one? Show work to justify your answer.

3. Let $f(x) = 10^x$. Show that $f(a + b) = f(a) + f(b)$ is true for $a = b = \log(2)$ and that it is not true for $a = b = 2$.

4. Let $f(x) = \frac{1}{x}$. Are there any real numbers a and b so that $f(a + b) = f(a) + f(b)$? Explain.

5. What do your findings from these Exercises illustrate about the linearity of these functions? Explain.

Problem Set

Examine the equations given in Problems 1–4, and show that the functions $f(x) = \cos x$ and $f(x) = \tan x$ are not linear transformations by demonstrating that they do not satisfy the conditions indicated for all real numbers. Then, find values of x and/or y for which the statement holds true.

1. $\cos(x + y) = \cos(x) + \cos(y)$

2. $\cos(2x) = 2\cos(x)$

3. $\tan(x + y) = \tan(x) + \tan(y)$

4. $\tan(2x) = 2\tan(x)$

5. Let $f(x) = \frac{1}{x^2}$, are there any real numbers a and b so that $f(a + b) = f(a) + f(b)$? Explain.

6. Let $f(x) = \log x$, find values of a such that $f(3a) = 3f(a)$.

7. Let $f(x) = \log x$, find values of a such that $f(ka) = kf(a)$.

8. Based on your results from the previous two problems, form a conjecture about whether $f(x) = \log x$ represents a linear transformation.

9. Let $f(x) = ax^2 + bx + c$.
   a. Describe the set of all values for a, b, and c that make $f(x + y) = f(x) + f(y)$ valid for all real numbers x and y.
   b. What does your result indicate about the linearity of quadratic functions?

Trigonometry Table

Angles Measure (x degrees)	Angle Measure (x radians)	$\sin(x)$	$\cos(x)$
0			
30			
	$\frac{\pi}{4}$		
	$\frac{\pi}{3}$		
90			

Lesson 3: Which Real Number Functions Define a Linear Transformation?

Classwork

Opening Exercises

Recall from the previous two lessons that a linear transformation is a function f that satisfies two conditions: (1) $f(x+y)=f(x)+f(y)$ and (2) $f(kx)=kf(x)$. Here, k refers to any real number, and x and y represent arbitrary elements in the domain of f.

1. Let $f(x)=x^2$. Is f a linear transformation? Explain why or why not.

2. Let $g(x)=\sqrt{x}$. Is g a linear transformation? Explain why or why not.

Problem Set

1. Suppose you have a linear transformation $f: \mathbb{R} \to \mathbb{R}$, where $f(2) = 1$ and $f(4) = 2$.
   a. Use the addition property to compute $f(6)$, $f(8)$, $f(10)$, and $f(12)$.
   b. Find $f(20)$, $f(24)$, and $f(30)$. Show your work.
   c. Find $f(-2)$, $f(-4)$, and $f(-8)$. Show your work.
   d. Find a formula for $f(x)$.
   e. Draw the graph of the function $f(x)$.

2. The symbol $\mathbb{Z}$ represents the set of integers, and so $g: \mathbb{Z} \to \mathbb{Z}$ represents a function that takes integers as inputs and produces integers as outputs. Suppose that a function $g: \mathbb{Z} \to \mathbb{Z}$ satisfies $g(a + b) = g(a) + g(b)$ for all integers a and b. Is there necessarily an integer k such that $g(n) = kn$ for all integer inputs n?
   a. Let $k = g(1)$. Compute $g(2)$ and $g(3)$.
   b. Let n be any positive integer. Compute $g(n)$.
   c. Now consider $g(0)$. Since $g(0) = g(0 + 0)$, what can you conclude about $g(0)$?
   d. Lastly, use the fact that $g(n + -n) = g(0)$ to learn something about $g(-n)$, where n is any positive integer.
   e. Use your work above to prove that $g(n) = kn$ for every integer n. Be sure to consider the fact that n could be positive, negative, or 0.

3. In the following problems, be sure to consider all kinds of functions: polynomial, rational, trigonometric, exponential, logarithmic, etc.
   a. Give an example of a function $f: \mathbb{R} \to \mathbb{R}$ that satisfies $f(x \cdot y) = f(x) + f(y)$.
   b. Give an example of a function $g: \mathbb{R} \to \mathbb{R}$ that satisfies $g(x + y) = g(x) \cdot g(y)$.
   c. Give an example of a function $h: \mathbb{R} \to \mathbb{R}$ that satisfies $h(x \cdot y) = h(x) \cdot h(y)$.

Lesson 4: An Appearance of Complex Numbers

Classwork

Opening Exercise

Is $R(x) = \frac{1}{x}$ a linear transformation? Explain how you know.

Exercises

1. Solve $5x^2 - 3x + 17 = 9$.

2. Use the fact that $i^2 = -1$ to show that $i^3 = -i$. Interpret this statement geometrically.

3. Calculate i^6.

4. Calculate i^5.

Problem Set

1. Solve the equation below.
   $5x^2 - 7x + 8 = 2$

2. Consider the equation $x^3 = 8$.
   a. What is the first solution that comes to mind?
   b. It may not be easy to tell at first, but this equation actually has three solutions. To find all three solutions, it is helpful to consider $x^3 - 8 = 0$, which can be rewritten as $(x-2)(x^2+2x+4) = 0$ (check this for yourself). Find all of the solutions to this equation.

3. Make a drawing that shows the first 5 powers of i (i.e., $i^1, i^2, \ldots, i^5$), and then confirm your results algebraically.

4. What is the value of i^{99}? Explain your answer using words or drawings.

5. What is the geometric effect of multiplying a number by $-i$? Does your answer make sense to you? Give an explanation using words or drawings.

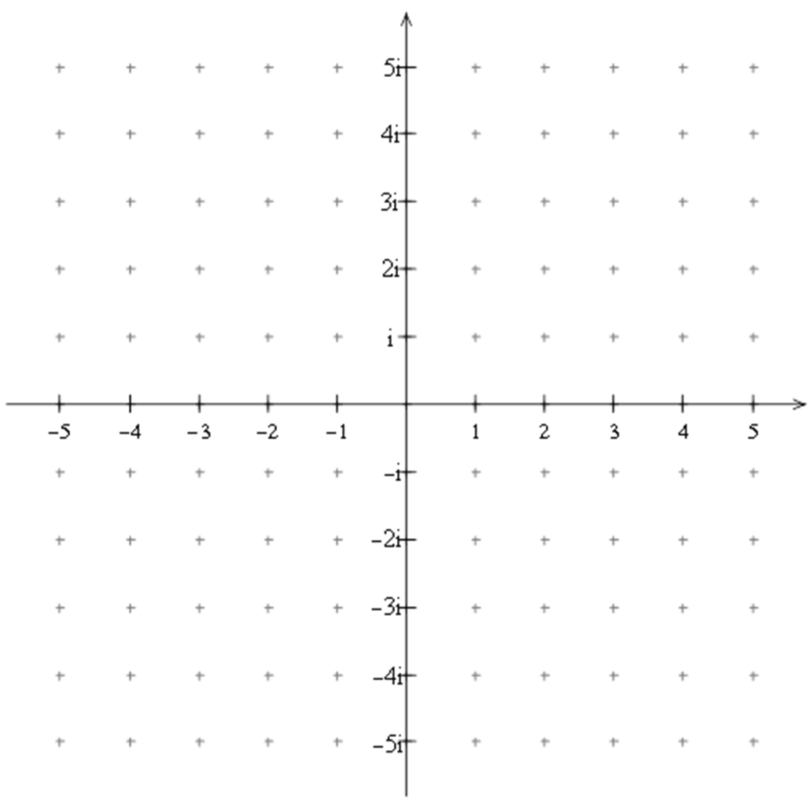
Lesson 5: An Appearance of Complex Numbers

Classwork

Opening Exercise

Write down two fundamental facts about i that you learned in the previous lesson.

Discussion: Visualizing Complex Numbers



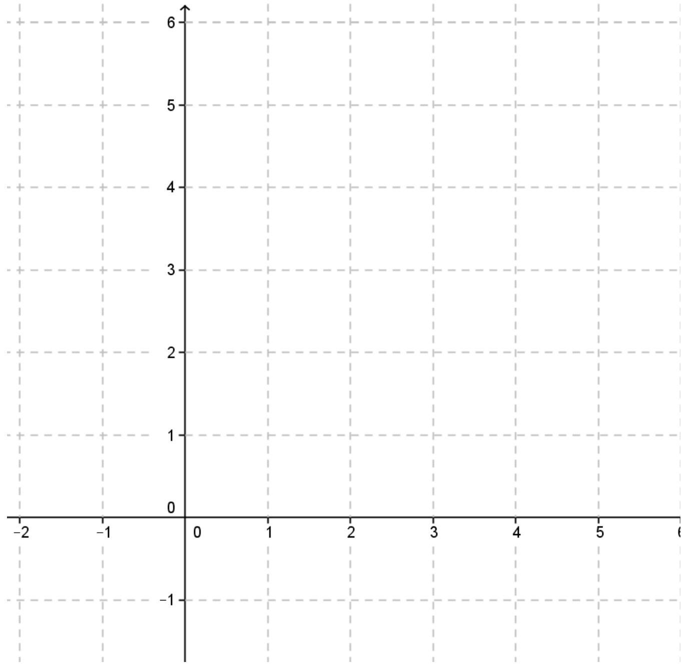
Exercises

1. Give an example of a real number, an imaginary number, and a complex number. Use examples that have not already been discussed in the lesson.

2. In the complex plane, what is the horizontal axis used for? What is the vertical axis used for?

3. How would you represent $-4+3i$ in the complex plane?

For Exercises 4–7, let $a = 1 + 3i$ and $b = 2 - i$.



4. Find $a + b$. Then plot a, b, and $a + b$ in the complex plane.

5. Find $a - b$. Then plot a, b, and $a - b$ in the complex plane.

6. Find $2a$. Then plot a and $2a$ in the complex plane.

7. Find $a \cdot b$. Then plot a, b, and $a \cdot b$ in the complex plane.

Problem Set

1. The number 5 is a real number. Is it also a complex number? Try to find values of a and b so that $5 = a + bi$.

2. The number $3i$ is an imaginary number and a multiple of i. Is it also a complex number? Try to find values of a and b so that $3i = a + bi$.

3. Daria says that "every real number is a complex number." Do you agree with her? Why or why not?

4. Colby says that "every imaginary number is a complex number." Do you agree with him? Why or why not?

In Problems 5–9, perform the indicated operations. Report each answer as a complex number $w = a + bi$, and graph it in a complex plane.

5. Given $z_1 = -9 + 5i$, $z_2 = -10 - 2i$, find $\mathrm{w} = z_1 + z_2$, and graph z_1, z_2, and w.

6. Given $z_1 = -4 + 10i$,$z_2 = -7 - 6i$, find $\mathrm{w} = z_1 - z_2$, and graph z_1, z_2, and w.

7. Given $z_1 = 3\sqrt{2} + 2i$, $z_2 = \sqrt{2} - i$, find $\mathrm{w} = z_1 - z_2$, and graph z_1, z_2, and w.

8. Given $z_1 = 3$, $z_2 = -4 + 8i$, find $\mathrm{w} = z_1 \cdot z_2$, and graph z_1, z_2, and w.

9. Given $z_1 = \frac{1}{4}$, $z_2 = 12 - 4i$, find $\mathrm{w} = z_1 \cdot z_2$, and graph z_1, z_2, and w.

10. Given $z_1 = -1$, $z_2 = 3 + 4i$, find $\mathrm{w} = z_1 \cdot z_2$, and graph z_1, z_2, and w.

11. Given $z_1 = 5 + 3i$, $z_2 = -4 - 2i$, find $\mathrm{w} = z_1 \cdot z_2$, and graph z_1, z_2, and w.

12. Given $z_1 = 1 + i$, $z_2 = 1 + i$, find $\mathrm{w} = z_1 \cdot z_2$, and graph z_1, z_2, and w.

13. Given $z_1 = 3$, $z_2 = i$, find $\mathrm{w} = z_1 \cdot z_2$, and graph z_1, z_2, and w.

14. Given $z_1 = 4 + 3i$, $z_2 = i$, find $\mathrm{w} = z_1 \cdot z_2$, and graph z_1, z_2, and w.

15. Given $z_1 = 2\sqrt{2} + 2\sqrt{2}i$, $z_2 = -\sqrt{2} + \sqrt{2}i$, find $\mathrm{w} = z_1 \cdot z_2$, and graph z_1, z_2, and w.

16. Represent $w = -4 + 3i$ as a point in the complex plane.

17. Represent $2w$ as a point in the complex plane. $2w = 2(-4 + 3i) = -8 + 6i$

18. Compare the positions of w and $2w$ from Problems 10 and 11. Describe what you see. (Hint: Draw a segment from the origin to each point.)

Lesson 6: Complex Numbers as Vectors

Classwork

Opening Exercises

Perform the indicated arithmetic operations for complex numbers $z = -4 + 5i$ and $w = -1 - 2i$.

a. $z + w$

b. $z - w$

c. $z + 2w$

d. $z - z$

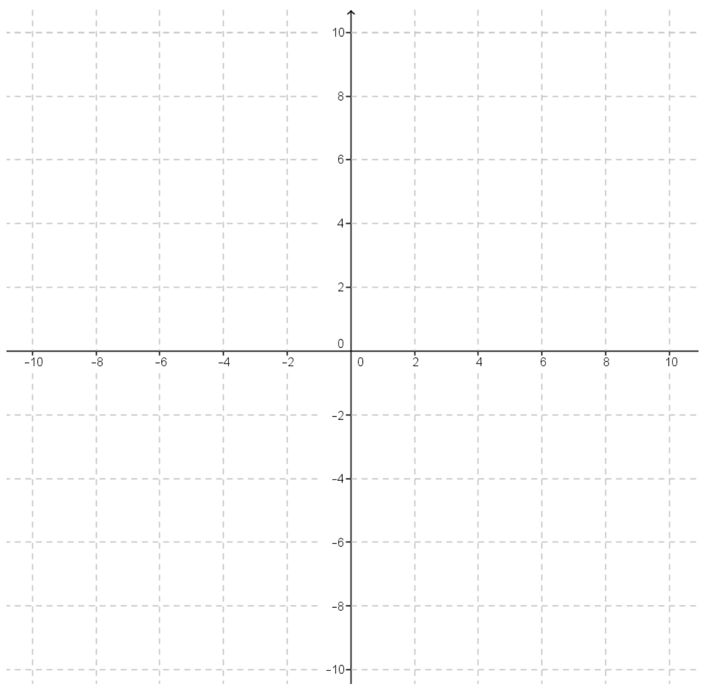
e. Explain how you add and subtract complex numbers.

Exercise 1

1. The length of the vector that represents $z_1 = 6 - 8i$ is 10 because $\sqrt{6^2 + (-8)^2} = \sqrt{100} = 10$.

   a. Find at least seven other complex numbers that can be represented as vectors that have length 10.

   b. Draw the vectors on the coordinate axes provided below.



   c. What do you observe about all of these vectors?

2. In the Opening Exercises, we computed $z + 2w$. Calculate this sum using vectors.

3. In the Opening Exercises, we also computed $z - z$. Calculate this sum using vectors.

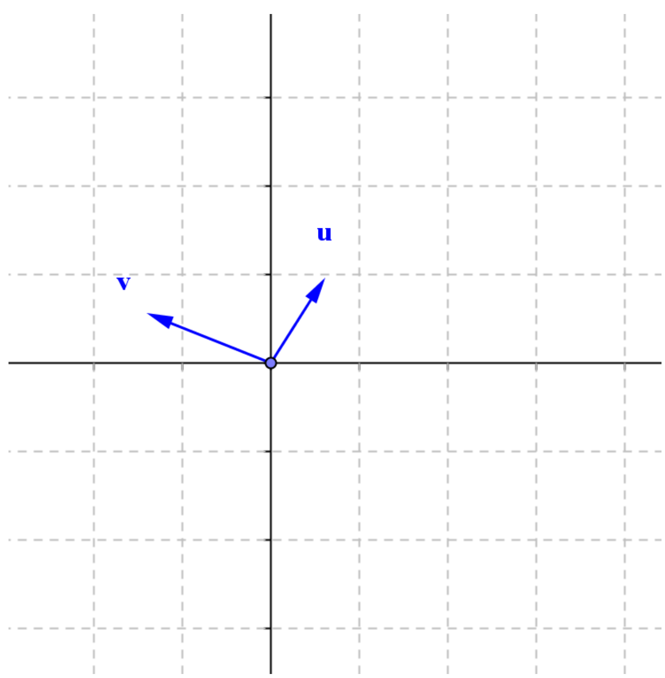
4. For the vectors u and v pictured below, draw the specified sum or difference on the coordinate axes provided.

   a. $u + v$

   b. $v - u$

   c. $2u - v$

   d. $-u - 3v$



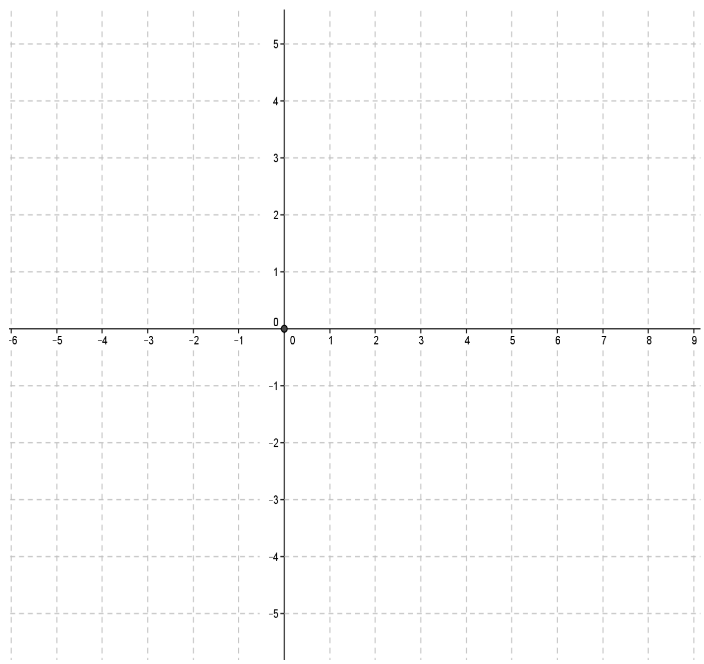
5. Find the sum of $4 + i$ and $-3 + 2i$ geometrically.

6. Show that $(7 + 2i) - (4 - i) = 3 + 3i$ by representing the complex numbers as vectors.

Problem Set

1. Let $z = 1 + i$ and $w = 1 - 3i$. Find the following. Express your answers in $a + bi$ form.
   a. $z + w$
   b. $z - w$
   c. $4w$
   d. $3z + w$
   e. $-w - 2z$
   f. What is the length of the vector representing z?
   g. What is the length of the vector representing w?

2. Let $u = 3 + 2i$, $v = 1 + i$, and $w = -2 - i$. Find the following. Express your answer in $a + bi$ form, and represent the result in the plane.
   a. $u - 2v$
   b. $u - 2w$
   c. $u + v + w$
   d. $u - v + w$
   e. What is the length of the vector representing u?
   f. What is the length of the vector representing $u - v + w$?



3. Find the sum of $-2 - 4i$ and $5 + 3i$ geometrically.

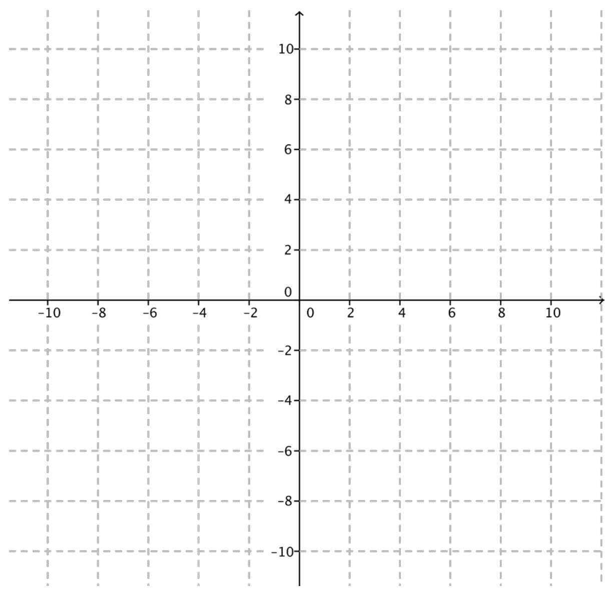
4. Show that $(-5 - 6i) - (-8 - 4i) = 3 - 2i$ by representing the complex numbers as vectors.

5. Let $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$, and $z_3 = a_3 + b_3 i$. Prove the following using algebra or by showing with vectors.
   a. $z_1 + z_2 = z_2 + z_1$
   b. $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

6. Let $z = -3 - 4i$ and $w = -3 + 4i$.
   a. Draw vectors representing z and w on the same set of axes.
   b. What are the lengths of the vectors representing z and w?
   c. Find a new vector, u_z, such that u_z is equal to z divided by the length of the vector representing z.
   d. Find u_w, such that u_w is equal to w divided by the length of the vector representing w.
   e. Draw vectors representing u_z and u_w on the same set of axes as part (a).
   f. What are the lengths of the vectors representing u_z and u_w?
   g. Compare the vectors representing u_z to z and u_w to w. What do you notice?
   h. What is the value of u_z times u_w?
   i. What does your answer to part (h) tell you about the relationship between u_z and u_w?

7. Let $z = a + bi$.
   a. Let u_z be represented by the vector in the direction of z with length 1. How can you find u_z? What is the value of u_z?
   b. Let u_w be the complex number that when multiplied by u_z, the product is 1. What is the value of u_w?
   c. What number could we multiply z by to get a product of 1?

8. Let $z = -3 + 5i$.
   a. Draw a picture representing $z + w = 8 + 2i$.
   b. What is the value of w?



Lesson 7: Complex Number Division

Classwork

Opening Exercise

Perform the indicated operations. Write your answer in $a + bi$ form. Identify the real part of your answer and the imaginary part of your answer.

a. $(2 + 3i) + (-7 - 4i)$

b. $i^2(-4i)$

c. $3i - (-2 + 5i)$

d. $(3 - 2i)(-7 + 4i)$

e. $(-4 - 5i)(-4 + 5i)$

Exercises

1. What is the multiplicative inverse of $2i$?

2. Find the multiplicative inverse of $5 + 3i$.

State the conjugate of each number, and then using the general formula for the multiplicative inverse of $z = a + bi$, find the multiplicative inverse.

3. $3 + 4i$

4. $7 - 2i$

5. i

6. 2

7. Show that $a = -1 + \sqrt{3}i$ and $b = 2$ satisfy $\frac{1}{a+b} = \frac{1}{a} + \frac{1}{b}$.

Problem Set

1. State the conjugate of each complex number. Then find the multiplicative inverse of each number, and verify by multiplying by $a + bi$ and solving a system of equations.
   a. $-5i$
   b. $5 - \sqrt{3}i$

2. Find the multiplicative inverse of each number, and verify using the general formula to find multiplicative inverses of numbers of the form $z = a + bi$.
   a. i^3
   b. $\frac{1}{3}$
   c. $\frac{\sqrt{3} - i}{4}$
   d. $1 + 2i$
   e. $4 - 3i$
   f. $2 + 3i$
   g. $-5 - 4i$
   h. $-3 + 2i$
   i. $\sqrt{2} + i$
   j. $3 - \sqrt{2} \cdot i$
   k. $\sqrt{5} + \sqrt{3} \cdot i$

3. Given $z_1 = 1 + i$ and $z_2 = 2 + 3i$.
   a. Let $w = z_1 \cdot z_2$. Find w and the multiplicative inverse of w.
   b. Show that the multiplicative inverse of w is the same as the product of the multiplicative inverses of z_1 and z_2.

Lesson 8: Complex Number Division

Classwork

Opening Exercises

Use the general formula to find the multiplicative inverse of each complex number.

a. $2 + 3i$

b. $-7 - 4i$

c. $-4 + 5i$

Exercises 1–4

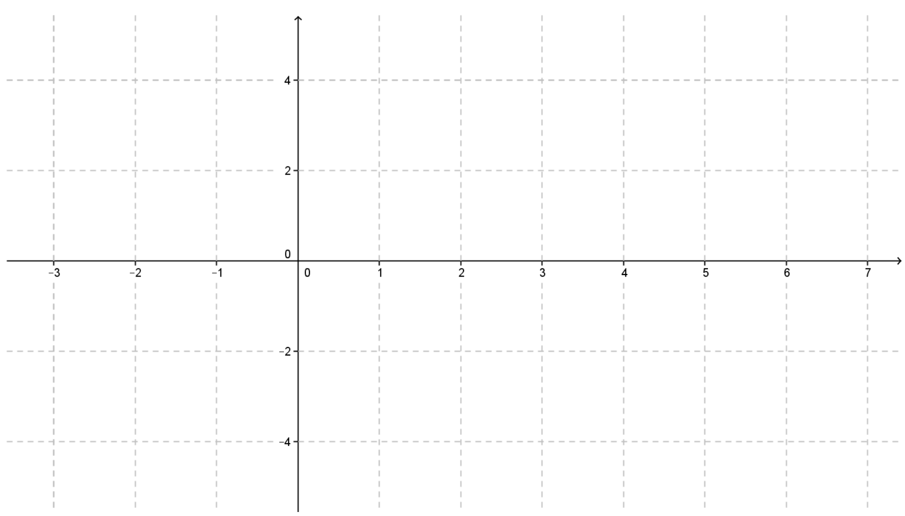
Find the conjugate, and plot the complex number and its conjugate in the complex plane. Label the conjugate with a prime symbol.

1. A: $3 + 4i$

2. B: $-2 - i$

3. C: 7

4. D: $4i$



Exercises 5–8

Find the modulus.

5. $3 + 4i$

6. $-2 - i$

7. 7

8. $4i$

Exercises 9–11

Given $z = a + bi$.

9. Show that for all complex numbers z, $|iz| = |z|$.

10. Show that for all complex numbers z, $z \cdot \bar{z} = |z|^2$.

11. Explain the following: Every nonzero complex number z has a multiplicative inverse. It is given by $\frac{1}{z} = \frac{\bar{z}}{|z|}$.

Example 1

$$\frac{2 - 6i}{2 + 5i}$$

Exercises 12–13

Divide.

12. $\frac{3 + 2i}{-2 - 7i}$

13. $\frac{3}{3 - i}$

Problem Set

1. Let $z = 4 - 3i$ and $w = 2 - i$. Show that
   a. $|z| = |\overline{z}|$
   b. $\left|\frac{1}{z}\right| = \frac{1}{|\overline{z}|}$
   c. If $|z| = 0$, must it be that $z = 0$?
   d. Give a specific example to show that $|z + w|$ usually does not equal $|z| + |w|$.

2. Divide.
   a. $\frac{1 - 2i}{2i}$
   b. $\frac{5 - 2i}{5 + 2i}$
   c. $\frac{\sqrt{3} - 2i}{-2 - \sqrt{3}i}$

3. Prove that $|zw| = |z| \cdot |w|$ for complex numbers z and w.

4. Given $z = 3 + i$, $w = 1 + 3$.
   a. Find $z + w$, and graph z, w, and $z + w$ on the same complex plane. Explain what you discover if you draw line segments from the origin to those points z,w, and $z + w$. Then draw line segments to connect w to $z + w$, and $z + w$ to z.
   b. Find $-w$, and graph z, w, and $z - w$ on the same complex plane. Explain what you discover if you draw line segments from the origin to those points z, w, and $z - w$. Then draw line segments to connect w to $z - w$, and $z - w$ to z.

5. Explain why $|z + w| \leq |z| + |w|$ and $|z - w| \leq |z| + |w|$ geometrically. (Hint: Triangle inequality theorem)

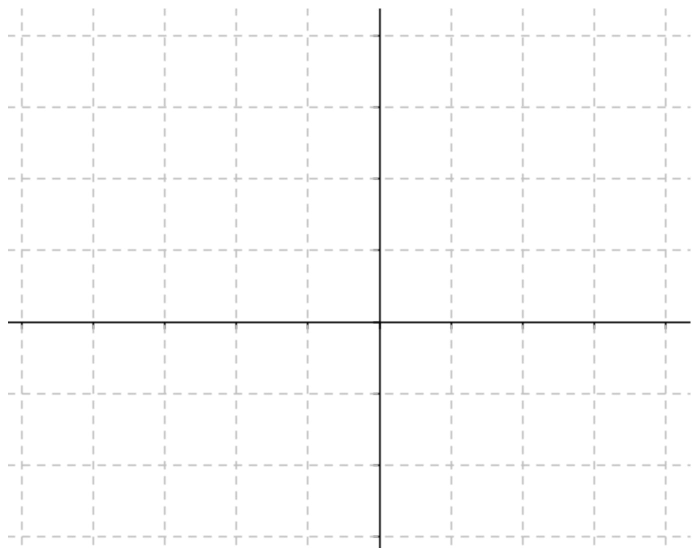
Lesson 9: The Geometric Effect of Some Complex Arithmetic

Classwork

Exercises

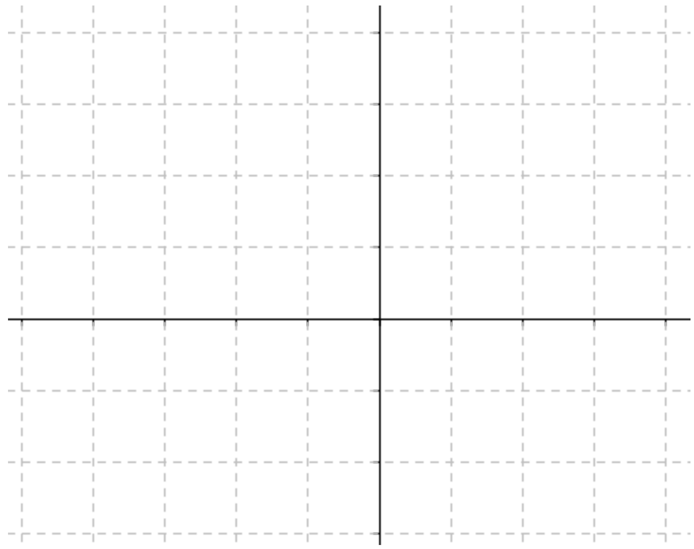
1. Taking the conjugate of a complex number corresponds to reflecting a complex number about the real axis. What operation on a complex number induces a reflection across the imaginary axis?

2. Given the complex numbers $w = -4 + 3i$ and $z = 2 - 5i$, graph each of the following:
   a. w
   b. z
   c. $w + 2$
   d. $z + 2$
   e. $w - 1$
   f. $z - 1$



3. Describe in your own words the geometric effect adding or subtracting a real number has on a complex number.

4. Given the complex numbers $w = -4 + 3i$ and $z = 2 - 5i$, graph each of the following:
   a. w
   b. z
   c. $w + i$
   d. $z + i$
   e. $w - 2i$
   f. $z - 2i$



5. Describe in your own words the geometric effect adding or subtracting an imaginary number has on a complex number.

Example 1

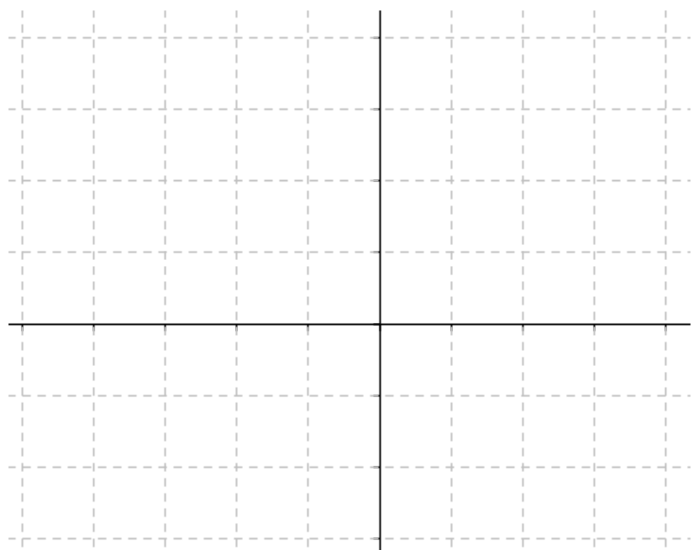
Given the complex number z, find a complex number w such that $z + w$ is shifted $\sqrt{2}$ units in a southwest direction.

Lesson Summary

- The conjugate, $\bar{z}$, of a complex number z, reflects the point across the real axis.
- The negative conjugate, $-\bar{z}$, of a complex number z, reflects the point across the imaginary axis.
- Adding or subtracting a real number to a complex number shifts the point left or right on the real (horizontal) axis.
- Adding or subtracting an imaginary number to a complex number shifts the point up or down on the imaginary (vertical) axis.

Problem Set

1. Given the complex numbers $w = 2 - 3i$ and $z = -3 + 2i$, graph each of the following:
   a. $w - 2$
   b. $z + 2$
   c. $w + 2i$
   d. $z - 3i$
   e. $w + z$
   f. $z - w$



2. Let $z = 5 - 2i$, find w for each case.
   a. z is a $90°$ counterclockwise rotation about the origin of w.
   b. z is reflected about the imaginary axis from w.
   c. z is reflected about the real axis from w.

3. Let $z = -1 + 2i, w = 4 - i$, simplify the following expressions.
   a. $z + \bar{w}$
   b. $|w - \bar{z}|$
   c. $2z - 3w$
   d. $\frac{z}{w}$

4. Given the complex number z, find a complex number w where $z + w$ is shifted
   a. $2\sqrt{2}$ units in a northeast direction.
   b. $5\sqrt{2}$ units in a southeast direction.

Lesson 10: The Geometric Effect of Some Complex Arithmetic

Classwork

Opening Exercises

1. Given $z = 3 - 2i$, plot and label the following and describe the geometric effect of the operation.
   a. z

   b. $z - 2$

   c. $z + 4i$

   d. $z + (-2 + 4i)$

2. Describe the geometric effect of the following:
   a. Multiplying by i.

   b. Taking the complex conjugate.

   c. What operation reflects a complex number across the imaginary axis?

Example 1

Plot the given points, then plot the image $L(z) = 2z$.

a. $z_1 = 3$

b. $z_2 = 2i$

c. $z_3 = 1 + i$

d. $z_4 = -4 + 3i$

e. $z_5 = 2 - 5i$

Exercises 1–7

Plot the given points, then plot the image $L(z) = iz$.

1. $z_1 = 3$

2. $z_2 = 2i$

3. $z_3 = 1 + i$

4. $z_4 = -4 + 3i$

5. $z_5 = 2 - 5i$

6. What is the geometric effect of the transformation? Confirm your conjecture using the slope of the segment joining the origin to the point and then to its image.

7. Is $L(z)$ a linear transformation? Explain how you know.

Example 2

Describe the geometric effect of $L(z) = (1 + i)z$ given the following. Plot the images on graph paper, and describe the geometric effect in words.

a. $z_1 = 1$

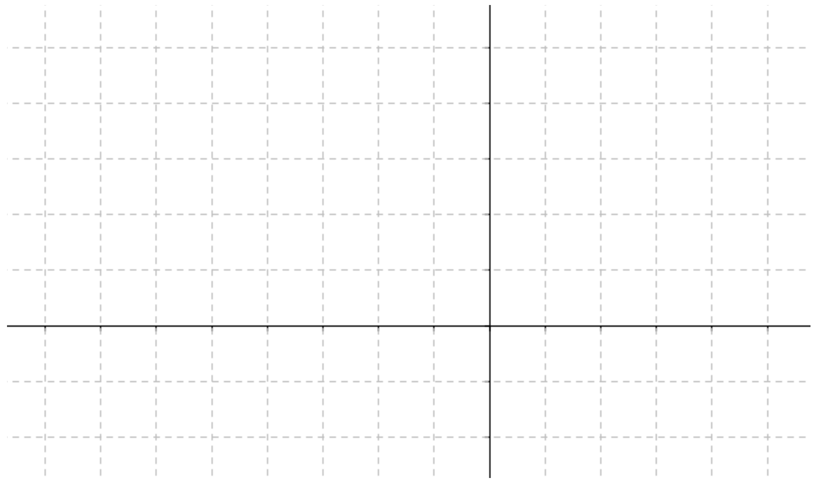
b. $z_2 = i$

c. $z_4 = 1 + i$

d. $z_5 = 4 + 6i$

Problem Set

1. Let $z = -4 + 2i$, simplify the following and describe the geometric effect of the operation. Plot the result in the complex plane.
   a. $z + 2 - 3i$
   b. $z - 2 - 3i$
   c. $z - (2 - 3i)$
   d. $2z$
   e. $\frac{z}{2}$



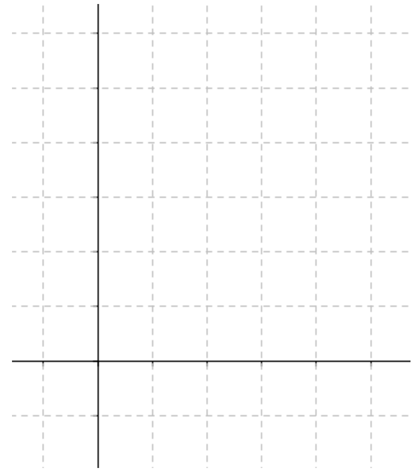
2. Let $z = 1 + 2i$, simplify the following and describe the geometric effect of the operation.
   a. iz
   b. i^2z
   c. $\bar{z}$
   d. $-\bar{z}$
   e. $i\bar{z}$
   f. $2iz$
   g. $iz + 5 - 3i$

3. Simplify the following expressions.
   a. $(4 - 2i)(5 - 3i)$
   b. $(-2 + 3i)(-2 - 3i)$
   c. $(1 + i)^2$
   d. $(1 + i)^{10}$ (Hint: $b^{nm} = (b^n)^m$)
   e. $\frac{-1 + 2i}{1 - 2i}$
   f. $\frac{x^2 + 4}{x - 2i}$, provided $x \neq 2i$.

4. Given $z = 2 + i$, describe the geometric effect of the following. Plot the result.

   a. $z(1 + i)$

   b. $z\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$



5. We learned that multiplying by i produces a $90°$ counterclockwise rotation about the origin. What do we need to multiply by to produce a $90°$ clockwise rotation about the origin?

6. Given z is a complex number $a + bi$, determine if $L(z)$ is a linear transformation. Explain why or why not.

   a. $L(z) = i^3 z$

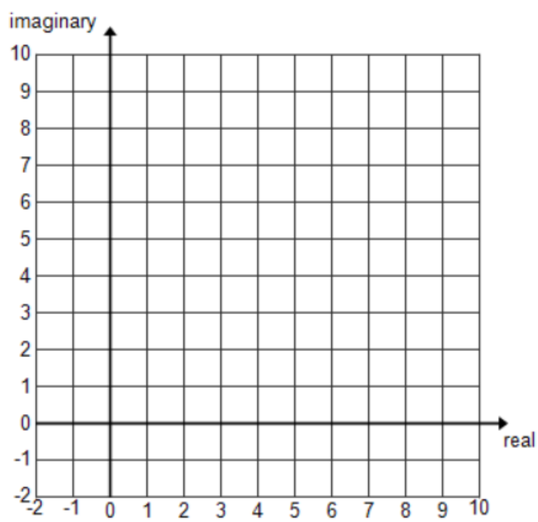
   b. $L(z) = z + 4i$

Lesson 11: Distance and Complex Numbers

Classwork

Opening Exercise

a. Plot the complex number $z = 2 + 3i$ on the complex plane. Plot the ordered pair $(2,3)$ on the coordinate plane.



b. In what way are complex numbers "points"?

c. What point on the coordinate plane corresponds to the complex number $-1 + 8i$?

d. What complex number corresponds to the point located at coordinate $(0, -9)$?

Exercises

1. The endpoints of a $\overline{AB}$ are $A(1, 8)$ and $B(-5, 3)$. What is the midpoint of $\overline{AB}$?

2.
   a. What is the midpoint of $A = 1 + 8i$ and $B = -5 + 3i$?

   b. Using $A = x_1 + y_1 i$ and $B = x_2 + y_2 i$, show that in general the midpoint of points A and B is $\frac{A+B}{2}$, the arithmetic average of the two numbers.

3. The endpoints of $\overline{AB}$ are $A(1, 8)$ and $B(-5, 3)$. What is the length of $\overline{AB}$?

4.
   a. What is the distance between $A = 1 + 8i$ and $B = -5 + 3i$?

   b. Show that, in general, the distance between $A = x_1 + y_1 i$ and $B = x_2 + y_2 i$ is the modulus of $A - B$.

5. Suppose $z = 2 + 7i$ and $w = -3 + i$.
   a. Find the midpoint m of z and w.

   b. Verify that $|z - m| = |w - m|$.

Lesson Summary

- Complex numbers can be thought of as points in a plane, and points in a plane can be thought of as complex numbers.
- For two complex numbers $A = x_1 + y_1 i$ and $B = x_2 + y_2 i$, the midpoint of points A and B is $\frac{A+B}{2}$.
- The distance between two complex numbers $A = x_1 + y_1 i$ and $B = x_2 + y_2 i$ is equal to $|A - B|$.

Problem Set

1. Find the midpoint between the two given points in the rectangular coordinate plane.
   a. $2 + 4i$ and $4 + 8i$
   b. $-3 + 7i$ and $5 - i$
   c. $-4 + 3i$ and $9 - 4i$
   d. $4 + i$ and $-12 - 7i$
   e. $-8 - 3i$ and $3 - 4i$
   f. $\frac{2}{3} - \frac{5}{2}i$ and $-0.2 + 0.4i$

2. Let $A = 2 + 4i$, $B = 14 + 8i$, and suppose that C is the midpoint of A and B, and that D is the midpoint of A and C.
   a. Find points C and D.
   b. Find the distance between A and B.
   c. Find the distance between A and C.
   d. Find the distance between C and D.
   e. Find the distance between D and B.
   f. Find a point one quarter of the way along the line segment connecting segment A and B, closer to A than to B.
   g. Terrence thinks the distance from B to C is the same as the distance from A to B. Is he correct? Explain why or why not.
   h. Using your answer from part (g), if E is the midpoint of C and B, can you find the distance from E to C? Explain.
   i. Without doing any more work, can you find point E? Explain.

Lesson 12: Distance and Complex Numbers

Classwork

Opening Exercise

a. Let $A = 2 + 3i$ and $B = -4 - 8i$. Find a complex number C so that B is the midpoint of A and C.

b. Given two complex numbers A and B, find a formula for a complex number C in terms of A and B so that B is the midpoint of A and C.

c. Verify that your formula is correct by using the result of part (a).

Exercise

Let $z = -100 + 100i$ and $w = 1000 - 1000i$.

a. Find a point one quarter of the way along the line segment connecting z and w closer to z than to w.

b. Write this point in the form $\alpha z + \beta w$ for some real numbers α and β. Verify that this does in fact represent the point found in part (a).

c. Describe the location of the point $\frac{2}{5}z + \frac{3}{5}w$ on this line segment.

Exploratory Challenge 1

a. Draw three points A, B, and C in the plane.

b. Start at any position P_0 and leapfrog over A to a new position P_1 so that A is the midpoint of $\overline{P_0P_1}$.

c. From P_1, leapfrog over B to a new position P_2 so that B is the midpoint $\overline{P_1P_2}$.

d. From P_2, leapfrog over C to a new position P_3 so that C is the midpoint $\overline{P_2P_3}$.

e. Continue alternately leapfrogging over A, then B, then C.

f. What eventually happens?

g. Using the formula from Opening Exercise part (b), show why this happens.

Exploratory Challenge 2

a. Plot a single point A in the plane.

b. What happens when you repeatedly jump over A?

c. Using the formula from Opening Exercise part (b), show why this happens.

d. Make a conjecture about what will happen if you leapfrog over two points, A and B, in the coordinate plane.

e. Test your conjecture by using the formula from Opening Exercise part (b).

f. Was your conjecture correct? If not, what is your new conjecture about what happens when you leapfrog over two points, A and B, in the coordinate plane?

g. Test your conjecture by actually conducting the experiment.

Problem Set

1. Find the distance between the following points.
   a. Point $A(2, 3)$ and point $B(6, 6)$
   b. $A = 2 + 3i$ and $B = 6 + 6i$
   c. $A = -1 + 5i$ and $B = 5 + 11i$
   d. $A = 1 - 2i$ and $B = -2 + 3i$
   e. $A = \frac{1}{2} - \frac{1}{2}i$ and $B = -\frac{2}{3} + \frac{1}{3}i$

2. Given three points A, B, C, where C is the midpoint of A and B.
   a. If $A = -5 + 2i$ and $C = 3 + 4i$, find B.
   b. If $B = 1 + 11i$ and $C = -5 + 3i$, find A.

3. Point C is the midpoint between $A = 4 + 3i$ and $B = -6 - 5i$. Find the distance between points C and D for each point D provided below.
   a. $2D = -6 + 8i$
   b. $D = -\overline{B}$

4. The distance between points $A = 1 + 1i$ and $B = a + bi$ is 5. Find the point B for each value provided below.
   a. $a = 4$
   b. $b = 6$

5. Draw five points in the plane A, B, C, D, E. Start at any position, P_0, and leapfrog over A to a new position, P_1 (so, A is the midpoint of $\overline{P_0P_1}$). Then leapfrog over B, then C, then D, then E, then A, then B, then C, then D, then E, then A again, and so on. How many jumps will it take to get back to the start position, P_0?

6. For the leapfrog puzzle problems in both Exploratory Challenge 1 and Problem 5, we are given an odd number of points to leapfrog over. What if we leapfrog over an even number of points? Let $A = 2$, $B = 2 + i$, and $P_0 = i$. Will P_n ever return to the starting position, P_0? Explain how you know.

Lesson 13: Trigonometry and Complex Numbers

Classwork

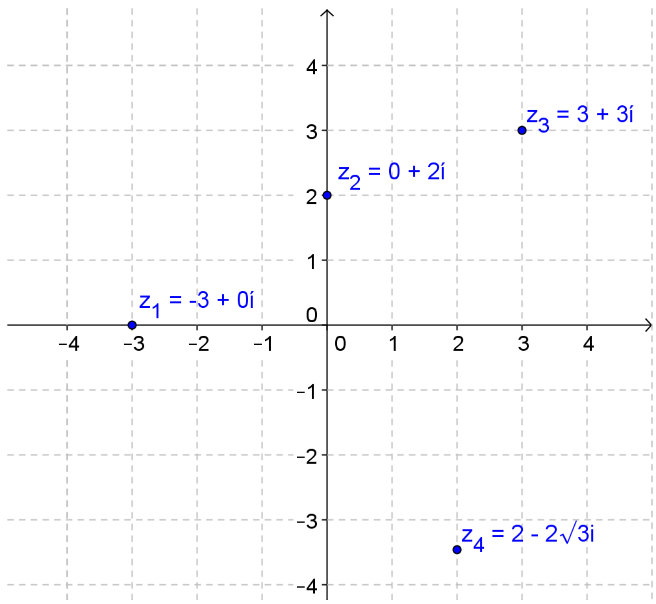
Opening Exercise

For each complex number shown below, answer the following questions. Record your answers in the table.

a. What are the coordinates (a, b) that correspond to this complex number?

b. What is the modulus of the complex number?

c. Suppose a ray from the origin that contains the real number 1 is rotated $\theta°$ so it passes through the point (a, b). What is a value of θ?



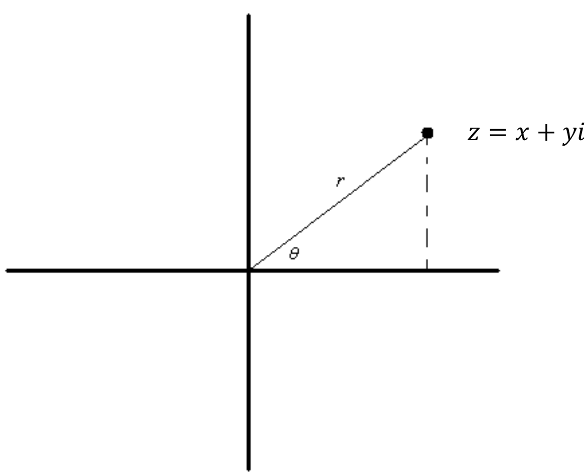
Complex Number	(a, b)	Modulus	Degrees of Rotation $\theta°$
$z_1 = -3 + 0i$			
$z_2 = 0 + 2i$			
$z_3 = 3 + 3i$			
$z_4 = 2 - 2\sqrt{3}i$			

Exercises 1–2

1. Can you find at least two additional rotations that would map a ray from the origin through the real number 1 to a ray from the origin passing through the point $(3, 3)$?

2. How are the rotations you found in Exercise 1 related?

Every complex number $z = x + yi$ appears as a point on the complex plane with coordinates (x, y) as a point in the coordinate plane.



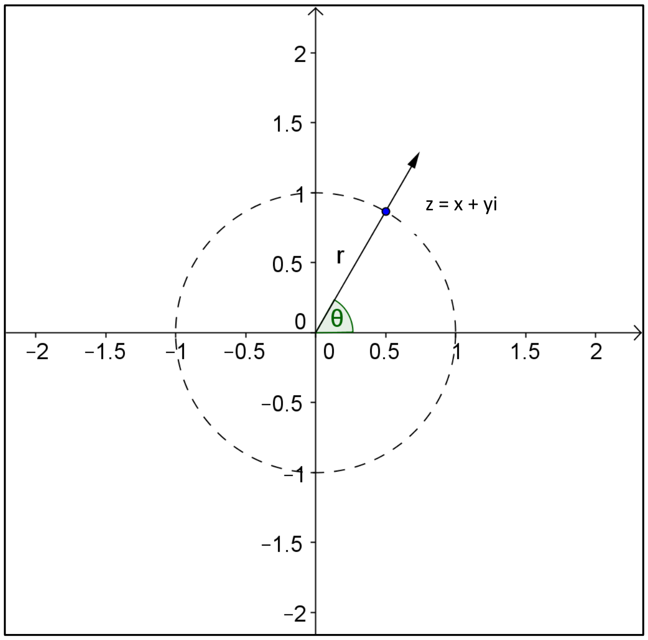
In the diagram above, notice that each complex number z has a distance r from the origin to the point (x, y) and a rotation of $\theta°$ that maps the ray from the origin along the positive real axis to the ray passing through the point (x, y).

ARGUMENT OF THE COMPLEX NUMBER z: The *argument of the complex number* z is the radian (or degree) measure of the counterclockwise rotation of the complex plane about the origin that maps the initial ray (i.e., the ray corresponding to the positive real axis) to the ray from the origin through the complex number z in the complex plane. The argument of z is denoted $\arg(z)$.

MODULUS OF A COMPLEX NUMBER z: The *modulus of a complex number z,* denoted $|z|$, is the distance from the origin to the point corresponding to z in the complex plane. If $z = a + bi$, then $|z| = \sqrt{a^2 + b^2}$.

Example 1: The Polar Form of a Complex Number

Derive a formula for a complex number in terms of its modulus r and argument θ.



Suppose that z has coordinates (x, y) that lie on the unit circle as shown.

a. What is the value of r and what are the coordinates of the point (x, y) in terms of θ? Explain how you know.

b. If $r = 2$, what would be the coordinates of the point (x, y)? Explain how you know.

c. If $r = 20$, what would be the coordinates of the point (x, y)? Explain how you know.

d. Use the definitions of sine and cosine to write coordinates of the point (x, y) in terms of cosine and sine for any $r \geq 0$ and real number θ.

e. Use your answers to part (d) to express $z = x + yi$ in terms of r and θ.

POLAR FORM OF A COMPLEX NUMBER: The *polar form of a complex number* z is $r(\cos(\theta) + i\sin(\theta))$, where $r = |z|$ and $\theta = \arg(z)$.

RECTANGULAR FORM OF A COMPLEX NUMBER: The *rectangular form of a complex number* z is $a + bi$, where z corresponds to the point (a, b) in the complex plane, and i is the imaginary unit. The number a is called the *real part* of $a + bi$, and the number b is called the *imaginary part* of $a + bi$.

General Form	Polar Form $z = r(\cos(\theta) + i\sin(\theta))$	Rectangular Form $z = a + bi$
Examples	$3(\cos(60°) + i\sin(60°))$	$0 + 2i$
Key Features	Modulus Argument Coordinate	Modulus Coordinate

Exercises 3–6

3. Write each complex number from the Opening Exercise in polar form.

Rectangular	Polar Form
$z_1 = -3 + 0i$	
$z_2 = 0 + 2i$	
$z_3 = 3 + 3i$	
$z_4 = 2 - 2\sqrt{3}i$	

4. Use a graph to help you answer these questions.
   a. What is the modulus of the complex number $2 - 2i$?

   b. What is the argument of the number $2 - 2i$?

   c. Write the complex number in polar form.

   d. Arguments can be measured in radians. Express your answer the answer to part (c) using radians.

e. Explain why the polar and rectangular forms of a complex number represent the same number.

5. State the modulus and argument of each complex number, and then graph it using the modulus and argument.

a. $4(\cos(120°) + i\sin(120°))$

b. $5\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$

c. $3(\cos(190°) + i\sin(190°))$

6. Evaluate the sine and cosine functions for the given values of θ, and then express each complex number in rectangular form, $z = a + bi$. Explain why the polar and rectangular forms represent the same number.

a. $4(\cos(120°) + i\sin(120°))$

b. $5\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$

c. $3(\cos(190°) + i\sin(190°))$

Example 2: Writing a Complex Number in Polar Form

a. Convert $3 + 4i$ to polar form.

b. Convert $3 - 4i$ to polar form.

Exercise 7

7. Express each complex number in polar form. Round arguments to the nearest thousandth.

   a. $2 + 5i$

   b. $-6 + i$

Lesson Summary

The polar form of a complex number $z = r\,(\cos(\theta) + i\sin(\theta))$ where θ is the argument of z and r is the modulus of z. The rectangular form of a complex number is $z = a + bi$.

The polar and rectangular forms of a complex number are related by the formulas $a = r\cos(\theta)$, $b = r\sin(\theta)$ and $r = \sqrt{a^2 + b^2}$.

The notation for modulus is $|z|$ and the notation for argument is $\arg(z)$.

Problem Set

1. Explain why the complex numbers $z_1 = 1 - \sqrt{3}i$, $z_2 = 2 - 2\sqrt{3}i$, and $z_3 = 5 - 5\sqrt{3}i$ can all have the same argument. Draw a diagram to support your answer.

2. What is the modulus of each of the complex numbers z_1, z_2, and z_3 given in Problem 1 above.

3. Write the complex numbers from Exercise 1 in polar form.

4. Explain why $1 - \sqrt{3}i$ and $2(\cos(300°) + i\sin(300°))$ represent the same number.

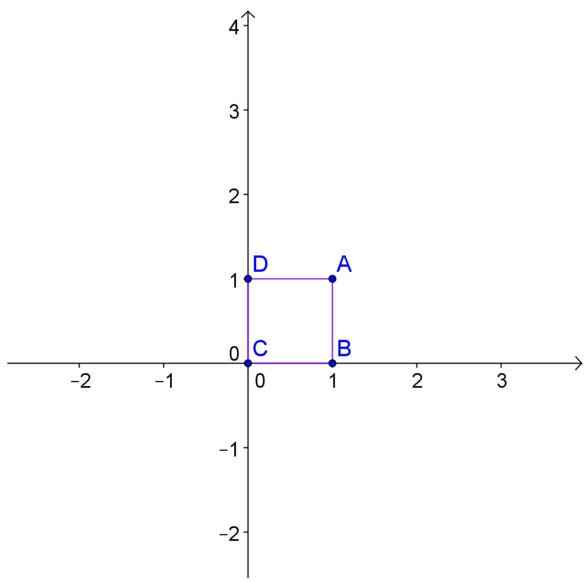
5. Julien stated that a given modulus and a given argument uniquely determine a complex number. Confirm or refute Julien's reasoning.

6. Identify the modulus and argument of the complex number in polar form, convert it to rectangular form and sketch the complex number in the complex plane. $0° \leq \arg(z) \leq 360°$ or $0 \leq \arg(z) \leq 2\pi$ (radians)
   a. $z = \cos(30°) + i\sin(30°)$
   b. $z = 2\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$
   c. $z = 4\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$
   d. $z = 2\sqrt{3}\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)$
   e. $z = 5(\cos(5.637) + i\sin(5.637))$
   f. $z = 5(\cos(2.498) + i\sin(2.498))$
   g. $z = \sqrt{34}(\cos(3.682) + i\sin(3.682))$
   h. $z = 4\sqrt{3}\left(\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right)$

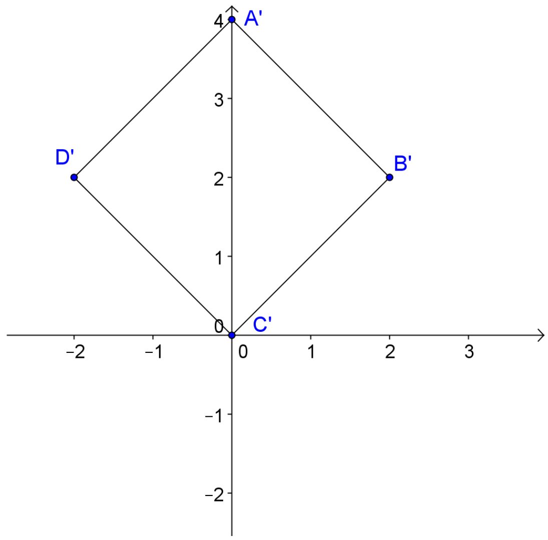
7. Convert the complex numbers in rectangular form to polar form. If the argument is a multiple of $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$, or $\frac{\pi}{2}$, express your answer exactly. If not, use $\arctan\left(\frac{b}{a}\right)$ to find $\arg(z)$ rounded to the nearest thousandth, $0 \leq \arg(z) \leq 2\pi$ (radians).
   a. $z = \sqrt{3} + i$
   b. $z = -3 + 3i$
   c. $z = 2 - 2\sqrt{3}i$
   d. $z = -12 - 5i$
   e. $z = 7 - 24i$

8. Show that the following complex numbers have the same argument.
   a. $z_1 = 3 + 3\sqrt{3}i$ and $z_2 = 1 + \sqrt{3}i$
   b. $z_1 = 1 + i$ and $z_2 = 4 + 4i$

9. A square with side length of one unit is shown below. Identify a complex number in polar form that corresponds to each point on the square.



10. Determine complex numbers in polar form whose coordinates are the vertices of the square shown below.



11. How do the modulus and argument of coordinate A in Problem 9, correspond to the modulus and argument of point A' in Problem 10? Does a similar relationship exist when you compare B to B', C to C', and D to D'? Explain why you think this relationship exists.

12. Describe the transformations that map $ABCD$ to $A'B'C'D'$.

General Form	Polar Form $z = r(\cos(\theta) + i\sin(\theta))$	Rectangular Form $z = a + bi$
Examples		
Key Features	Modulus Argument Coordinate	Modulus Coordinate
Examples		
Key Features	Modulus Argument Coordinate	Modulus Coordinate
Examples		
Key Features	Modulus Argument Coordinate	Modulus Coordinate

Lesson 14: Discovering the Geometric Effect of Complex Multiplication

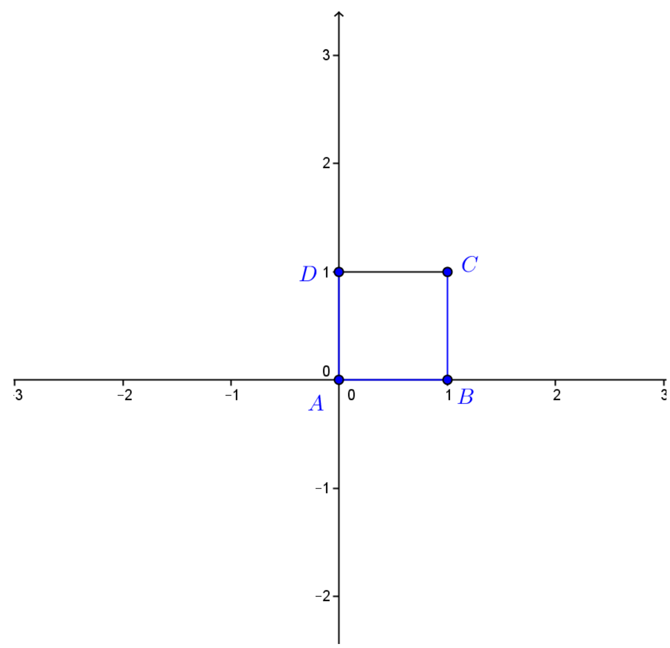
Classwork

Exercises

The vertices $A(0,0)$, $B(1,0)$, $C(1,1)$, and $D(0,1)$ of a unit square can be represented by the complex numbers $A = 0$, $B = 1$, $C = 1 + i$, and $D = i$.

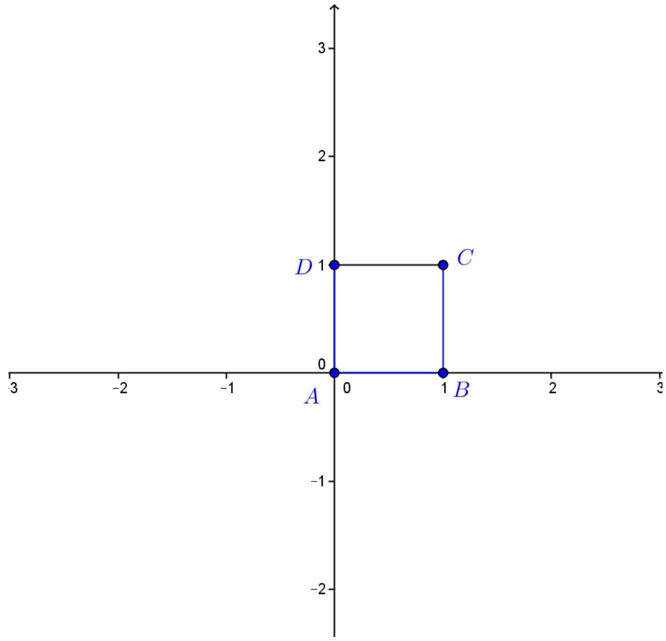
1. Let $L_1(z) = -z$.
   a. Calculate $A' = L_1(A)$, $B' = L_1(B)$, $C' = L_1(C)$, and $D' = L_1(D)$. Plot these four points on the axes.

   b. Describe the geometric effect of the linear transformation $L_1(z) = -z$ on the square $ABCD$.



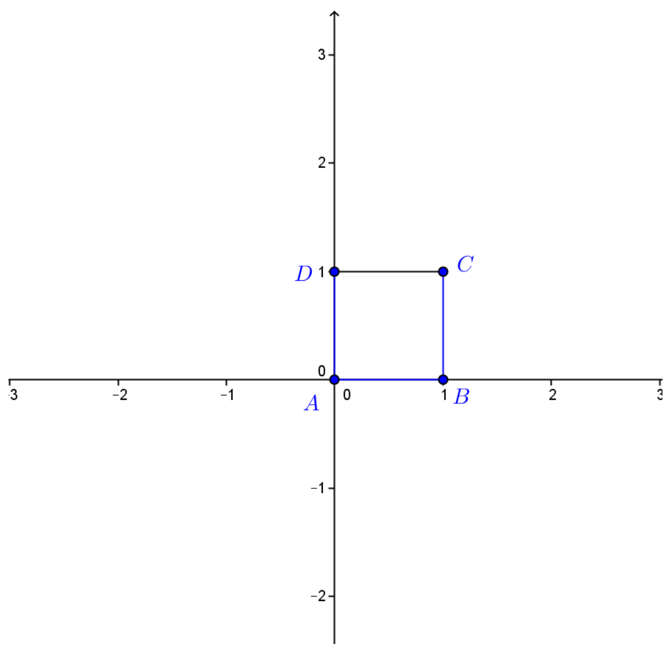
2. Let $L_2(z) = 2z$.
   a. Calculate $A' = L_2(A)$, $B' = L_2(B)$, $C' = L_2(C)$, and $D' = L_2(D)$. Plot these four points on the axes.

   b. Describe the geometric effect of the linear transformation $L_2(z) = 2z$ on the square $ABCD$.

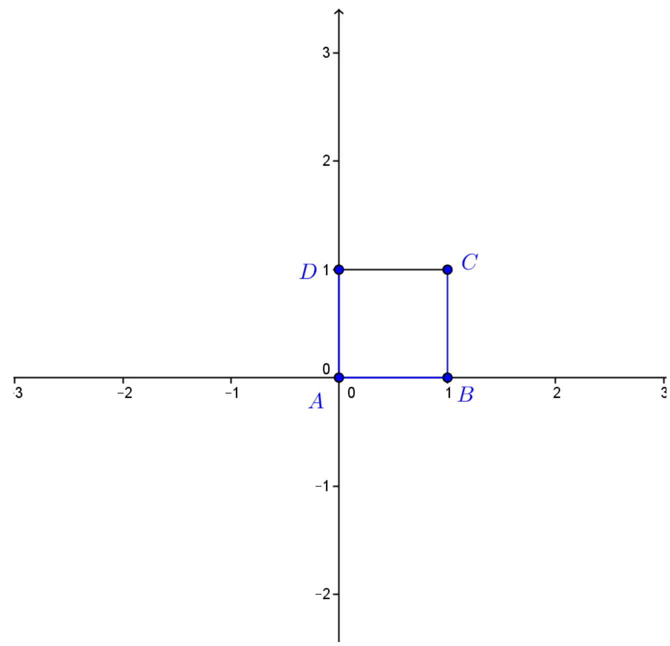


3. Let $L_3(z) = iz$.
   a. Calculate $A' = L_3(A)$, $B' = L_3(B)$, $C' = L_3(C)$, and $D' = L_3(D)$. Plot these four points on the axes.

   b. Describe the geometric effect of the linear transformation $L_3(z) = iz$ on the square $ABCD$.



4. Let $L_4(z) = (2i)z$.
   a. Calculate $A' = L_4(A)$, $B' = L_4(B)$, $C' = L_4(C)$, and $D' = L_4(D)$. Plot these four points on the axes.



   b. Describe the geometric effect of the linear transformation $L_4(z) = (2i)z$ on the square $ABCD$.

5. Explain how transformations L_2, L_3, and L_4 are related.

6. We will continue to use the unit square $ABCD$ with $A = 0$, $B = 1$, $C = 1 + I$, $D = i$ for this exercise.
   a. What is the geometric effect of the transformation $L(z) = 5z$ on the unit square?

   b. What is the geometric effect of the transformation $L(z) = (5i)z$ on the unit square?

c. What is the geometric effect of the transformation $L(z) = (5i^2)z$ on the unit square?

d. What is the geometric effect of the transformation $L(z) = (5i^3)z$ on the unit square?

e. What is the geometric effect of the transformation $L(z) = (5i^4)z$ on the unit square?

f. What is the geometric effect of the transformation $L(z) = (5i^5)z$ on the unit square?

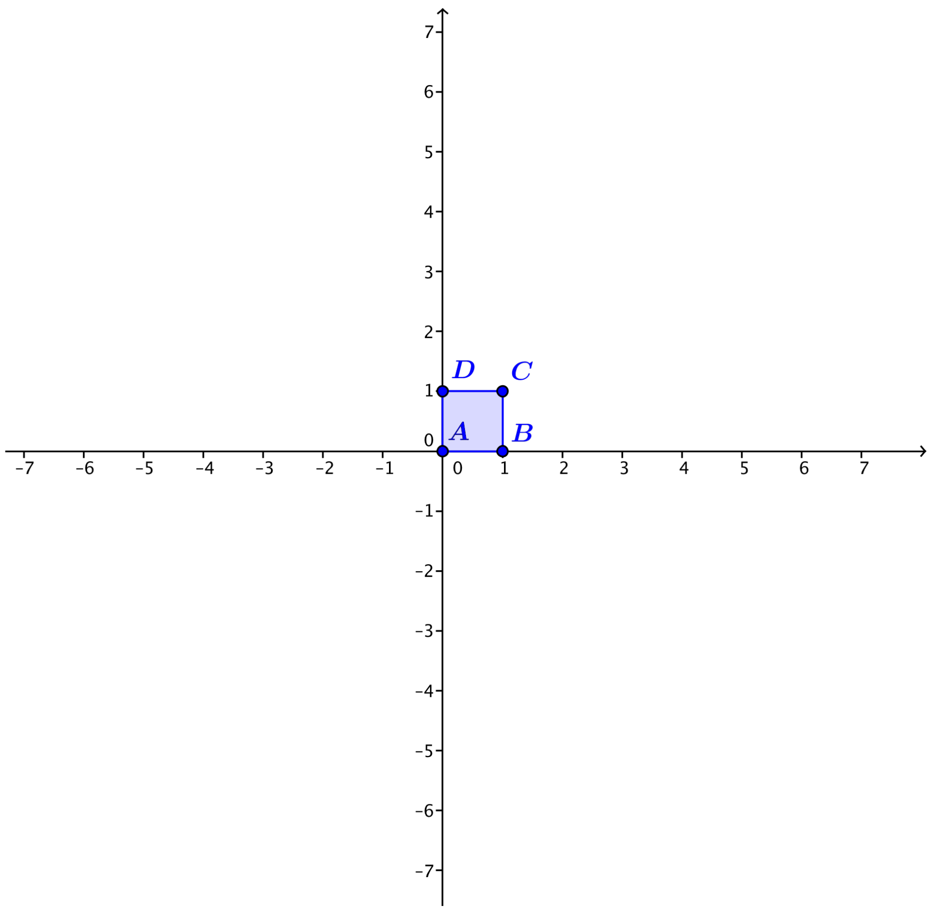
g. What is the geometric effect of the transformation $L(z) = (5i^n)z$ on the unit square, for some integer $n \geq 0$?

Exploratory Challenge

Your group has been assigned either to the 1-team, 2-team, 3-team, or 4-team. Each team will answer the questions below for the transformation that corresponds to their team number:

$$L_1(z) = (3 + 4i)z$$
$$L_2(z) = (-3 + 4i)z$$
$$L_3(z) = (-3 - 4i)z$$
$$L_4(z) = (3 - 4i)z.$$

The unit square unit square $ABCD$ with $A = 0$, $B = 1$, $C = 1 + i$, $D = i$ is shown below. Apply your transformation to the vertices of the square $ABCD$ and plot the transformed points A', B', C', and D' on the same coordinate axes.



For the 1-team: a. Why is $B' = 3 + 4i$? b. What is the argument of $3 + 4i$? c. What is the modulus of $3 + 4i$?	For the 2-team: a. Why is $B' = -3 + 4i$? b. What is the argument of $-3 + 4i$? c. What is the modulus of $-3 + 4i$?
For the 3-team: a. Why is $B' = -3 - 4i$? b. What is the argument of $-3 - 4i$? c. What is the modulus of $-3 - 4i$?	For the 4-team: a. Why is $B' = 3 - 4i$? b. What is the argument of $3 - 4i$? c. What is the modulus of $3 - 4i$?

All groups should also answer the following:

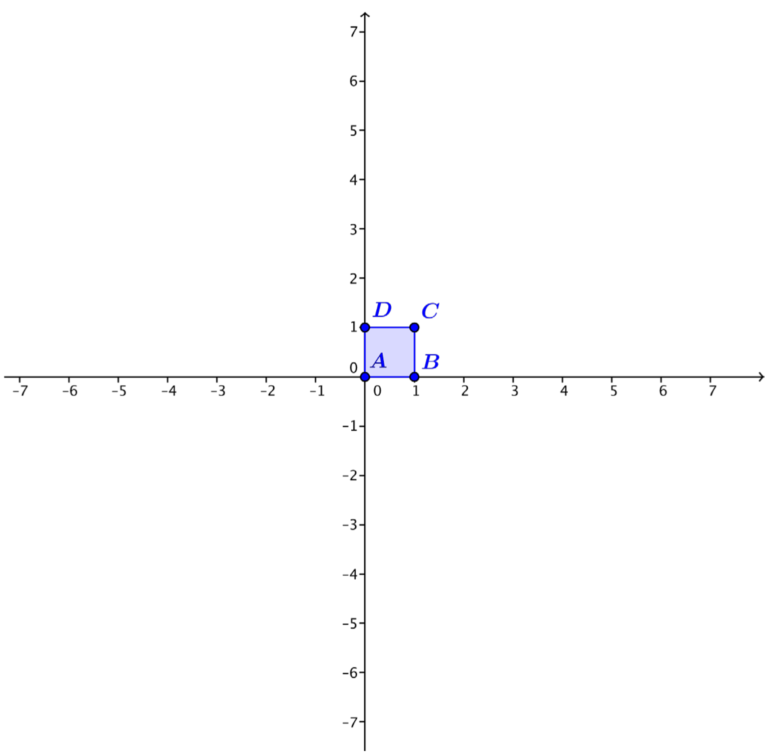
a. Describe the amount the square has been rotated counterclockwise.

b. What is the dilation factor of the square? Explain how you know.

c. What is the geometric effect of your transformation L_1, L_2, L_3, or L_4 on the unit square $ABCD$?

d. Make a conjecture: What do you expect to be the geometric effect of the transformation $L(z) = (2+i)z$ on the unit square $ABCD$?

e. Test your conjecture with the unit square on the axes below.



Problem Set

1. Find the modulus and argument for each of the following complex numbers.
   a. $z_1 = \frac{\sqrt{3}}{2} + \frac{1}{2}i$
   b. $z_2 = 2 + 2\sqrt{3}i$
   c. $z_3 = -3 + 5i$
   d. $z_4 = -2 - 2i$
   e. $z_5 = 4 - 4i$
   f. $z_6 = 3 - 6i$

2. For parts (a)–(c), determine the geometric effect of the specified transformation.
   a. $L(z) = -3z$
   b. $L(z) = -100z$
   c. $L(z) = -\frac{1}{3}z$
   d. Describe the geometric effect of the transformation $L(z) = az$ for any negative real number a.

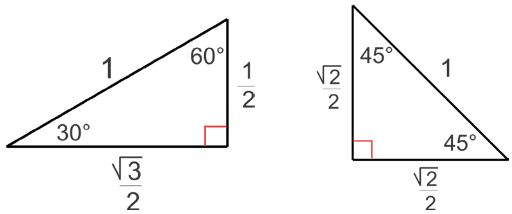
3. For parts (a)–(c), determine the geometric effect of the specified transformation.
   a. $L(z) = (-3i)z$
   b. $L(z) = (-100i)z$
   c. $L(z) = \left(-\frac{1}{3}i\right)z$
   d. Describe the geometric effect of the transformation $L(z) = (bi)z$ for any negative real number b.

4. Suppose that we have two linear transformations $L_1(z) = 3z$ and $L_2(z) = (5i)z$.
   a. What is the geometric effect of first performing transformation L_1, and then performing transformation L_2?
   b. What is the geometric effect of first performing transformation L_2, and then performing transformation L_1?
   c. Are your answers to parts (a) and (b) the same or different? Explain how you know.

5. Suppose that we have two linear transformations $L_1(z) = (4 + 3i)z$ and $L_2(z) = -z$. What is the geometric effect of first performing transformation L_1, and then performing transformation L_2?

6. Suppose that we have two linear transformations $L_1(z) = (3 - 4i)z$ and $L_2(z) = -z$. What is the geometric effect of first performing transformation L_1, and then performing transformation L_2?

7. Explain the geometric effect of the linear transformation $L(z) = (a - bi)z$, where a and b are positive real numbers.



8. In Geometry, we learned the special angles of a right triangle whose hypotenuse is 1 unit. The figures are shown above. Describe the geometric effect of the following transformations.

   a. $L_1(z) = \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) z$

   b. $L_2(z) = \left(2 + 2\sqrt{3}i\right)z$

   c. $L_3(z) = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) z$

   d. $L_4(z) = (4 + 4i)z$

9. Recall that a function L is a linear transformation if all z and w in the domain of L and all constants a meet the following two conditions:

   i. $L(z + w) = L(z) + L(w)$

   ii. $L(az) = aL(z)$

   Show that the following functions meet the definition of a linear transformation.

   a. $L_1(z) = 4z$

   b. $L_2(z) = iz$

   c. $L_3(z) = (4 + i)z$

10. The vertices $A(0,0)$, $B(1,0)$, $C(1,1)$, $D(0,1)$ of a unit square can be represented by the complex numbers $A = 0$, $B = 1$, $C = 1 + i$, $D = i$. We learned that multiplication of those complex numbers by i rotates the unit square by $90°$ counterclockwise. What do you need to multiply by so that the unit square will be rotated by $90°$ clockwise?

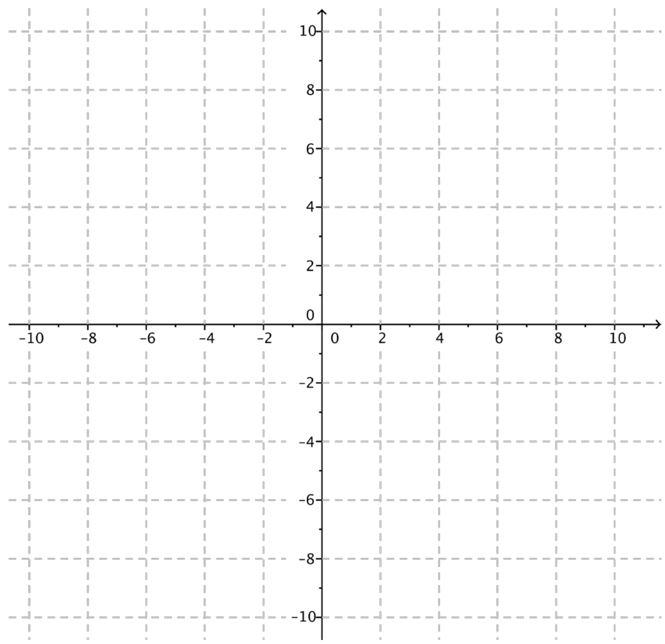
Lesson 15: Justifying the Geometric Effect of Complex Multiplication

Classwork

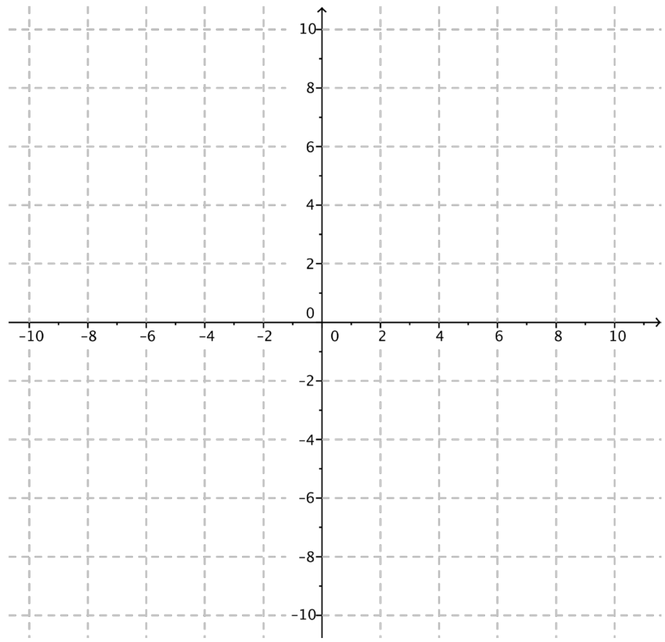
Opening Exercise

For each exercise below, compute the product wz. Then, plot the complex numbers z, w, and wz on the axes provided.

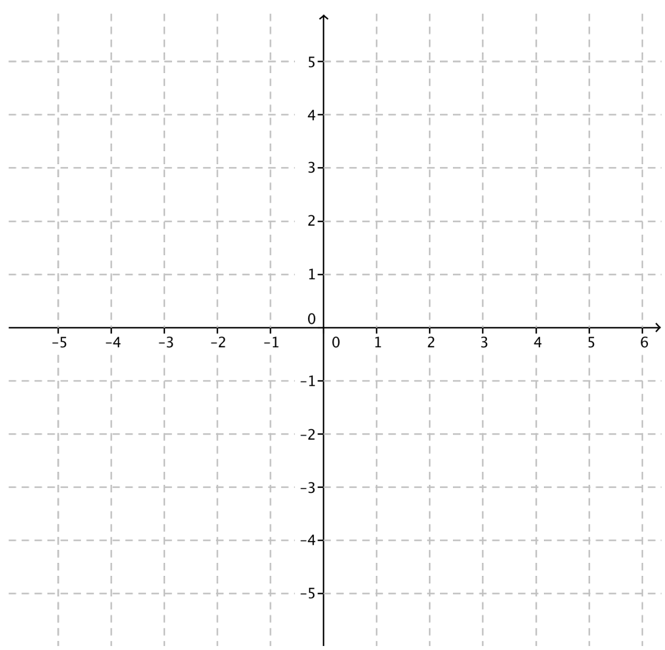
a. $z = 3 + i$, $w = 1 + 2i$



b. $z = 1 + 2i, w = -1 + 4i$



c. $z = -1 + i, w = -2 - i$



d. For each part (a), (b), and (c), draw line segments connecting each point z, w and wz to the origin. Determine a relationship between the arguments of the complex numbers z, w, and wz.

Exercises

1. Let $w = a + bi$ and $z = c + di$.
   a. Calculate the product wz.

   b. Calculate the moduli $|w|$, $|z|$, and $|wz|$.

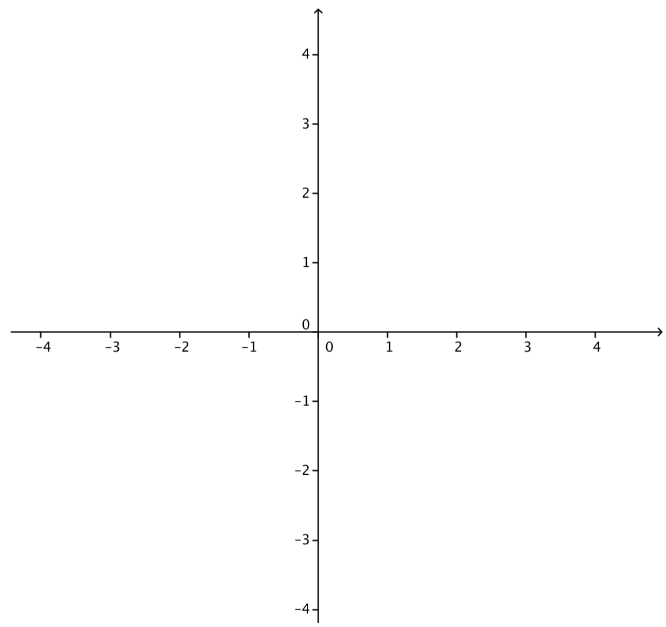
   c. What can you conclude about the quantities $|w|$, $|z|$, and $|wz|$?

2. What does the result of Exercise 1 tell us about the geometric effect of the transformation $L(z) = wz$?

3. If z and w are the complex numbers with the specified arguments and moduli, locate the point that represents the product wz on the provided coordinate axes.

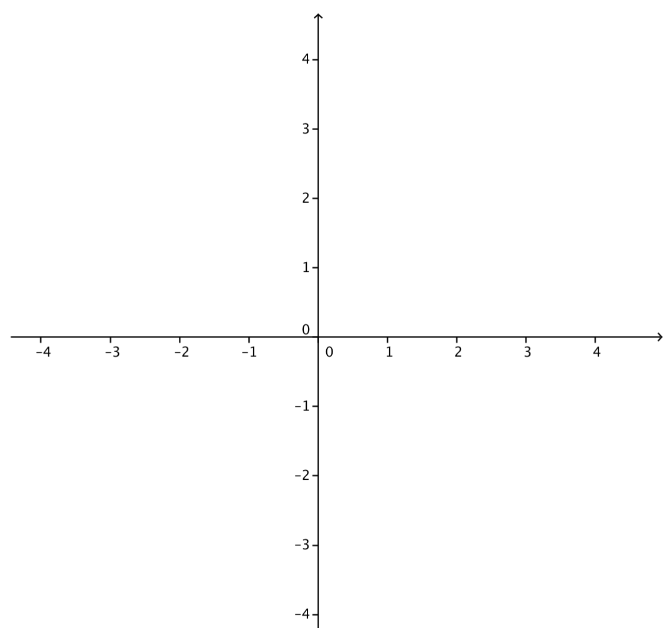
   a. $|w| = 3,\ \arg(w) = \frac{\pi}{4}$

      $|z| = \frac{2}{3},\ \arg(z) = -\frac{\pi}{2}$



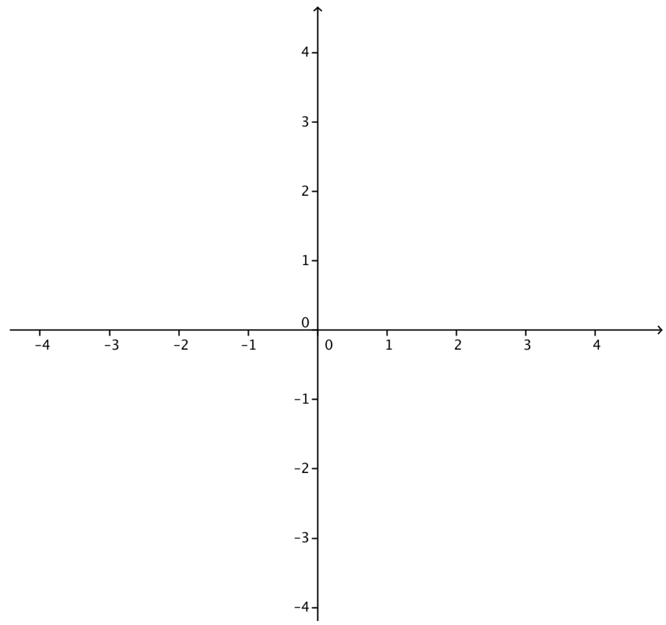
   b. $|w| = 2,\ \arg(w) = \pi$

      $|z| = 1,\ \arg(z) = \frac{\pi}{4}$



c. $|w| = \frac{1}{2}$, $\arg(w) = \frac{4\pi}{3}$

$|z| = 4$, $\arg(z) = -\frac{\pi}{6}$



Lesson Summary

For complex numbers z and w,

- The modulus of the product is the product of the moduli:

$$|wz| = |w| \cdot |z|,$$

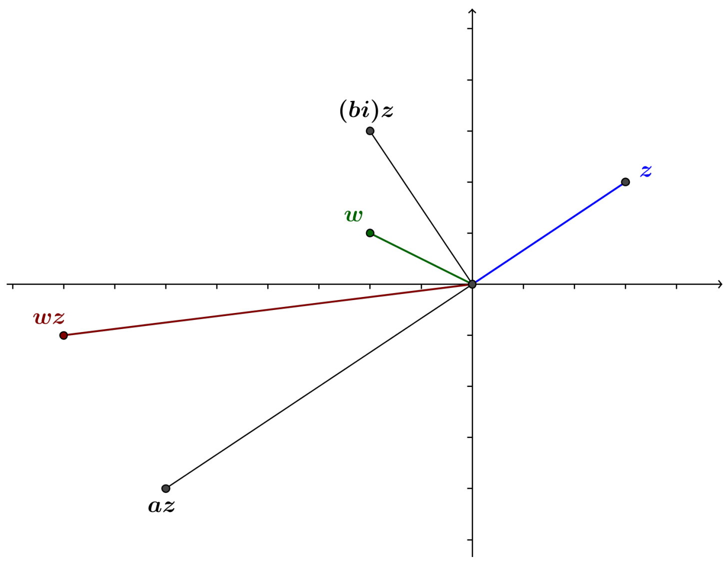
- The argument of the product is the sum of the arguments:

$$\arg(wz) = \arg(w) + \arg(z).$$

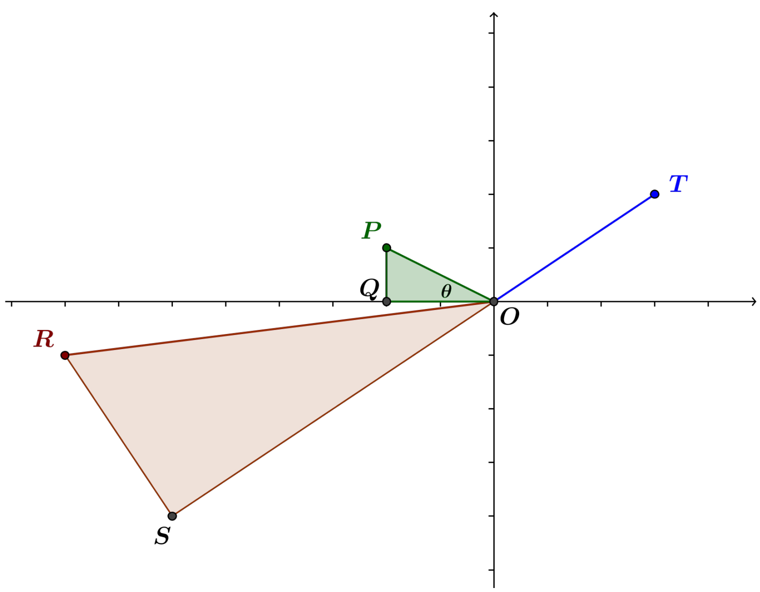
Problem Set

1. In the lesson, we justified our observation that the geometric effect of a transformation $L(z) = wz$ is a rotation by $\arg(w)$ and a dilation by $|w|$ for a complex number w that is represented by a point in the first quadrant of the coordinate plane. In this exercise, we will verify that this observation is valid for any complex number w. For a complex number $w = a + bi$, we only considered the case where $a > 0$ and $b > 0$. There are eight additional possibilities we need to consider.
   a. Case 1: The point representing w is the origin. That is, $a = 0$ and $b = 0$.
      In this case, explain why $L(z) = (a + bi)z$ has the geometric effect of rotation by $\arg(a + bi)$ and dilation by $|a + bi|$.
   b. Case 2: The point representing w lies on the positive real axis. That is, $a > 0$ and $b = 0$.
      In this case, explain why $L(z) = (a + bi)z$ has the geometric effect of rotation by $\arg(a + bi)$ and dilation by $|a + bi|$.
   c. Case 3: The point representing w lies on the negative real axis. That is, $a < 0$ and $b = 0$.
      In this case, explain why $L(z) = (a + bi)z$ has the geometric effect of rotation by $\arg(a + bi)$ and dilation by $|a + bi|$.
   d. Case 4: The point representing w lies on the positive imaginary axis. That is, $a = 0$ and $b > 0$.
      In this case, explain why $L(z) = (a + bi)z$ has the geometric effect of rotation by $\arg(a + bi)$ and dilation by $|a + bi|$.
   e. Case 5: The point representing w lies on the negative imaginary axis. That is, $a = 0$ and $b < 0$.
      In this case, explain why $L(z) = (a + bi)z$ has the geometric effect of rotation by $\arg(a + bi)$ and dilation by $|a + bi|$.

f. Case 6: The point representing $w = a + bi$ lies in the second quadrant. That is, $a < 0$ and $b > 0$. Points representing , z, az, $(bi)z$, and $wz = az + (bi)z$ are shown in the figure below.



For convenience, rename the origin O and let $P = w$, $Q = a$, $R = wz$, $S = az$, and $T = z$, as shown below. Let $m(\angle POQ) = \theta$.



i. Argue that $\triangle OPQ \sim \triangle ORS$.

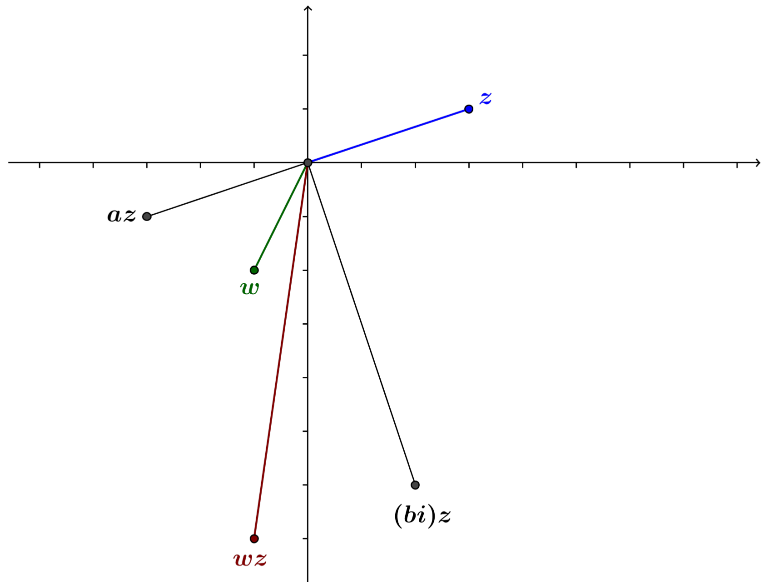
ii. Express the argument of az in terms of $\arg(z)$.

iii. Express $\arg(w)$ in terms of θ, where $\theta = m(\angle POQ)$.

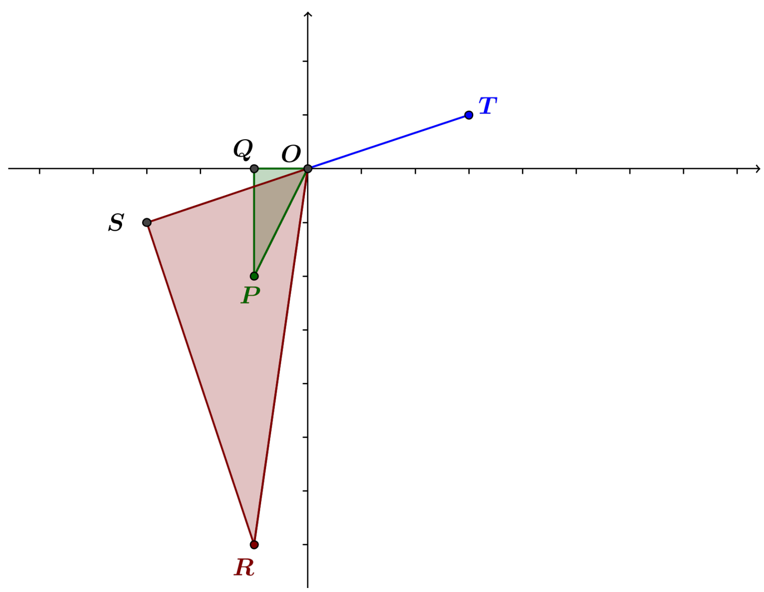
iv. Explain why $\arg(wz) = \arg(az) - \theta$.

v. Combine your responses from parts (ii), (iii) and (iv) to express $\arg(wz)$ in terms of $\arg(z)$ and $\arg(w)$.

g. Case 7: The point representing $w = a + bi$ lies in the third quadrant. That is, $a < 0$ and $b < 0$.
Points representing , z, az, $(bi)z$, and $wz = az + (bi)z$ are shown in the figure below.

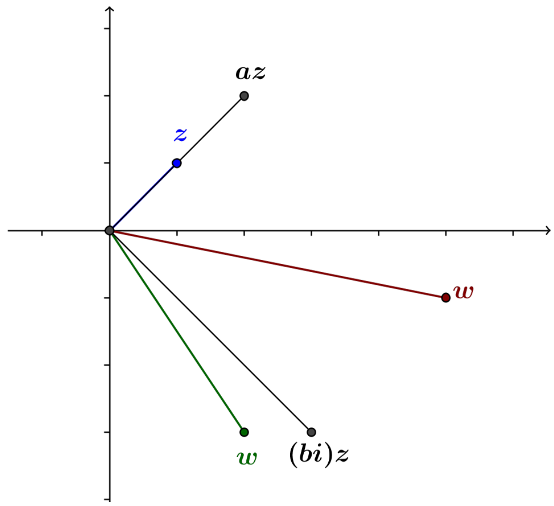


For convenience, rename the origin O and let $P = w$, $Q = a$, $R = wz$, $S = az$, and $T = z$, as shown below. Let $m(\angle POQ) = \theta$.

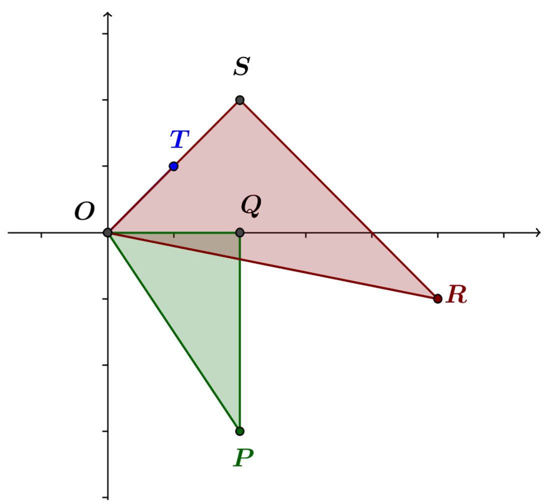


i. Argue that $\triangle OPQ \sim \triangle ORS$.
ii. Express the argument of az in terms of $\arg(z)$.
iii. Express $\arg(w)$ in terms of θ, where $\theta = m(\angle POQ)$.
iv. Explain why $\arg(wz) = \arg(az) + \theta$.
v. Combine your responses from parts (ii), (iii), and (iv) to express $\arg(wz)$ in terms of $\arg(z)$ and $\arg(w)$.

h. Case 8: The point representing $w = a + bi$ lies in the fourth quadrant. That is, $a > 0$ and $b < 0$.
Points representing , z, az, $(bi)z$, and $wz = az + (bi)z$ are shown in the figure below.



For convenience, rename the origin O, and let $P = w$, $Q = a$, $R = wz$, $S = az$, and $T = z$, as shown below.
Let $m(\angle POQ) = \theta$.



i. Argue that $\triangle\, OPQ \sim \triangle\, ORS$.
ii. Express $\arg(w)$ in terms of θ, where $\theta = m(\angle POQ)$.
iii. Explain why $m(\angle QOR) = \theta - \arg(z)$.
iv. Express $\arg(wz)$ in terms of $m(\angle QOR)$
v. Combine your responses from parts (ii), (iii), and (iv) to express $\arg(wz)$ in terms of $\arg(z)$ and $\arg(w)$.

2. Summarize the results of Problem 1, parts (a)–(h) and the lesson.

3. Find a linear transformation L that will have the geometric effect of rotation by the specified amount without dilating.
   a. $45°$ counterclockwise
   b. $60°$ counterclockwise
   c. $180°$ counterclockwise
   d. $120°$ counterclockwise
   e. $30°$ clockwise
   f. $90°$ clockwise
   g. $180°$ clockwise
   h. $135°$ clockwise

4. Suppose that we have linear transformations L_1 and L_2 as specified below. Find a formula for $L_2\left(L_1(z)\right)$ for complex numbers z.
   a. $L_1(z) = (1+i)z$ and $L_2(z) = (1-i)z$
   b. $L_1(z) = (3-2i)z$ and $L_2(z) = (2+3i)z$
   c. $L_1(z) = (-4+3i)z$ and $L_2(z) = (-3-i)z$
   d. $L_1(z) = (a+bi)z$ and $L_2(z) = (c+di)z$ for real numbers a, b, c and d.

Lesson 16: Representing Reflections with Transformations

Classwork

Opening Exercise

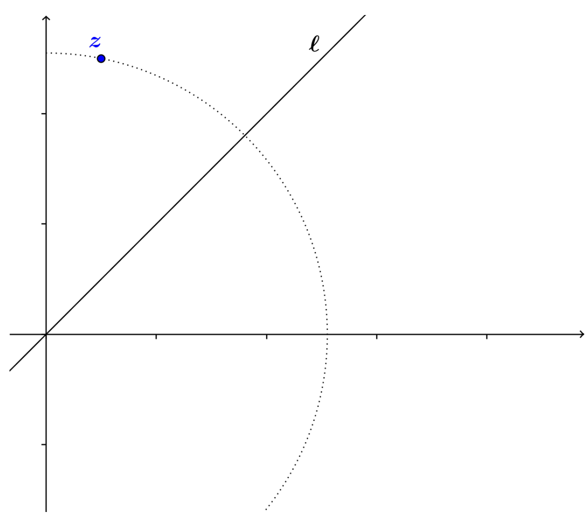
a. Find a transformation $R_{(0,45^\circ)}: \mathbb{C} \to \mathbb{C}$ that rotates a point represented by the complex number z by 45° counterclockwise in the coordinate plane, but does not produce a dilation.

b. Find a transformation $R_{(0,-45^\circ)}: \mathbb{C} \to \mathbb{C}$ that rotates a point represented by the complex number z by 45° clockwise in the coordinate plane, but does not produce a dilation.

c. Find a transformation $r_{x\text{-axis}}: \mathbb{C} \to \mathbb{C}$ that reflects a point represented by the complex number z across the x-axis.

Discussion

We want to find a transformation $r_\ell: \mathbb{C} \to \mathbb{C}$ that reflects a point representing a complex number z across the diagonal line ℓ with equation $y = x$.



Exercises

1. The number z in the figure used in the discussion above is the complex number $1 + 5i$. Compute $r_\ell(1 + 5i)$ and plot it below.

2. We know from previous courses that the reflection of a point (x, y) across the line with equation $y = x$ is the point (y, x). Does this agree with our result from the previous discussion?

3. We now want to find a formula for the transformation of reflection across the line ℓ that makes a $60°$ angle with the positive x-axis. Find formulas to represent each component of the transformation, and use them to find one formula that represents the overall transformation.

Lesson Summary

Let ℓ be a line through the origin that contains the terminal ray of a rotation of the x-axis by θ. Then reflection across line ℓ can be done by the following sequence of transformations:

- Rotation by $-\theta$ about the origin.
- Reflection across the x-axis.
- Rotation by θ about the origin.

Problem Set

1. Find a formula for the transformation of reflection across the line ℓ with equation $y = -x$.

2. Find the formula for the sequence of transformations comprising reflection across the line with equation $y = x$ and then rotation by $180°$ about the origin.

3. Compare your answers to Problems 1 and 2. Explain what you find.

4. Find a formula for the transformation of reflection across the line ℓ that makes a $-30°$ angle with the positive x-axis.

5. Max observed that when reflecting a complex number, $z = a + bi$ about the line $y = x$, that a and b are reversed, which is similar to how we learned to find an inverse function. Will Max's observation also be true when the line $y = -x$ is used, where $a = -b$ and $b = -a$? Give an example to show his assumption is either correct or incorrect.

6. For reflecting a complex number, $z = a + bi$ about the line $y = 2x$, will Max's idea work if he makes $b = 2a$ and $a = \frac{b}{2}$? Use $z = 1 + 4i$ as an example to show whether or not it works.

7. What would the formula look like if you want to reflect a complex number about the line $y = mx$, where $m > 0$?

Lesson 17: The Geometric Effect of Multiplying by a Reciprocal

Classwork

Opening Exercise

Given $w = 1 + i$. What is $\arg(w)$ and $|w|$? Explain how you got your answer.

Exploratory Challenge 1/Exercises 1–9

1. Describe the geometric effect of the transformation $L(z) = (1 + i)z$.

2. Describe a way to undo the effect of the transformation $L(z) = (1 + i)z$.

3. Given that $0 \leq \arg(z) < 2\pi$ for any complex number, how could you describe any clockwise rotation of θ as an argument of a complex number?

4. Write a complex number in polar form that describes a rotation and dilation that will undo multiplication by $(1 + i)$, and then convert it to rectangular form.

5. In a previous lesson you learned that to undo multiplication by $1+i$, you would multiply by the reciprocal $\frac{1}{1+i}$. Write the complex number $\frac{1}{1+i}$ in rectangular form $z = a + bi$ where a and b are real numbers.

6. How do your answers to Exercises 4 and 5 compare? What does that tell you about the geometric effect of multiplication by the reciprocal of a complex number?

7. Jimmy states the following:

   Multiplication by $\frac{1}{a+bi}$ *has the reverse geometric effect of multiplication by* $+bi$*.*

   Do you agree or disagree? Use your work on the previous exercises to support your reasoning.

8. Show that the following statement is true when $z = 2 - 2\sqrt{3}i$:

   The reciprocal of a complex number z *with modulus* r *and argument* θ *is* $\frac{1}{z}$ *with modulus* $\frac{1}{r}$ *and argument* $2\pi - \theta$*.*

9. Explain using transformations why $z \cdot \frac{1}{z} = 1$.

Exploratory Challenge 2/Exercise 10

10. Complete the graphic organizer below to summarize your work with complex numbers so far.

Operation	**Geometric Transformation**	**Example. Illustrate algebraically and geometrically** **Let $z = 3 - 3i$ and $w = -2i$**
Addition $z + w$		
Subtraction $z - w$		
Conjugate of z		
Multiplication $w \cdot z$		
Reciprocal of z		
Division $\frac{w}{z}$		

Exercises 11–13

Let $z = -1 + i$ and let $w = 2i$. Describe each complex number as a transformation of z and then write the number in rectangular form.

11. $w\bar{z}$

12. $\frac{1}{\bar{z}}$

13. $\overline{w + z}$

Problem Set

1. Describe the geometric effect of multiplying z by the reciprocal of each complex number listed below.
   a. $w_1 = 3\mathrm{i}$
   b. $w_2 = -2$
   c. $w_3 = \sqrt{3} + i$
   d. $w_4 = 1 - \sqrt{3}i$

2. Let $z = -2 - 2\sqrt{3}i$. Show that the geometric transformations you described in Problem 1 really produce the correct complex number by performing the indicated operation and determining the argument and modulus of each number.
   a. $\frac{-2-2\sqrt{3}i}{w_1}$
   b. $\frac{-2-2\sqrt{3}i}{w_2}$
   c. $\frac{-2-2\sqrt{3}i}{w_3}$
   d. $\frac{-2-2\sqrt{3}i}{w_4}$

3. In Exercise 12 of this lesson you described the complex number $\frac{1}{\bar{z}}$ as a transformation of z for a specific complex number z. Show that this transformation always produces a dilation of $z = a + bi$.

4. Does $L(z) = \frac{1}{z}$ satisfy the conditions that $L(z + w) = L(z) + L(w)$ and $L(mz) = mL(z)$ which makes it a linear transformation? Justify your answer.

5. Show that $L(z) = w\left(\overline{\frac{1}{w}z}\right)$ describes a reflection of z about the line containing the origin and w for $z = 3i$ and $w = 1 + i$.

6. Describe the geometric effect of each transformation function on z where z, w, and a are complex numbers.
   a. $L_1(z) = \frac{z-w}{a}$
   b. $L_2(z) = \overline{\left(\frac{z-w}{a}\right)}$
   c. $L_3(z) = a\overline{\left(\frac{z-w}{a}\right)}$
   d. $L_3(z) = a\overline{\left(\frac{z-w}{a}\right)} + w$

7. Verify your answers to Problem 6 if $z = 1 - i$, $w = 2i$, and $a = -1 - i$.
   a. $L_1(z) = \frac{z-w}{a}$

b. $L_2(z) = \overline{\left(\frac{z-w}{a}\right)}$

c. $L_3(z) = a\overline{\left(\frac{z-w}{a}\right)}$

d. $L_3(z) = a\overline{\left(\frac{z-w}{a}\right)} + w$

Lesson 18: Exploiting the Connection to Trigonometry

Classwork

Opening Exercise

a. Identify the modulus and argument of each complex number, and then rewrite it in rectangular form.

i. $2\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$

ii. $5\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$

iii. $3\sqrt{2}\left(\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)\right)$

iv. $3\left(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right)$

v. $1(\cos(\pi) + i\sin(\pi))$

b. What is the argument and modulus of each complex number? Explain how you know.

i. $2 - 2i$

ii. $3\sqrt{3} + 3i$

iii. $-1 - \sqrt{3}i$

iv. $-5i$

v. 1

Exploratory Challenge /Exercises 1–12

1. Rewrite each expression as a complex number in rectangular form.
   a. $(1 + i)^2$

   b. $(1 + i)^3$

   c. $(1 + i)^4$

2. Complete the table below showing the rectangular form of each number and its modulus and argument.

Power of $(1+i)$	Rectangular Form	Modulus	Argument
$(1+i)^0$			
$(1+i)^1$			
$(1+i)^2$			
$(1+i)^3$			
$(1+i)^4$			

3. What patterns do you notice each time you multiply by another factor of $(1+i)$?

4. Graph each power of $1+i$ shown in the table on the same coordinate grid. Describe the location of these numbers in relation to one another using transformations.

5. Predict what the modulus and argument of $(1+i)^5$ would be without actually performing the multiplication. Explain how you made your prediction.

6. Graph $(1+i)^5$ in the complex plane using the transformations you described in Exercise 5.

7. Write each number in polar form using the modulus and argument you calculated in Exercise 4.

   $(1+i)^0$

   $(1+i)^1$

   $(1+i)^2$

   $(1+i)^3$

   $(1+i)^4$

8. Use the patterns you have observed to write $(1+i)^5$ in polar form, and then convert it to rectangular form.

9. What is the polar form of $(1+i)^{20}$? What is the modulus of $(1+i)^{20}$? What is its argument? Explain why $(1+i)^{20}$ is a real number.

10. If z has modulus r and argument θ, what is the modulus and argument of z^2? Write the number z^2 in polar form.

11. If z has modulus r and argument θ, what is the modulus and argument of z^n where n is a nonnegative integer? Write the number z^n in polar form. Explain how you got your answer.

12. Recall that $\frac{1}{z} = \frac{1}{r}(\cos(-\theta) + i\sin(-\theta))$. Explain why it would make sense that formula holds for all integer values of n.

Exercises 13–14

13. Compute $\left(\frac{1-i}{\sqrt{2}}\right)^7$ and write it as a complex number in the form $a + bi$ where a and b are real numbers.

14. Write $\left(1 + \sqrt{3}i\right)^6$, and write it as a complex number in the form $a + bi$ where a and b are real numbers.

Lesson Summary

Given a complex number z with modulus r and argument θ, the nth power of z is given by $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$ where n is an integer.

Problem Set

1. Write the complex number in $a + bi$ form where a and b are real numbers.
   a. $2\left(\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right)$
   b. $3(\cos(210°) + i\sin(210°))$
   c. $\left(\sqrt{2}\right)^{10}\left(\cos\left(\frac{15\pi}{4}\right) + i\sin\left(\frac{15\pi}{4}\right)\right)$
   d. $\cos(9\pi) + i\sin(9\pi)$
   e. $4^3\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$
   f. $6(\cos(480°) + i\sin(480°))$

2. Use the formula discovered in this lesson to compute each power of z. Verify that the formula works by expanding and multiplying the rectangular form and rewriting it in the form $a + bi$ where a and b are real numbers.
   a. $\left(1 + \sqrt{3}i\right)^3$
   b. $(-1 + i)^4$
   c. $(2 + 2i)^5$
   d. $(2 - 2i)^{-2}$
   e. $\left(\sqrt{3} - i\right)^4$
   f. $\left(3\sqrt{3} - 3i\right)^6$

3. Given $z = -1 - i$, graph the first five powers of z by applying your knowledge of the geometric effect of multiplication by a complex number. Explain how you determined the location of each in the coordinate plane.

4. Use your work from Problem 3 to determine three values of n for which $(-1 - i)^n$ is a multiple of $-1 - i$.

5. Find the indicated power of the complex number, and write your answer in form $a + bi$ where a and b are real numbers.
   a. $\left[2\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)\right]^3$
   b. $\left[\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)\right]^{10}$

c. $\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)^6$

d. $\left[\frac{1}{3}\left(\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right)\right]^4$

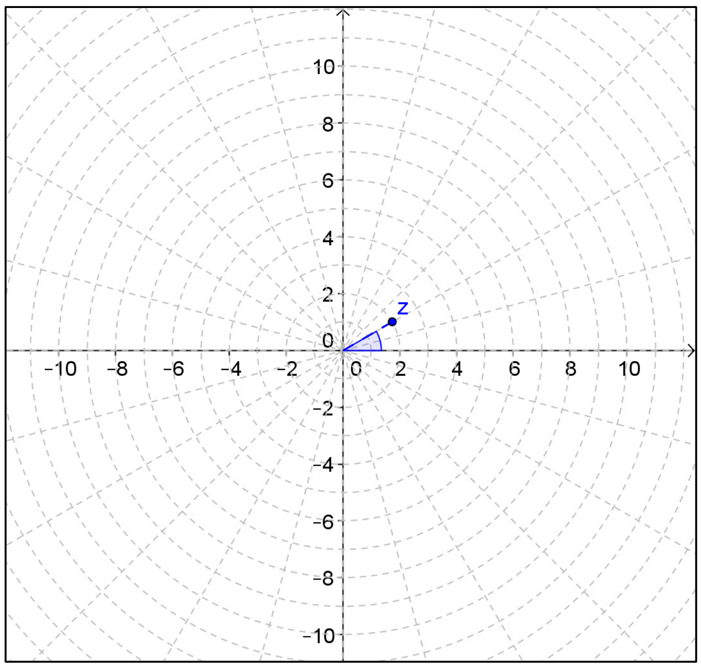
e. $\left[4\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right)\right]^{-4}$

Lesson 19: Exploiting the Connection to Trigonometry

Classwork

Opening Exercise

A polar grid is shown below. The grid is formed by rays from the origin at equal rotation intervals and concentric circles centered at the origin. The complex number $z = \sqrt{3} + i$ is graphed on this polar grid.



a. Use the polar grid to identify the modulus and argument of z.

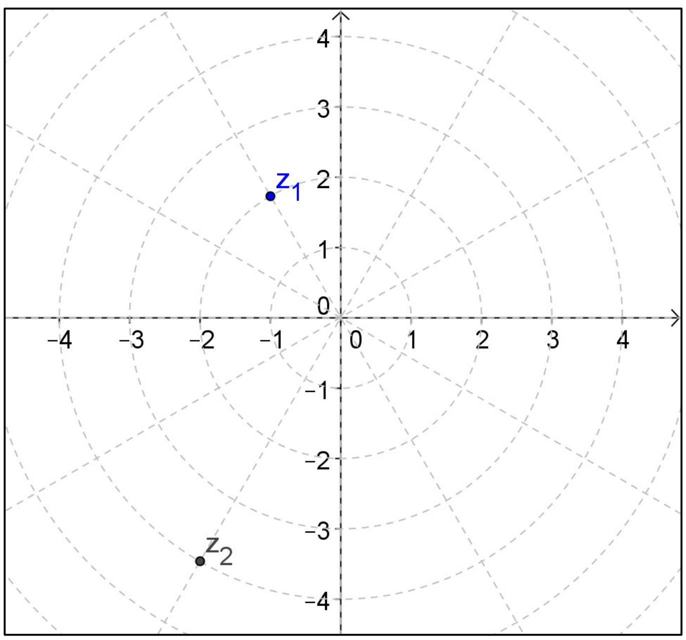
b. Graph the next three powers of z on the polar grid. Explain how you got your answers.

c. Write the polar form of the number in the table below, and then rewrite it in rectangular form.

Power of z	Polar Form	Rectangular Form
$\sqrt{3}+i$		
$\left(\sqrt{3}+i\right)^2$		
$\left(\sqrt{3}+i\right)^3$		
$\left(\sqrt{3}+i\right)^4$		

Exercises 1–3

The complex numbers $z_2 = \left(-1+\sqrt{3}i\right)^2$ and z_1 are graphed below.



1. Use the graph to help you write the numbers in polar and rectangular form.

2. Describe how the modulus and argument of $z_1 = -1+\sqrt{3i}$ are related to the modulus and argument of $z_2 = \left(-1+\sqrt{3}i\right)^2$.

3. Why could we call $-1 + \sqrt{3}i$ a square root of $-2 - 2\sqrt{3}i$?

Example 1: Find the Two Square Roots of a Complex Number

Find both of the square roots of $-2 - 2\sqrt{3}i$.

Exercises 4–6

4. Find the cube roots of $-2 = 2\sqrt{3}i$.

5. Find the square roots of $4i$.

6. Find the cube roots of 8.

Lesson Summary

Given a complex number z with modulus r and argument θ, the n^{th} roots of z are given by

$$\sqrt[n]{r}\left(\cos\left(\frac{\theta}{n}+\frac{2\pi k}{n}\right)+i\sin\left(\frac{\theta}{n}+\frac{2\pi k}{n}\right)\right)$$

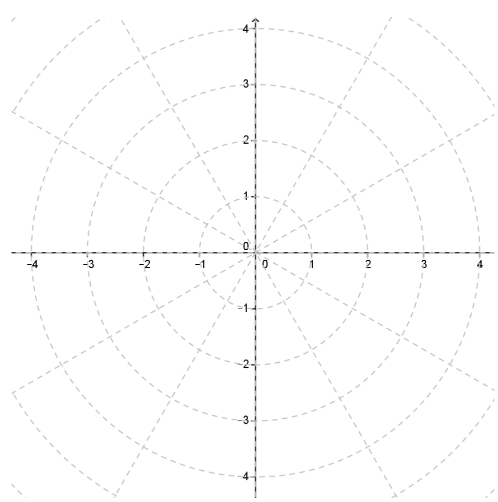
for integers k and n such that $n > 0$ and $0 \le k < n$.

Problem Set

1. For each complex number what is z^2?
   a. $1+\sqrt{3}i$
   b. $3-3i$
   c. $4i$
   d. $-\frac{\sqrt{3}}{2}+\frac{1}{2}i$
   e. $\frac{1}{9}+\frac{1}{9}i$
   f. -1

2. For each complex number, what are the square roots of z?
   a. $1+\sqrt{3}i$
   b. $3-3i$
   c. $4i$
   d. $-\frac{\sqrt{3}}{2}+\frac{1}{2}i$
   e. $\frac{1}{9}+\frac{1}{9}i$
   f. -1

3. For each complex number, graph z, z^2, and z^3 on a polar grid.
   a. $2\left(\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right)$
   b. $3(\cos(210°)+i\sin(210°))$
   c. $2\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right)$
   d. $\cos(\pi)+i\sin(\pi)$
   e. $4\left(\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right)$
   f. $\frac{1}{2}(\cos(60°)+i\sin(60°))$



4. What are the cube roots of $-3i$?

5. What are the fourth roots of 64?

6. What are the square roots of $-4-4i$?

7. Find the square roots of -5. Show that the square roots satisfy the equation $x^2+5=0$.

8. Find the cube roots of 27. Show that the cube roots satisfy the equation $x^3-27=0$.

Lesson 20: Exploiting the Connection to Cartesian Coordinates

Classwork

Opening Exercise

a. Find a complex number w so that the transformation $L_1(z) = wz$ produces a clockwise rotation by $1°$ about the origin with no dilation.

b. Find a complex number w so that the transformation $L_2(z) = wz$ produces a dilation with scale factor 0.1 with no rotation.

Exercises 1–4

1.

a. Find values of a and b so that $L_1(x, y) = (ax - by, bx + ay)$ has the effect of dilation with scale factor 2 and no rotation.

b. Evaluate $L_1(L_1(x,y))$, and identify the resulting transformation.

2.

a. Find values of a and b so that $L_2(x,y) = (ax - by, bx + ay)$ has the effect of rotation about the origin by $180°$ counterclockwise and no dilation.

b. Evaluate $L_2(L_2(x,y))$, and identify the resulting transformation.

3.

a. Find values of a and b so that $L_3(x,y) = (ax - by, bx + ay)$ has the effect of rotation about the origin by $90°$ counterclockwise and no dilation.

b. Evaluate $L_3\big(L_3(x,y)\big)$, and identify the resulting transformation.

4.

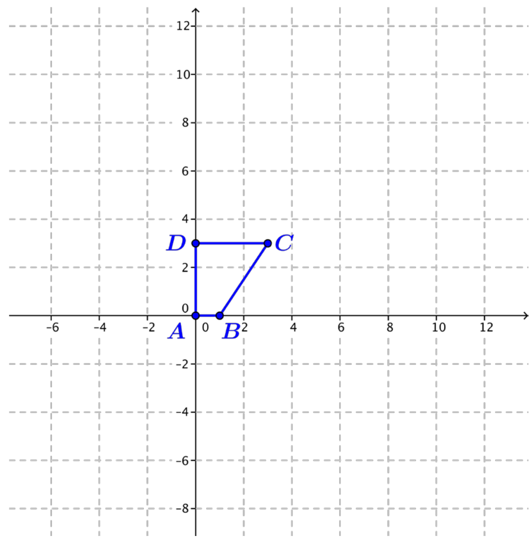
a. Find values of a and b so that $L_3(x,y) = (ax - by, bx + ay)$ has the effect of rotation about the origin by $45°$ counterclockwise and no dilation.

b. Evaluate $L_4\big(L_4(x,y)\big)$, and identify the resulting transformation.

Exercises 5–6

5. The figure below shows a quadrilateral with vertices $A(0,0)$, $B(1,0)$, $C(3,3)$, and $D(0,3)$.

   a. Transform each vertex under $L_5 = (3x + y, 3y - x)$, and plot the transformed vertices on the figure.

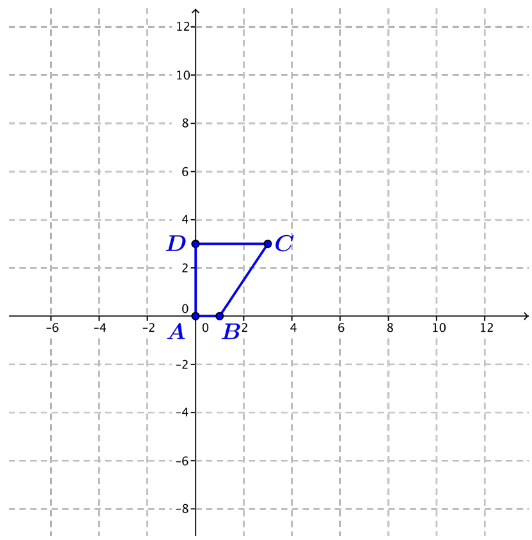


   b. Does L_5 represent a rotation and dilation? If so, estimate the amount of rotation and the scale factor from your figure.

   c. If L_5 represents a rotation and dilation, calculate the amount of rotation and the scale factor from the formula for L_5. Do your numbers agree with your estimate in part (b)? If not, explain why there are no values of a and b so that $L_5(x, y) = (ax - by, bx + ay)$.

6. The figure below shows a figure with vertices $A(0,0)$, $B(1,0)$, $C(3,3)$, and $D(0,3)$.

   a. Transform each vertex under $L_6 = (2x + 2y, 2x - 2y)$, and plot the transformed vertices on the figure.



   b. Does L_6 represent a rotation and dilation? If so, estimate the amount of rotation and the scale factor from your figure.

   c. If L_5 represents a rotation and dilation, calculate the amount of rotation and the scale factor from the formula for L_6. Do your numbers agree with your estimate in part (b)? If not, explain why there are no values of a and b so that $L_6(x, y) = (ax - by, bx + ay)$.

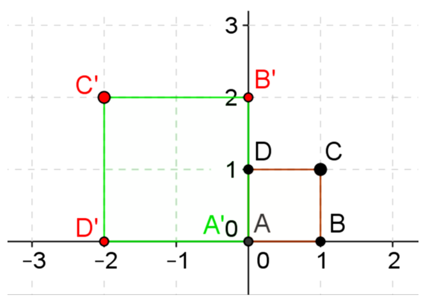
Lesson Summary

For real numbers a and b, the transformation $L(x, y) = (ax - by, bx + ay)$ corresponds to a counterclockwise rotation by $\arg(a + bi)$ about the origin and dilation with scale factor $\sqrt{a^2 + b^2}$.

Problem Set

1. Find real numbers a and b so that the transformation $L(x, y) = (ax - by, bx + ay)$ produces the specified rotation and dilation.
   a. Rotation by $270°$ counterclockwise and dilation by scale factor $\frac{1}{2}$.
   b. Rotation by $135°$ counterclockwise and dilation by scale factor $\sqrt{2}$.
   c. Rotation by $45°$ clockwise and dilation by scale factor 10.
   d. Rotation by $540°$ counterclockwise and dilation by scale factor 4.

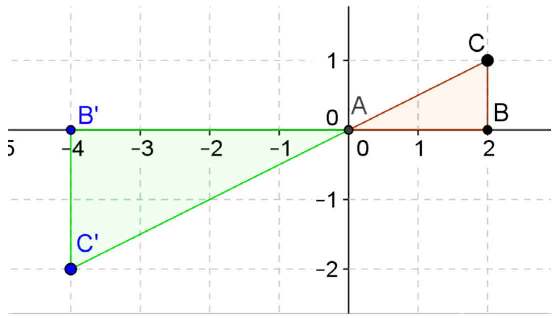
2. Determine if the following transformations represent a rotation and dilation. If so, identify the scale factor and the amount of rotation.



   a. $L(x, y) = (3x + 4y, 4x + 3y)$
   b. $L(x, y) = (-5x + 12y, -12x - 5y)$
   c. $L(x, y) = (3x + 3y, -3y + 3x)$

3. Grace and Lily have a different point of view about the transformation on cube $ABCD$ that is shown above. Grace states that it is a reflection about the imaginary axis and a dilation of factor of 2. However, Lily argues it should be a $90°$ counterclockwise rotation about the origin with a dilation of a factor of 2.
   a. Who is correct? Justify your answer.
   b. Represent the above transformation in the form $L(x, y) = (ax - by, bx + ay)$.

4. Grace and Lily still have a different point of view on this transformation on triangle ABC shown above. Grace states that it is reflected about the real axis first, then reflected about the imaginary axis, and then is dilated with a factor of 2. However, Lily asserts that it is a $180°$ counterclockwise rotation about the origin with a dilation of a factor of 2.



   a. Who is correct? Justify your answer.
   b. Represent the above transformation in the form $L(x, y) = (ax - by, bx + ay)$.

5. Given $z = \sqrt{3} + i$.
   a. Find the complex number w that will cause a rotation with the same number of degrees as z without a dilation.
   b. Can you come up with a general formula $L(x, y) = (ax - by, bx + ay)$ for any complex number $z = x + yi$ to represent this condition?

Lesson 21: The Hunt for Better Notation

Classwork

Opening Exercise

Suppose that $L_1(x, y) = (2x - 3y, 3x + 2y)$ and $L_2(x, y) = (3x + 4y, -4y + 3x)$.
Find the result of performing L_1 and then L_2 on a point (p, q). That is, find $L_2\big(L_1(p, q)\big)$.

Exercises 1–2

1. Calculate each of the following products.

   a. $\begin{pmatrix} 3 & -2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 5 \end{pmatrix}$

   b. $\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ -4 \end{pmatrix}$

   c. $\begin{pmatrix} 2 & -4 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

2. Find a value of k so that $\begin{pmatrix} 1 & 2 \\ k & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 11 \end{pmatrix}$.

Exercises 3–9

3. Find a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ so that we can represent the transformation $L(x, y) = (2x - 3y, 3x + 2y)$ by $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

4. If a transformation $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ has the geometric effect of rotation and dilation, do you know about the values a, b, c, and d?

5. Describe the form of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ so that the transformation $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ has the geometric effect of only dilation by a scale factor r.

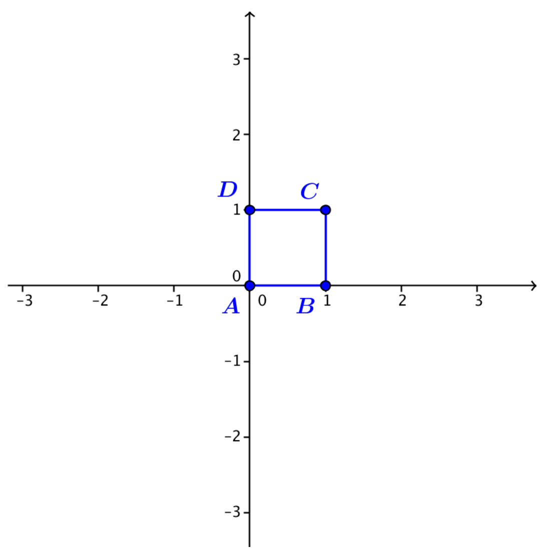
6. Describe the form of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ so that the transformation $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ has the geometric effect of only rotation by θ. Describe the matrix in terms of θ.

7. Describe the form of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ so that the transformation $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ has the geometric effect of rotation by θ and dilation with scale factor r. Describe the matrix in terms of θ and r.

8. Suppose that we have a transformation $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

   a. Does this transformation have the geometric effect of rotation and dilation?

   b. Transform each of the points $A = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $D = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and plot the images in the plane shown.



9. Describe the geometric effect of the transformation $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

Lesson Summary

For real numbers a, b, c, and d, the transformation $L(x, y) = (ax + by, cx + dy)$ can be represented using matrix multiplication by $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$ and the $\begin{pmatrix} x \\ y \end{pmatrix}$ represents the point (x, y) in the plane.

- The transformation is a counterclockwise rotation by θ if and only if the matrix representation is $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.
- The transformation is a dilation with scale factor k if and only if the matrix representation is $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.
- The transformation is a counterclockwise rotation by $\arg(a + bi)$ and dilation with scale factor $|a + bi|$ if and only if the matrix representation is $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. If we let $r = |a + bi|$ and $\theta = \arg(a + bi)$, then the matrix representation is $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos(\theta) & -r\sin(\theta) \\ r\sin(\theta) & r\cos(\theta) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

Problem Set

1. Perform the indicated multiplication.
   a. $\begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix}$
   b. $\begin{pmatrix} 3 & 5 \\ -2 & -6 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \end{pmatrix}$
   c. $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 6 \\ 8 \end{pmatrix}$
   d. $\begin{pmatrix} 5 & 7 \\ 4 & 9 \end{pmatrix}\begin{pmatrix} 10 \\ 100 \end{pmatrix}$
   e. $\begin{pmatrix} 4 & 2 \\ 3 & 7 \end{pmatrix}\begin{pmatrix} -3 \\ 1 \end{pmatrix}$
   f. $\begin{pmatrix} 6 & 4 \\ 9 & 6 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \end{pmatrix}$
   g. $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
   h. $\begin{pmatrix} \pi & 1 \\ 1 & -\pi \end{pmatrix}\begin{pmatrix} 10 \\ 7 \end{pmatrix}$

2. Find a value of k so that $\begin{pmatrix} k & 3 \\ 4 & k \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \end{pmatrix}$.

3. Find values of k and m so that $\begin{pmatrix} k & 3 \\ -2 & m \end{pmatrix}\begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 7 \\ -10 \end{pmatrix}$.

4. Find values of k and m so that $\begin{pmatrix} 1 & 2 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} k \\ m \end{pmatrix} = \begin{pmatrix} 0 \\ -9 \end{pmatrix}$.

5. Write the following transformations using matrix multiplication.
   a. $L(x, y) = (3x - 2y, 4x - 5y)$
   b. $L(x, y) = (6x + 10y, -2x + y)$
   c. $L(x, y) = (25x + 10y, 8x - 64y)$
   d. $L(x, y) = (\pi x - y, -2x + 3y)$
   e. $L(x, y) = (10x, 100x)$
   f. $L(x, y) = (2y, 7x)$

6. Identify whether or not the following transformations have the geometric effect of rotation only, dilation only, rotation and dilation only, or none of these.
   a. $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 4 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
   b. $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 42 & 0 \\ 0 & 42 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
   c. $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 & -2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
   d. $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
   e. $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -7 & 1 \\ 1 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
   f. $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

7. Create a matrix representation of a linear transformation that has the specified geometric effect.
   a. Dilation by a factor of 4 and no rotation.
   b. Rotation by $180°$ and no dilation.
   c. Rotation by $-\frac{\pi}{2}$ rad and dilation by a scale factor of 3.
   d. Rotation by $30°$ and dilation by a scale factor of 4.

8. Identify the geometric effect of the following transformations. Justify your answer.
   a. $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
   b. $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -5 \\ 5 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
   c. $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -10 & 0 \\ 0 & -10 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
   d. $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 & 6\sqrt{3} \\ -6\sqrt{3} & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Lesson 22: Modeling Video Game Motion with Matrices

Classwork

Opening Exercise

Let $D\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

a. Plot the point $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

b. Find $D\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, and plot it.

c. Describe the geometric effect of performing the transformation $\begin{pmatrix} x \\ y \end{pmatrix} \to D\begin{pmatrix} x \\ y \end{pmatrix}$.

Exercises 1–2

1. Let $f(t) = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}\begin{pmatrix} 2 \\ 4 \end{pmatrix}$, where t represents time, measured in seconds. $P = f(t)$ represents the position of a moving object at time t. If the object starts at the origin, how long would it take to reach $(12, 24)$?

2. Let $g(t) = \begin{pmatrix} kt & 0 \\ 0 & kt \end{pmatrix}\begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

   a. Find the value of k that moves an object from the origin to $(12,24)$ in just 2 seconds.

   b. Find the value of k that moves an object from the origin to $(12, 24)$ in 30 seconds.

Exercises 3–4

3. Let $f(t) = \begin{pmatrix} 2+t & 0 \\ 0 & 2+t \end{pmatrix}\begin{pmatrix} 5 \\ 7 \end{pmatrix}$, where t represents time, measured in seconds, and $f(t)$ represents the position of a moving object at time t.

   a. Find the position of the object at $t = 0, t = 1$, and $t = 2$.

   b. Write $f(t)$ in the form $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$.

4. Write the transformation $g(t) = \begin{pmatrix} 15 + 5t \\ -6 - 2t \end{pmatrix}$ as a matrix transformation.

Exercise 5

5. An object is moving in a straight line from $(18,12)$ to the origin over a 6-second period of time. Find a function $f(t)$ that gives the position of the object after t seconds. Write your answer in the form $f(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, then express $f(t)$ as a matrix transformation.

Exercises 6–9

6. Write a rule for the function that shifts every point in the plane 6 units to the left.

7. Write a rule for the function that shifts every point in the plane 9 units upward.

8. Write a rule for the function that shifts every point in the plane 10 units down and 4 units to the right.

9. Consider the rule $\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x-7 \\ y+2 \end{pmatrix}$. Describe the effect this transformation has on the plane.

Problem Set

1. Let $D\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2&0\\0&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$ find and plot the following.
   a. Plot the point: $\begin{pmatrix}-1\\2\end{pmatrix}$ and find $D\begin{pmatrix}-1\\2\end{pmatrix}$, and plot it.
   b. Plot the point: $\begin{pmatrix}3\\4\end{pmatrix}$ and find $D\begin{pmatrix}3\\4\end{pmatrix}$, and plot it.
   c. Plot the point: $\begin{pmatrix}5\\2\end{pmatrix}$ and find $D\begin{pmatrix}5\\2\end{pmatrix}$, and plot it.

2. Let $f(t)=\begin{pmatrix}t&0\\0&t\end{pmatrix}\begin{pmatrix}-1\\2\end{pmatrix}$, find $f(0), f(1), f(2), f(3)$, and plot them on the same graph.

3. Let $f(t)=\begin{pmatrix}t&0\\0&t\end{pmatrix}\begin{pmatrix}3\\2\end{pmatrix}$ represent the location of an object at time t that is measured in seconds.
   a. How long does it take the object to travel from the origin to the point $\begin{pmatrix}12\\8\end{pmatrix}$?
   b. Find the speed of the object in the horizontal direction and in the vertical direction.

4. Let $f(t)=\begin{pmatrix}0.2t&0\\0&0.2t\end{pmatrix}\begin{pmatrix}3\\2\end{pmatrix}$, $h(t)=\begin{pmatrix}2t&0\\0&2t\end{pmatrix}\begin{pmatrix}3\\2\end{pmatrix}$. Which one will reach the point $\begin{pmatrix}12\\8\end{pmatrix}$ first? The time t is measured in seconds.

5. Let $f(t)=\begin{pmatrix}kt&0\\0&kt\end{pmatrix}\begin{pmatrix}3\\2\end{pmatrix}$, find the value of k that moves the object from the origin to $\begin{pmatrix}-45\\-30\end{pmatrix}$ in 5 seconds.

6. Write $f(t)$ in the form $\begin{pmatrix}x(t)\\y(t)\end{pmatrix}$ if
   a. $f(t)=\begin{pmatrix}t&0\\0&t\end{pmatrix}\begin{pmatrix}2\\5\end{pmatrix}$.
   b. $f(t)=\begin{pmatrix}2t+1&0\\0&2t+1\end{pmatrix}\begin{pmatrix}3\\2\end{pmatrix}$.
   c. $f(t)=\begin{pmatrix}\frac{t}{2}-3&0\\0&\frac{t}{2}-3\end{pmatrix}\begin{pmatrix}4\\-6\end{pmatrix}$.

7. Let $f(t)=\begin{pmatrix}t&0\\0&t\end{pmatrix}\begin{pmatrix}2\\5\end{pmatrix}$ represent the location of an object after t seconds.
   a. If the object starts at $\begin{pmatrix}6\\15\end{pmatrix}$, how long would it take to reach $\begin{pmatrix}34\\85\end{pmatrix}$?
   b. Write the new function $f(t)$ that gives the position of the object after t seconds.
   c. Write $f(t)$ as a matrix transformation.

8. Write the following functions as a matrix transformation.
   a. $f(t) = \begin{pmatrix} 10 + 2t \\ 15 + 3t \end{pmatrix}$
   b. $f(t) = \begin{pmatrix} -6t + 15 \\ 8t - 20 \end{pmatrix}$

9. Write a function rule that represents the change in position of the point $\begin{pmatrix} x \\ y \end{pmatrix}$ for the following.
   a. 5 units to the right and 3 units downward.
   b. 2 units downward and 3 units to the left
   c. 3 units upward, 5 units to the left, and then it dilates by 2
   d. 3 units upward, 5 units to the left, and then it rotates by $\frac{\pi}{2}$ counterclockwise.

10. Annie is designing a video game and wants her main character to be able to move from any given point $\begin{pmatrix} x \\ y \end{pmatrix}$ in the following ways: right 1 unit, jump up 1 unit, and both jump up and move right 1 unit each.
   a. What function rules can she use to represent each time the character moves?
   b. Annie is also developing a ski slope stage for her game and wants to model her character's position using matrix transformations. Annie wants the player to start at $\begin{pmatrix} -20 \\ 10 \end{pmatrix}$ and eventually pass through the origin moving 5 units per second down. How fast does the player need to move to the right in order to pass through the origin? What matrix transformation can Annie use to describe the movement of the character? If the far-right of the screen is at $x = 20$, how long until the player moves off the screen traveling this path?

11. Remy thinks that he has developed matrix transformations to model the movements of Annie's characters in Problem 10 from any given point $\begin{pmatrix} x \\ y \end{pmatrix}$, and he has tested them on the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. This is the work Remy did on the transformations:
   $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\qquad$ $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\qquad$ $\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

   Do these matrix transformations accomplish the movements that Annie wants to program into the game? Explain why or why not.

12. Nolan has been working on how to know when the path of a point can be described with matrix transformations and how to know when it requires translations and cannot be described with matrix transformations. So far he has been focusing on the following two functions which both pass through the point $(2,5)$:
   $f(t) = \begin{pmatrix} 2t + 6 \\ 5t + 15 \end{pmatrix}$ and $g(t) = \begin{pmatrix} t + 2 \\ t + 5 \end{pmatrix}$
   a. If we simplify these functions algebraically, how does the rule for f differ from the rule for g? What does this say about which function can be expressed with matrix transformations?
   b. Nolan has noticed functions that can be expressed with matrix transformations always pass through the origin; does either f or g pass through the origin, and does this support or contradict Nolan's reasoning?
   c. Summarize the results of parts (a) and (b) to describe how we can tell from the equation for a function or from the graph of a function that it can be expressed with matrix transformations.

Lesson 23: Modeling Video Game Motion with Matrices

Classwork

Opening Exercise

Let $R\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\cos\left(\frac{\pi}{3}\right) & -\sin\left(\frac{\pi}{3}\right)\\ \sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right)\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$.

a. Describe the geometric effect of performing the transformation $\begin{pmatrix}x\\y\end{pmatrix}\to R\begin{pmatrix}x\\y\end{pmatrix}$.

b. Plot the point $\begin{pmatrix}1\\0\end{pmatrix}$, then find $R\begin{pmatrix}1\\0\end{pmatrix}$ and plot it.

c. If we want to show that R has been applied twice to $(1,0)$, we can write $R^2\begin{pmatrix}1\\0\end{pmatrix}$, which represents $R\left(R\begin{pmatrix}1\\0\end{pmatrix}\right)$.

Find $R^2\begin{pmatrix}1\\0\end{pmatrix}$, and plot it. Then find $R^3\begin{pmatrix}1\\0\end{pmatrix}=R\left(R\left(R\begin{pmatrix}1\\0\end{pmatrix}\right)\right)$, and plot it.

d. Describe the matrix transformation $\begin{pmatrix}x\\y\end{pmatrix}\to R^2\begin{pmatrix}x\\y\end{pmatrix}$ using a single matrix.

Exercises

1. Let $f(t) = \begin{pmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and let $g(t) = \begin{pmatrix} \cos\left(\frac{t}{2}\right) & -\sin\left(\frac{t}{2}\right) \\ \sin\left(\frac{t}{2}\right) & \cos\left(\frac{t}{2}\right) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

   a. Suppose $f(t)$ represents the position of a moving object that starts at $(1,0)$. How long does it take for this object to return to its starting point? When the argument of the trigonometric function changes from t to $2t$, what effect does this have?

   b. If the position is given instead by $g(t)$, how long would it take the object to return to its starting point? When the argument of the trigonometric functions changes from t to $\frac{t}{2}$, what effect does this have?

2. Let $G(t) = \begin{pmatrix} \cos\left(\frac{\pi}{2} \cdot t\right) & -\sin\left(\frac{\pi}{2} \cdot t\right) \\ \sin\left(\frac{\pi}{2} \cdot t\right) & \cos\left(\frac{\pi}{2} \cdot t\right) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

   a. Draw the path that $P = G(t)$ traces out as t varies within the interval $0 \leq t \leq 1$.

   b. Where will the object be at $t = 3$ seconds?

   c. How long will it take the object to reach $(0, -1)$?

3. Let $H(t) = \begin{pmatrix} \cos\left(\frac{\pi}{2} \cdot t\right) & -\sin\left(\frac{\pi}{2} \cdot t\right) \\ \sin\left(\frac{\pi}{2} \cdot t\right) & \cos\left(\frac{\pi}{2} \cdot t\right) \end{pmatrix} \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

   a. Draw the path that $P = H(t)$ traces out as t varies within the interval $0 \leq t \leq 2$.

   b. Where will the object be at $t = 1$ seconds?

   c. How long will it take the object to return to its starting point?

4. Suppose you want to write a program that takes the point $(3, 5)$ and rotates it about the origin to the point $(-3, -5)$ over a 1-second interval. Write a function $P = f(t)$ that encodes this rotation.

5. If instead you wanted the rotation to take place over a 1.5-second interval, how would your function change?

Problem Set

1. Let $R\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, find the following.

   a. $R^2\begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix}$

   b. How many transformations do you need to take so that the image returns to where it started?

   c. Describe the matrix transformation $\begin{pmatrix} x \\ y \end{pmatrix} \to R^2\begin{pmatrix} x \\ y \end{pmatrix}$, and $R^n\begin{pmatrix} x \\ y \end{pmatrix}$ using a single matrix.

2. For $f(t) = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, it takes 2π to transform the object at $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ back to where it starts. How long does it take the following functions to return to their starting point?

   a. $f(t) = \begin{pmatrix} \cos(3t) & -\sin(3t) \\ \sin(3t) & \cos(3t) \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

   b. $f(t) = \begin{pmatrix} \cos\left(\frac{t}{3}\right) & -\sin\left(\frac{t}{3}\right) \\ \sin\left(\frac{t}{3}\right) & \cos\left(\frac{t}{3}\right) \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

   c. $f(t) = \begin{pmatrix} \cos\left(\frac{2t}{5}\right) & -\sin\left(\frac{2t}{5}\right) \\ \sin\left(\frac{2t}{5}\right) & \cos\left(\frac{2t}{5}\right) \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

3. Let $F(t) = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, where t is measured in radians. Find the following:

   a. $F\left(\frac{3\pi}{2}\right)$, $F\left(\frac{7\pi}{6}\right)$ and the radius of the path.

   b. Show that the radius is always $\sqrt{x^2 + y^2}$ for the path of this transformation $(\mathrm{t}) = \begin{pmatrix} \cos(\mathrm{t}) & -\sin(\mathrm{t}) \\ \sin(\mathrm{t}) & \cos(\mathrm{t}) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

4. Let $F(t) = \begin{pmatrix} \cos\left(\frac{\pi t}{2}\right) & -\sin\left(\frac{\pi t}{2}\right) \\ \sin\left(\frac{\pi t}{2}\right) & \cos\left(\frac{\pi t}{2}\right) \end{pmatrix}\begin{pmatrix} 4 \\ 4 \end{pmatrix}$, where t is a real number.

   a. Draw the path that $P = F(t)$ traces out as t varies within each of the following intervals:

      i. $0 \le t \le 1$

      ii. $1 \le t \le 2$

      iii. $2 \le t \le 3$

      iv. $3 \le t \le 4$

   b. Where will the object be located at $t = 2.5$ seconds?

   c. How long does it take the object to reach $\begin{pmatrix} -8\sqrt{6} \\ 8\sqrt{2} \end{pmatrix}$

5. Let $F(t) = \begin{pmatrix} \cos\left(\frac{\pi t}{3}\right) & -\sin\left(\frac{\pi t}{3}\right) \\ \sin\left(\frac{\pi t}{3}\right) & \cos\left(\frac{\pi t}{3}\right) \end{pmatrix} \begin{pmatrix} -1 \\ -\sqrt{3} \end{pmatrix}$

   a. Draw the path that $P = F(t)$ traces out as t varies within the interval $0 \le t \le 1$.

   b. How long does it take the object to reach $\left(\sqrt{3}, 0\right)$

   c. How long does it take the object to return to its starting point?

6. Find the function that will rotate the point $(4, 2)$ about the origin to the point $(-4, -2)$over the following time intervals.

   a. Over a 1-second interval

   b. Over a 2-second interval

   c. Over a $\frac{1}{3}$-second interval

   d. How about rotating it back to where it starts over a $\frac{4}{5}$-second interval?

7. Summarize the geometric effect of the following function at the given point and the time interval.

   a. $F(t) = \begin{pmatrix} 5\cos\left(\frac{\pi t}{4}\right) & -5\sin\left(\frac{\pi t}{4}\right) \\ 5\sin\left(\frac{\pi t}{4}\right) & 5\cos\left(\frac{\pi t}{4}\right) \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix}, 0 \le t \le 1$

   b. $F(t) = \begin{pmatrix} \frac{1}{2}\cos\left(\frac{\pi t}{6}\right) & -\frac{1}{2}\sin\left(\frac{\pi t}{6}\right) \\ \frac{1}{2}\sin\left(\frac{\pi t}{6}\right) & \frac{1}{2}\cos\left(\frac{\pi t}{6}\right) \end{pmatrix} \begin{pmatrix} 6 \\ 2 \end{pmatrix}, 0 \le t \le 1$

8. In programming a computer video game, Grace coded the changing location of a rocket as follows:

   At the time t second between $t = 0$ seconds and $t = 4$ seconds, the location $\begin{pmatrix} x \\ y \end{pmatrix}$ of the rocket is given by

   $\begin{pmatrix} \cos\left(\frac{\pi}{4}t\right) & -\sin\left(\frac{\pi}{4}t\right) \\ \sin\left(\frac{\pi}{4}t\right) & \cos\left(\frac{\pi}{4}t\right) \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix}$.

   At a time of t seconds between $t = 4$ and $t = 8$ seconds, the location of the rocket is given by

   $\begin{pmatrix} -\sqrt{2} + \frac{\sqrt{2}}{2}(t-4) \\ -\sqrt{2} + \frac{\sqrt{2}}{2}(t-4) \end{pmatrix}$.

   a. What is the location of the rocket at time $t = 0$? What is its location at time $t = 8$?

   b. Mason is worried that Grace may have made a mistake and the location of the rocket is unclear at time $t = 4$ seconds. Explain why there is no inconsistency in the location of the rocket at this time.

   c. What is the area of the region enclosed by the path of the rocket from time $t = 0$ to $t = 8$?

Lesson 24: Matrix Notation Encompasses New Transformations!

Classwork

Example 1

Determine the following:

a. $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}3\\-2\end{bmatrix}$

b. $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}-7\\12\end{bmatrix}$

c. $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}3 & 5\\-2 & 1\end{bmatrix}$

d. $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}-1 & -3\\-7 & 6\end{bmatrix}$

e. $\begin{bmatrix}9 & 12\\-3 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

f. $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}a & c\\b & d\end{bmatrix}$

g. $\begin{bmatrix}x & y\\z & w\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

Example 2

Can the reflection about the real axis $L(z) = \bar{z}$ be expressed in matrix notation?

Exercises 1–3

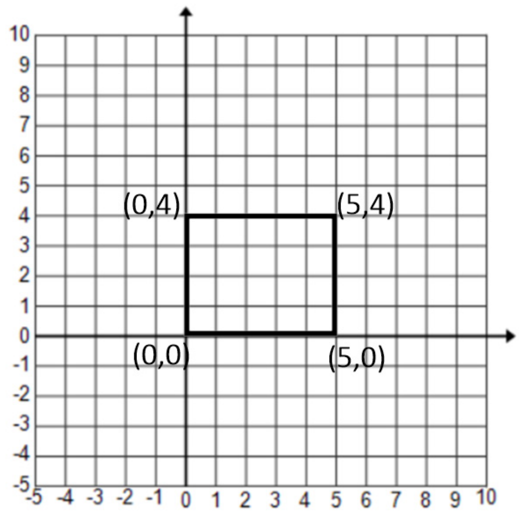
1. Express a reflection about the vertical axis in matrix notation. Prove that it produces the desired reflection by using matrix multiplication.

2. Express a reflection about both the horizontal and vertical axes in matrix notation. Prove that it produces the desired reflection by using matrix multiplication.

3. Express a reflection about the vertical axis and a dilation with a scale factor of 6 in matrix notation. Prove that it produces the desired reflection by using matrix multiplication.

Exercises 4–8

Explore the transformation given by each matrix below. Use the graph of the rectangle provided to assist in the exploration. Describe the effect on the graph of the rectangle, and then show the general effect of the transformation by using matrix multiplication.



Matrix	Transformation of the rectangle	General effect of the matrix
4. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$		
5. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$		
6. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$		
7. $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$		
8. $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$		

Lesson Summary

All matrices in the form $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ correspond to a transformation of some kind.

- The matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ reflects all coordinates about the horizontal axis.
- The matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ reflects all coordinates about the vertical axis.
- The matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity matrix and corresponds to a transformation that leaves points alone.
- The matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is the zero matrix and corresponds to a dilation of scale factor 0.

Problem Set

1. What matrix do you need to use to reflect the following points about the y-axis? What is the resulting matrix when this is done? Show all work and sketch it.

   a. $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$

   b. $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

   c. $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$

   d. $\begin{pmatrix} -3 \\ -2 \end{pmatrix}$

   e. $\begin{pmatrix} 3 \\ -3 \end{pmatrix}$

   f. $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$

2. What matrix do you need to use to reflect the following points about the x-axis? What is the resulting matrix when this is done? Show all work and sketch it.

   a. $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$

   b. $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

   c. $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$

   d. $\begin{pmatrix} -3 \\ -2 \end{pmatrix}$

   e. $\begin{pmatrix} 3 \\ -3 \end{pmatrix}$

   f. $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$

3. What matrix do you need to use to dilate the following points by a given factor? What is the resulting matrix when this is done? Show all work and sketch it.

   a. $\begin{pmatrix}1\\0\end{pmatrix}$, a factor of 3

   b. $\begin{pmatrix}3\\2\end{pmatrix}$, a factor of 2

   c. $\begin{pmatrix}1\\-2\end{pmatrix}$, a factor of 1

   d. $\begin{pmatrix}-4\\-6\end{pmatrix}$, a factor of $\frac{1}{2}$

   e. $\begin{pmatrix}9\\3\end{pmatrix}$, a factor of $\frac{1}{3}$

   f. $\begin{pmatrix}\sqrt{3}\\\sqrt{11}\end{pmatrix}$, a factor of $\sqrt{2}$

4. What matrix will rotate the given point by the angle? What is the resulting matrix when this is done? Show all work and sketch it.

   a. $\begin{pmatrix}1\\0\end{pmatrix}$, $\frac{\pi}{2}$ radians

   b. $\begin{pmatrix}1\\0\end{pmatrix}$, $\frac{\pi}{3}$ radians

   c. $\begin{pmatrix}1\\0\end{pmatrix}$, $\frac{\pi}{6}$ radians

   d. $\begin{pmatrix}1\\0\end{pmatrix}$, $\frac{\pi}{4}$ radians

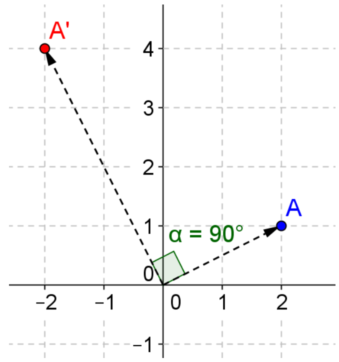
   e. $\begin{pmatrix}\frac{\sqrt{3}}{2}\\\frac{1}{2}\end{pmatrix}$, $\frac{\pi}{6}$ radians

   f. $\begin{pmatrix}\frac{\sqrt{2}}{2}\\\frac{\sqrt{2}}{2}\end{pmatrix}$, $\frac{\pi}{4}$ radians

   g. $\begin{pmatrix}\frac{\sqrt{3}}{2}\\\frac{1}{2}\end{pmatrix}$, π radians

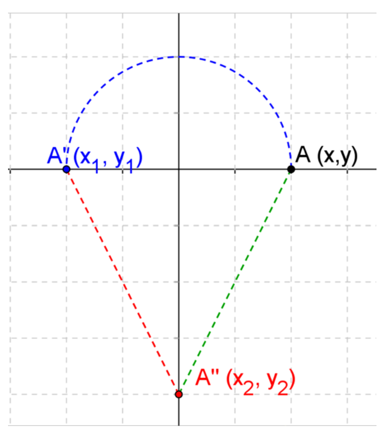
   h. $\begin{pmatrix}1\\0\end{pmatrix}$, $-\frac{\pi}{6}$ radians

5. For the transformation shown below, find the matrix that will transform point A to A', and verify your answer.

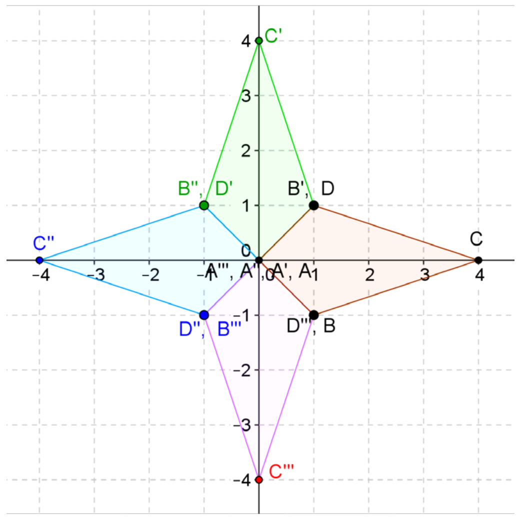


6. In this lesson, we learned $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ will produce a reflection about the line $y = x$. What matrix will produce a reflection about the line $y = -x$? Verify your answers by testing the given point $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and graphing them on the coordinate plane.

7. Describe the transformation and the translations in the diagram below. Write the matrices that will perform the tasks. What is the area that these transformations and translations have enclosed?



8. Given the kite figure $ABCD$ below, answer the following questions.

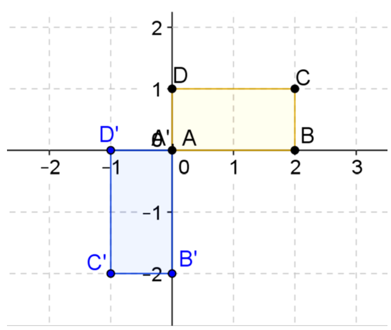


a. Explain how you would create the star figure above using only rotations.

b. Explain how to create the star figure above using reflections and rotation.

c. Explain how to create the star figure above using only reflections. Explain your answer.

9. Given the rectangle $ABCD$ below, answer the following questions.



a. Can you transform the rectangle $A'B'C'D'$ above using only rotations? Explain your answer.

b. Describe a way to create the rectangle $A'B'C'D'$.

c. Can you make the rectangle $A'B'C'D'$ above using only reflections? Explain your answer.

Lesson 25: Matrix Multiplication and Addition

Classwork

Opening Exercise

Consider the point $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ that undergoes a series of two transformations: a dilation of scale factor 4 followed by a reflection about the horizontal axis.

a. What matrix produces the dilation of scale factor 4? What is the coordinate of the point after the dilation?

b. What matrix produces the reflection about the horizontal axis? What is the coordinate of the point after the reflection?

c. Could we have produced both the dilation and the reflection using a single matrix? If so, what matrix would both dilate by a scale factor of 4 and produce a reflection about the horizontal axis? Show that the matrix you came up with combines these two matrices.

Example 1: Is Matrix Multiplication Commutative?

a. Take the point $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ through the following transformations: a rotation of $\frac{\pi}{2}$ and a reflection across the y-axis.

b. Will the resulting point be the same if the order of the transformations is reversed?

c. Are transformations commutative?

d. Let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Find AB and then BA.

e. Is matrix multiplication commutative?

f. If we apply matrix AB to the point $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, in what order are the transformations applied.

g. If we apply matrix BA to the point $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, in what order are the transformations applied.

h. Can we apply $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ to matrix BA?

Exercises 1–3

1. Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $M = \begin{pmatrix} 4 & -6 \\ 3 & -2 \end{pmatrix}$.

   a. Find IM.

   b. Find MI.

   c. Do these results make sense based on what you know about the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$?

2. Calculate AB, then BA. Is matrix multiplication commutative?

   a. $A = \begin{pmatrix} 2 & 0 \\ -2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$

   b. $A = \begin{pmatrix} -10 & 1 \\ 3 & 7 \end{pmatrix}$, $B = \begin{pmatrix} -3 & 2 \\ 4 & -1 \end{pmatrix}$

3. Write a matrix that would perform the following transformations in this order: a rotation of $180°$, a dilation by a scale factor of 4, and a reflection across the horizontal axis. Use the point $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ to illustrate that your matrix is correct.

Example 2: More Operations on Matrices

Find the sum. $\begin{pmatrix} 2 & 0 \\ -2 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$

Find the difference. $\begin{pmatrix} 2 & 0 \\ -2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$

Find the sum. $\begin{pmatrix} 2 & 0 \\ -2 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Exercises 4–5

4. Express each of the following as a single matrix.

   a. $\begin{pmatrix} 6 & -3 \\ 10 & -1 \end{pmatrix} + \begin{pmatrix} -2 & 8 \\ 3 & -12 \end{pmatrix}$

b. $\begin{pmatrix} -2 & 7 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 5 \end{pmatrix}$

c. $\begin{pmatrix} 8 & 5 \\ 0 & 15 \end{pmatrix} - \begin{pmatrix} 4 & -6 \\ -3 & 18 \end{pmatrix}$

5. In arithmetic, the additive identity says that for some number a, $a + 0 = 0 + a = 0$. What would be an additive identity in matrix arithmetic?

Lesson Summary

- If L is given by $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and M is given by $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$, then $ML\begin{pmatrix} x \\ y \end{pmatrix}$ is the same as applying the matrix $\begin{pmatrix} pa+rb & pc+rd \\ qa+sb & qc+sd \end{pmatrix}$ to $\begin{pmatrix} x \\ y \end{pmatrix}$.
- If L is given by $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and I is given by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then I acts as a multiplicative identity and $IL = LI = L$.
- If L is given by $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and O is given by $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then O acts as an additive identity and $O + L = L + O = L$.

Problem Set

1. What type of transformation is shown in the following examples? What is the resulting matrix?
   a. $\begin{pmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$
   b. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$
   c. $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$
   d. $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$
   e. $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$
   f. $\begin{pmatrix} \cos 2\pi & -\sin 2\pi \\ \sin 2\pi & \cos 2\pi \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$
   g. $\begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$
   h. $\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

2. Calculate each of the following products.
   a. $\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$
   b. $\begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$
   c. $\begin{pmatrix} -1 & -3 \\ -2 & -4 \end{pmatrix}\begin{pmatrix} -3 \\ -1 \end{pmatrix}$
   d. $\begin{pmatrix} -3 & -1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} -2 & -1 \\ 0 & -3 \end{pmatrix}$

e. $\begin{pmatrix} 5 & 0 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 3 \end{pmatrix}$

f. $\begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & -3 \end{pmatrix}$

3. Calculate each sum or difference.

a. $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix}$

b. $\begin{pmatrix} -4 & -5 \\ -6 & -7 \end{pmatrix} + \begin{pmatrix} -2 & 3 \\ -1 & 4 \end{pmatrix}$

c. $\begin{pmatrix} -5 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

d. $\begin{pmatrix} 3 \\ -5 \end{pmatrix} - \begin{pmatrix} 7 \\ 9 \end{pmatrix}$

e. $\begin{pmatrix} -4 & -5 \\ -6 & -7 \end{pmatrix} - \begin{pmatrix} -2 & 3 \\ -1 & 4 \end{pmatrix}$

4. In video game programming, Fahad translates a car, whose coordinate is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, 2 units up and 4 units to the right, rotates it $\frac{\pi}{2}$ radians counterclockwise, reflects it about the x-axis, reflects it about the y-axis rotates it $\frac{\pi}{2}$ radians counterclockwise, and finally translates it 4 units down and 2 units to the left. What point represents the final location of the car?

Lesson 26: Getting a Handle on New Transformations

Classwork

Opening Exercise

Perform the following matrix operations:

a. $\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

b. $\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix}$

d. $\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

e. $\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}+\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

f. $\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}+\begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix}$

g. $\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}+\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

h. Can you add the two matrices in part (a)? Why or why not?

Exercises 1–3

1. Perform the transformation $\begin{bmatrix} 109 & 3 \\ 1 & -2 \end{bmatrix}$ on the unit square.

   a. Sketch the image. What is the shape of the image?

   b. What are the coordinates of the vertices of the image?

   c. What is the area of the image? Show your work.

2. In the Exploratory Challenge, we drew the image of a general rotation/dilation of the unit square $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

   a. Calculate the area of the image by enclosing the image in a rectangle and subtracting the area of surrounding right triangles. Show your work.

   b. Confirm the area using the determinant of the resulting matrix.

3. We have looked at several general matrix transformations in Module 1. Answer the questions below about these familiar matrices and explain your answers.

   a. What effect does the identity transformation have on the unit square? What is the area of the image? Confirm your answer using the determinant.

   b. How does a dilation with a scale factor of k change the area of the unit square? Calculate the determinant of a matrix representing a pure dilation of k.

   c. Does a rotation with no dilation change the area of the unit square? Confirm your answer by calculating the determinant of a pure rotation matrix and explain.

Lesson Summary

Definition

- The area of the image of the unit square under the linear transformation represented by a 2×2 matrix is called the *determinant* of that matrix.

Problem Set

1. Perform the following transformation on the unit square: sketch the image and the area of the image.
   a. $\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$
   b. $\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$
   c. $\begin{bmatrix} 4 & -2 \\ 2 & 4 \end{bmatrix}$
   d. $\begin{bmatrix} 2 & -4 \\ 4 & 2 \end{bmatrix}$

2. Perform the following transformation on the unit square: sketch the image, find the determinant of the given matrix, and find the area the image.
   a. $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$
   b. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
   c. $\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$
   d. $\begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$
   e. The determinants in parts (a), (b), (c), and (d) have positive or negative values. What is the value of the determinants if the vertices (b, c) and (c, d) are switched?

3. Perform the following transformation on the unit square: sketch the image, find the determinant of the given matrix, and find the area the image.
   a. $\begin{bmatrix} -1 & -3 \\ -2 & -4 \end{bmatrix}$
   b. $\begin{bmatrix} -1 & -3 \\ 2 & 4 \end{bmatrix}$
   c. $\begin{bmatrix} 1 & 3 \\ -2 & -4 \end{bmatrix}$
   d. $\begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$
   e. $\begin{bmatrix} 1 & -3 \\ 2 & -4 \end{bmatrix}$

f. $\begin{bmatrix} -1 & 3 \\ 2 & -4 \end{bmatrix}$

g. $\begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$

Lesson 27: Getting a Handle on New Transformations

Classwork

Opening Exercise

Explain the geometric effect of each matrix.

a. $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

b. $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

c. $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$

d. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

e. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

f. $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$

Example 1

Given the transformation $\begin{bmatrix} k & 0 \\ k & 1 \end{bmatrix}$ with $k > 0$:

a. Perform this transformation on the vertices of the unit square. Sketch the image and label the vertices.

b. Calculate the area of the image using the dimensions of the image parallelogram.

c. Confirm the area of the image using the determinant.

d. Perform the transformation on $\begin{bmatrix} x \\ y \end{bmatrix}$.

e. In order for two matrices to be equivalent, each of the corresponding elements must be equivalent. Given that, if the image of this transformation is $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$, find $\begin{bmatrix} x \\ y \end{bmatrix}$.

f. Perform the transformation on $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Write the image matrix.

g. Perform the transformation on the image again, and then repeat until the transformation has been performed four times on the image of the preceding matrix.

Exercise 1

1. Perform the transformation $\begin{bmatrix} k & 0 \\ 1 & k \end{bmatrix}$ with $k > 1$ on the vertices of the unit square.

   a. What are the vertices of the image?

   b. Calculate the area of the image.

   c. If the image of the transformation on $\begin{bmatrix} x \\ y \end{bmatrix}$ is $\begin{bmatrix} -2 \\ -1 \end{bmatrix}$, find $\begin{bmatrix} x \\ y \end{bmatrix}$ in terms of k.

Example 2

Consider the matrix $L = \begin{bmatrix} 2 & 5 \\ -1 & 3 \end{bmatrix}$. For each real number $0 \le t \le 1$ consider the point $(3 + t, 10 + 2t)$.

a. Find point A when $t = 0$.

b. Find point B when $t = 1$.

c. Show that for $t = \frac{1}{2}$, $(3 + t, 10 + 2t)$ is the midpoint of $\overline{AB}$.

d. Show that for each value of t, $(3 + t, 10 + 2t)$ is a point on the line through A and B.

e. Find LA and LB.

f. What is the equation of the line through LA and LB?

g. Show that $L\begin{bmatrix} 3+t \\ 10+2t \end{bmatrix}$ lies on the line through LA and LB.

Problem Set

1. Perform the following transformation on the vertices of the unit square. Sketch the image, label the vertices, and find the area of the image parallelogram.
   a. $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$
   b. $\begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}$
   c. $\begin{bmatrix} 3 & 0 \\ 3 & 1 \end{bmatrix}$
   d. $\begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$
   e. $\begin{bmatrix} 3 & 0 \\ 3 & 3 \end{bmatrix}$
   f. $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$
   g. $\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$
   h. $\begin{bmatrix} 2 & 0 \\ 2 & 3 \end{bmatrix}$
   i. $\begin{bmatrix} 3 & 0 \\ 3 & 5 \end{bmatrix}$

2. Given $\begin{bmatrix} k & 0 \\ k & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ kx + y \end{bmatrix}$. Find $\begin{bmatrix} x \\ y \end{bmatrix}$ if the image of the transformation is the following:
   a. $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$
   b. $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$
   c. $\begin{bmatrix} 5 \\ -6 \end{bmatrix}$

3. Given $\begin{bmatrix} k & 0 \\ k & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ kx + y \end{bmatrix}$. Find value of k so that:
   a. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and the image is $\begin{bmatrix} 24 \\ 22 \end{bmatrix}$
   b. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 27 \\ 3 \end{bmatrix}$ and the image is $\begin{bmatrix} 18 \\ 21 \end{bmatrix}$

4. Given $\begin{bmatrix} k & 0 \\ 1 & k \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ x + ky \end{bmatrix}$. Find value of k so that:
   a. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ 5 \end{bmatrix}$ and the image is $\begin{bmatrix} -12 \\ 11 \end{bmatrix}$
   b. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{2}{9} \end{bmatrix}$ and image is $\begin{bmatrix} -15 \\ -\frac{1}{3} \end{bmatrix}$

5. Perform the following transformation on the vertices of the unit square. Sketch the image, label the vertices, and find the area of the image parallelogram.

   a. $\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$

   b. $\begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}$

   c. $\begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$

   d. $\begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix}$

   e. $\begin{bmatrix} 4 & 0 \\ 2 & 4 \end{bmatrix}$

   f. $\begin{bmatrix} 3 & 0 \\ 5 & 3 \end{bmatrix}$

6. Consider the matrix $L = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$. For each real number $o \leq t \leq 1$, consider the point $(3 + 2t, 12 + 2t)$.

   a. Find the point A when $t = 0$.

   b. Find the point B when $t = 1$.

   c. Show that for $t = \frac{1}{2}$, $(3 + 2t, 12 + 2t)$ is the midpoint of $\overline{AB}$.

   d. Show that for each value of t, $(3 + 2t, 12 + 2t)$ is a point on the line through A and B.

   e. Find LA and LB.

   f. What is the equation of the line through LA and LB?

   g. Show that $L\begin{bmatrix} 3 + 2t \\ 12 + 2t \end{bmatrix}$ lies on the line through LA and LB.

Lesson 28: When Can We Reverse a Transformation?

Classwork

Opening Exercise

Perform the operation $\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$ on the unit square.

a. State the vertices of the transformation.

b. Explain the transformation in words.

c. Find the area of the transformed figure.

d. If the original square was 2×2 instead of a unit square, how would the transformation change?

e. What is the area of the image? Explain how you know.

Example 1

What transformation reverses a pure dilation from the origin with a scale factor of k?

a. Write the pure dilation matrix and multiply it by $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$.

b. What values of a, b, c, and d would produce the identity matrix? (Hint: Write and solve a system of equations.)

c. Write the matrix and confirm that it reverses the pure dilation with a scale factor of k.

Exercises 1–3

Find the inverse matrix and verify.

1. $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

2. $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$

3. $\begin{bmatrix} -2 & -5 \\ 1 & 2 \end{bmatrix}$

Problem Set

1. In this lesson, we learned $R_\theta R_{-\theta} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Chad was saying that he found an easy way to find the inverse matrix, which is: $R_{-\theta} = \frac{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}{R_\theta}$. His argument is that if we have $2x = 1$, then $x = \frac{1}{2}$.
   a. Is Chad correct? Explain your reason.
   b. If Chad is not correct, what is the correct way to find the inverse matrix?

2. Find the inverse matrix and verify it.
   a. $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$
   b. $\begin{bmatrix} -2 & -1 \\ 3 & 1 \end{bmatrix}$
   c. $\begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix}$
   d. $\begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}$
   e. $\begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}$

3. Find the starting point $\begin{bmatrix} x \\ y \end{bmatrix}$ if
   a. the point $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ is the image of a pure dilation with a factor of 2.
   b. the point $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ is the image of a pure dilation with a factor of $\frac{1}{2}$.
   c. the point $\begin{bmatrix} -10 \\ 35 \end{bmatrix}$ is the image of a pure dilation with a factor of 5.
   d. the point $\begin{bmatrix} \frac{4}{9} \\ \frac{16}{21} \end{bmatrix}$ is the image of a pure dilation with a factor of $\frac{2}{3}$.

4. Find the starting point if
   a. $3 + 2i$ is the image of a reflection about the real axis.
   b. $3 + 2i$ is the image of a reflection about the imaginary axis.
   c. $3 + 2i$ is the image of a reflection about the real axis and then the imaginary axis.
   d. $-3 - 2i$ is the image of a π radians counterclockwise rotation.

5. Let's call the pure counterclockwise rotation of the matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ as R_θ, and the "undo" of the pure rotation is $\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$ as $R_{-\theta}$.
   a. Simplify $\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$.

b. What would you get if you multiply R_θ to $R_{-\theta}$?

c. Write the matrix if you want to rotate $\frac{\pi}{2}$ radians counterclockwise.

d. Write the matrix if you want to rotate $\frac{\pi}{2}$ radians clockwise.

e. Write the matrix if you want to rotate $\frac{\pi}{6}$ radians counterclockwise.

f. Write the matrix if you want to rotate $\frac{\pi}{4}$ radians counterclockwise.

g. If the point $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ is the image of $\frac{\pi}{4}$ radians counterclockwise rotation, find the starting point $\begin{bmatrix} x \\ y \end{bmatrix}$.

h. If the point $\begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$ is the image of $\frac{\pi}{6}$ radians counterclockwise rotation, find the starting point $\begin{bmatrix} x \\ y \end{bmatrix}$.

Lesson 29: When Can We Reverse a Transformation?

Classwork

Opening Exercise

Find the inverse of $\begin{bmatrix} -7 & -2 \\ 4 & 1 \end{bmatrix}$. Show your work. Confirm that the matrices are inverses.

Exercises

1. Find the inverse of $\begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix}$. Confirm your answer.

Find the inverse matrix and verify.

2. $\begin{bmatrix} 3 & -3 \\ 1 & 4 \end{bmatrix}$

3. $\begin{bmatrix} 5 & -2 \\ 4 & -3 \end{bmatrix}$

4. $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$

Example 1

Find the determinant of $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

Problem Set

Find the inverse matrix of the following.

a. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

e. $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

f. $\begin{bmatrix} -2 & 2 \\ -5 & 4 \end{bmatrix}$

g. $\begin{bmatrix} 4 & 6 \\ 5 & 8 \end{bmatrix}$

h. $\begin{bmatrix} 6 & -9 \\ 5 & -7 \end{bmatrix}$

i. $\begin{bmatrix} \frac{1}{2} & -\frac{2}{3} \\ -6 & 4 \end{bmatrix}$

j. $\begin{bmatrix} 0.8 & 0.4 \\ -0.75 & -0.5 \end{bmatrix}$

Lesson 30: When Can We Reverse a Transformation?

Classwork

Opening Exercise

a. What is the geometric effect of the following matrices?

i. $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$

ii. $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

iii. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

b. Jadavis says that the identity matrix is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Sophie disagrees and states that the identity matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

i. Their teacher, Mr. Kuzy, says they are both correct and asks them to explain their thinking about matrices to each other, but also use a similar example in the real number system. Can you state each of their arguments?

ii. Mr. Kuzy then asks each of them to explain the geometric effect that their matrix would have on the unit square.

c. Given the matrices below, answer the following:

$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 5 & 2 \\ 10 & 4 \end{bmatrix}$

i. Which matrix does not have an inverse? Explain how you know algebraically and geometrically.

ii. If a matrix has an inverse, find it.

Example 1

Given $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$.

a. Perform this transformation on the unit square, and sketch the results on graph paper. Label the vertices.

b. Explain the transformation that occurred to the unit square.

c. Find the area of the image.

d. Find the inverse of this transformation.

e. Explain the meaning of the inverse transformation on the unit square.

Exercises 1–8

1. Given $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$.

   a. Perform this transformation on the unit square, and sketch the results on graph paper. Label the vertices.

   b. Explain the transformation that occurred to the unit square.

   c. Find the area of the image.

   d. Find the inverse of this transformation.

   e. Explain the meaning of the inverse transformation on the unit square.

   f. If any matrix produces a dilation with a scale factor of k, what would the inverse matrix produce?

2. Given $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

   a. Perform this transformation on the unit square, and sketch the results on graph paper. Label the vertices.

b. Explain the transformation that occurred to the unit square.

c. Find the area of the image.

d. Find the inverse of the transformation.

e. Explain the meaning of the inverse transformation on the unit square.

f. Rewrite the original matrix if it also included a dilation with a scale factor of 2.

g. What is the inverse of this matrix?

3. Find a transformation that would create a $90°$ counterclockwise rotation about the origin. Set up a system of equations and solve to find the matrix.

4.

a. Find a transformation that would create a $180°$ counterclockwise rotation about the origin. Set up a system of equations and solve to find the matrix.

b. Rewrite the matrix to also include a dilation with a scale factor of 5.

5. For which values of a does $\begin{bmatrix} 3 & -100 \\ 900 & a \end{bmatrix}$ have an inverse matrix?

6. For which values of a does $\begin{bmatrix} a & a+4 \\ 2 & a \end{bmatrix}$ have an inverse matrix?

7. For which values of a does $\begin{bmatrix} a+2 & a-4 \\ a-3 & a+3 \end{bmatrix}$ have an inverse matrix?

8. Chethan says that the matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ produces a rotation $\theta°$ counterclockwise. He justifies his work by showing that when $\theta = 60°$, the rotation matrix is $\begin{bmatrix} \cos(60) & -\sin(60) \\ \sin(60) & \cos(60) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$. Shayla disagrees and says that the matrix $\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$ produces a $60°$ rotation counterclockwise. Tyler says that he has found that the matrix $\begin{bmatrix} 2 & -2\sqrt{3} \\ 2\sqrt{3} & 2 \end{bmatrix}$ produces a $60°$ rotation counterclockwise, too.

a. Who is correct? Explain.

b. Which matrix has the largest scale factor? Explain.

c. Create a matrix with a scale factor less than 1 that would produce the same rotation.

Problem Set

1. Find a transformation that would create a $30°$ counterclockwise rotation about the origin and then its inverse.

2. Find a transformation that would create a $30°$ counterclockwise rotation about the origin, a dilation with a scale factor of 4, and then its inverse.

3. Find a transformation that would create a $270°$ counterclockwise rotation about the origin. Set up a system of equations and solve to find the matrix.

4. Find a transformation that would create a $270°$ counterclockwise rotation about the origin, a dilation with a scale factor of 3, and its inverse.

5. For which values of a does $\begin{bmatrix} 8 & a \\ a & 2 \end{bmatrix}$ have an inverse matrix?

6. For which values of a does $\begin{bmatrix} a & a-4 \\ a+4 & a \end{bmatrix}$ have an inverse matrix?

7. For which values of a does $\begin{bmatrix} 3a & 2a-6 \\ 6a & 4a-12 \end{bmatrix}$ have an inverse matrix?

8. In Lesson 27, we learned the effect of a transformation on a unit square by multiplying a matrix. For example, $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
   a. Sasha says that we can multiply the inverse of A to those resultants of the square after the transformation to get back to the unit square. Is her conjecture correct? Justify your answer.
   b. From part (a), what would you say about the inverse matrix with regard to the geometric effect of transformations?
   c. A pure rotation matrix is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Prove the inverse matrix for a pure rotation of $\frac{\pi}{4}$ radians counterclockwise is $\begin{bmatrix} \cos(-\frac{\pi}{4}) & -\sin(-\frac{\pi}{4}) \\ sin(-\frac{\pi}{4}) & \cos(-\frac{\pi}{4}) \end{bmatrix}$, which is the same as $\begin{bmatrix} \frac{d}{ad-bc} & \frac{-c}{ad-bc} \\ \frac{-b}{ad-bc} & \frac{d}{ad-bc} \end{bmatrix}$.
   d. Prove that the inverse matrix of a pure dilation with a factor of 4 is $\begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$, which is the same as $\begin{bmatrix} \frac{d}{ad-bc} & \frac{-c}{ad-bc} \\ \frac{-b}{ad-bc} & \frac{d}{ad-bc} \end{bmatrix}$.

e. Prove that the matrix used to undo a $\frac{\pi}{3}$ radians clockwise rotation and a dilation of a factor of 2 is $\frac{1}{2}\begin{bmatrix}\cos\left(\frac{\pi}{3}\right) & -\sin\left(\frac{\pi}{3}\right)\\ \sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right)\end{bmatrix}$, which is the same as $\begin{bmatrix}\frac{d}{ad-bc} & \frac{-c}{ad-bc}\\ \frac{-b}{ad-bc} & \frac{d}{ad-bc}\end{bmatrix}$.

f. Prove that any matrix whose determinant is not 0 will have an inverse matrix to "undo" a transformation. For example, use the matrix $A = \begin{bmatrix}a & c\\ b & d\end{bmatrix}$ and the point $\begin{bmatrix}x\\ y\end{bmatrix}$.

9. Perform the transformation $\begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix}$ on the unit square.

a. Can you find the inverse matrix that will "undo" the transformation? Explain your reasons arithmetically.

b. When all four vertices of the unit square are transformed and collapsed onto a straight line, what can be said about the inverse?

c. Find the equation of the line that all four vertices of the unit square collapsed onto.

d. Find the equation of the line that all four vertices of the unit square collapsed onto using the matrix $\begin{bmatrix}1 & 3\\ 2 & 6\end{bmatrix}$.

e. A function has an inverse function if and only if it is a one-to-one function. By applying this concept, explain why we do not have an inverse matrix when the transformation is collapsed onto a straight line.

10. The determinants of the following matrices are 0. Describe what pattern you can find among them.

a. $\begin{bmatrix}1 & 2\\ 1 & 2\end{bmatrix}$, $\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}$, $\begin{bmatrix}1 & 2\\ 4 & 8\end{bmatrix}$, and $\begin{bmatrix}1 & -2\\ 2 & -4\end{bmatrix}$

b. $\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}$, $\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$, $\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}$, $\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$, and $\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$