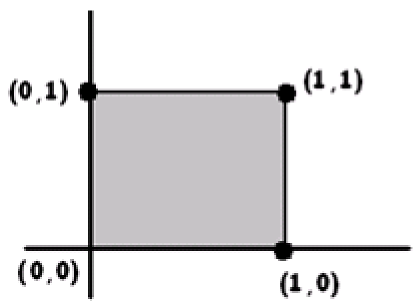
Name ______________________________ Date ______________

1. Consider the transformation on the plane given by the 2×2 matrix $\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}$ for a fixed positive number $k > 1$.

   a. Draw a sketch of the image of the unit square under this transformation (the unit square has vertices$(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$). Be sure to label all four vertices of the image figure.



The Unit Square

b. What is the area of the image parallelogram?

c. Find the coordinates of a point $\begin{pmatrix} x \\ y \end{pmatrix}$ whose image under the transformation is $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

d. The transformation $\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}$ is applied once to the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then once to the image point, then once to the image of the image point, and then once to the image of the image of the image point, and so on. What are the coordinates of a tenfold image of the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, that is, the image of the point after the transformation has been applied 10 times?

2. Consider the transformation given by $\begin{pmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{pmatrix}$.

a. Describe the geometric effect of applying this transformation to a point $\begin{pmatrix} x \\ y \end{pmatrix}$ in the plane.

b. Describe the geometric effect of applying this transformation to a point $\begin{pmatrix} x \\ y \end{pmatrix}$ in the plane twice: once to the point and then once to its image.

c. Use part (b) to prove $\cos(2) = \cos^2(1) - \sin^2(1)$ and $\sin(2) = 2\sin(1)\cos(1)$.

3.

a. Explain the geometric representation of multiplying a complex number by $1 + i$.

b. Write $(1 + i)^{10}$ as a complex number of the form $a + bi$ for real numbers a and b.

c. Find a complex number $a + bi$, with a and b positive real numbers, such that $(a + bi)^3 = i$.

d. If z is a complex number, is there sure to exist, for any positive integer n, a complex number w such that $w^n = z$? Explain your answer.

e. If z is a complex number, is there sure to exist, for any negative integer n, a complex number w such that $w^n = z$? Explain your answer.

4. Let $P = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

a. Give an example of a 2×2 matrix A, not with all entries equal to zero, such that $PA = O$.

b. Give an example of a 2×2 matrix B with $PB \neq O$.

c. Give an example of a 2×2 matrix C such that $CR = R$ for all 2×2 matrices R.

d. If a 2×2 matrix D has the property that $D + R = R$ for all 2×2 matrices R, must D be the zero matrix O? Explain.

e. Let $E = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}$. Is there 2×2 matrix F so that $EF = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $FE = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$? If so, find one. If not, explain why no such matrix F can exist.

5. In programming a computer video game, Mavis coded the changing location of a space rocket as follows: At a time t seconds between $t = 0$ seconds and $t = 2$ seconds, the location $\begin{pmatrix} x \\ y \end{pmatrix}$ of the rocket is given by

$$\begin{pmatrix} \cos\left(\frac{\pi}{2}t\right) & -\sin\left(\frac{\pi}{2}t\right) \\ \sin\left(\frac{\pi}{2}t\right) & \cos\left(\frac{\pi}{2}t\right) \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

At a time of t seconds between $t = 2$ seconds and $t = 4$ seconds, the location of the rocket is given by $\begin{pmatrix} 3 - t \\ 3 - t \end{pmatrix}$.

a. What is the location of the rocket at time $t = 0$? What is its location at time $t = 4$?

b. Petrich is worried that Mavis may have made a mistake and the location of the rocket is unclear at time $t = 2$ seconds. Explain why there is no inconsistency in the location of the rocket at this time.

c. What is the area of the region enclosed by the path of the rocket from time $t = 0$ to time $t = 4$?

d. Mavis later decided that the moving rocket should be shifted five places farther to the right. How should she adjust her formulations above to accomplish this translation?

A Progression Toward Mastery

Assessment Task Item		STEP 1 Missing or incorrect answer and little evidence of reasoning or application of mathematics to solve the problem.	STEP 2 Missing or incorrect answer but evidence of some reasoning or application of mathematics to solve the problem.	STEP 3 A correct answer with some evidence of reasoning or application of mathematics to solve the problem, <u>OR</u> an incorrect answer with substantial evidence of solid reasoning or application of mathematics to solve the problem.	STEP 4 A correct answer supported by substantial evidence of solid reasoning or application of mathematics to solve the problem.
1	**a** N-VM.C.11 N-VM.C.12	Student provides a solution that does not apply matrix multiplication or transformations to determine the coordinates of the resulting image. The sketch is missing.	Student computes two or more coordinates of the image incorrectly, and the sketch of the image is incomplete or poorly labeled, or the image is a parallelogram with no work shown and no vertices labeled.	Student computes coordinates of the image correctly, but the sketch of the image may be slightly inaccurate. Work to support the calculation of the image coordinates is limited. <u>OR</u> Student computes three out of four coordinates correctly and the sketch accurately reflects the student's coordinates.	Student applies matrix multiplication to each coordinate of the unit square to get the image coordinates and draws a fairly accurate sketch of a parallelogram with vertices correctly labeled. Values for k will vary, but the resulting image should look like a parallelogram, and the distance k in the vertical and horizontal direction should appear equal.
	b N-VM.C.12	Student does not compute the area of a parallelogram or his sketched figure correctly.	Student computes the area of his sketched figure correctly but does not use determinant of the 2×2 matrix in his calculation.	Student computes the area of his figure using the determinant of the 2×2 matrix, but the solution may contain minor errors.	Student computes area of the parallelogram correctly using a determinant. Work shows understanding that the area of the image is the product of the area of the original figure and the absolute value of the determinant of the transformation matrix.

	c **N-VM.C.10** **N-VM.C.11**	Student does not provide a solution. OR Student provides work that is unrelated to the standards addressed in this problem.	Student computes an incorrect solution or setup of the original matrix equation. Limited evidence is evident that the student understands that the solution to the matrix equation will find the point in question. OR Student creates a correct matrix equation, and no additional work is given. OR Student creates the correct system of linear equations, and no additional work Is given.	Student creates a correct matrix equation to solve for the point and translates the equation to a system of linear equations. Work shown may be incomplete, and final answer may contain minor errors. OR Student has the correct solution, but the matrix equation or the system of equations is missing from the solution. Very little work is shown to provide evidence of student thinking.	Student creates a correct matrix equation to solve for the point. Student translates the equation to a system of linear equations and solves the system correctly. Work shown is organized in a manner that is easy to follow and uses proper mathematical notation.
	d **N-VM.C.11** **N-VM.C.12**	Student provides a solution that does not correctly apply the transformation one time. AND Student does not attempt a generalization for the tenfold image.	Student provides a solution that does not correctly apply the transformation more than one time. Student may attempt to generalize to the tenfold image, but the answer contains major conceptual errors.	Student provides a solution that includes evidence that the student understood the problem and observed patterns, but minor errors prevent a correct solution for the tenfold image OR Student provides a solution that shows correct repeated application of the transformation at least three times, but the student is unable to extend the pattern to the tenfold image.	Student gives correct solution for the tenfold image. Student solution provides enough evidence and explanation to clearly illustrate how she observed and extended the pattern.
2	**a** **N-VM.C.11** **N-VM.C.12**	Student does not recognize the transformation as a rotation of the point about the origin.	Student identifies the transformation as a rotation but cannot correctly state the direction or the angle measure.	Student correctly identifies the transformation as a rotation about the origin, but the answer contains an error, such as the wrong direction or the wrong angle measurement.	Student correctly identifies the transformation as a counterclockwise rotation about the origin through an angle of 1 radian.

	b **N-VM.C.11** **N-VM.C.12**	Student does not identify the repeated transformation as a rotation.	Student identifies the transformation as a rotation, but the solution does not make it clear that the second rotation applies to the image of the original point OR Student identifies the transformation as an additional rotation, but the answer contains two or more errors.	Student correctly identifies the repeated transformation as an additional rotation, but the answer contains no more than one error.	Student correctly identifies the repeated transformation as a rotation of the image of the point another 1 radian clockwise about the origin for a total of 2 radians.
	c **N-VM.C.8**	Student makes little or no attempt at multiplying the point (x, y) by either of the rotation matrices.	Student sets up and attempts the necessary matrix multiplications, but solution has too many major errors. OR Student provides too little work to make significant progress on the proof.	Student provides a solution that includes multiplication of (x, y) by the original rotation matrix twice and multiplication of (x, y) by the 2-radian rotation matrix. Student fails to equate the two answers to finish the proof. The solution may contain minor computation errors.	Student provides a solution that details multiplication by the original rotation matrix twice, compares that result to multiplication by the 2-radian rotation matrix, and equates the two answers to verify the identities. Student uses correct notation, and the solution illustrates her thinking clearly. The solution is free from minor errors.
3	**a** **N-CN.B.5**	Student makes little or no attempt to explain the geometric relationship of multiplying by $1 + i$.	Student attempts to explain the geometric relationship of multiplying by $1 + i$ but makes mistakes.	Student attempts to explain the geometric relationship of multiplying by $1 + i$ but mentions either the dilation or rotation, not both.	Student fully explains the geometric relationship of multiplying by $1 + i$ in terms of a dilation and a rotation.
	b **N-CN.B.4**	Student makes little or no attempt to find the modulus and argument.	Student attempts to find the modulus and argument, but solution has major errors that lead to an incorrect answer.	Student has the correct answer but may not be in proper form or makes minor computational errors in finding the modulus and argument.	Student writes the correct answer in the proper form and correctly solves for the modulus and argument of the expression, showing all steps.

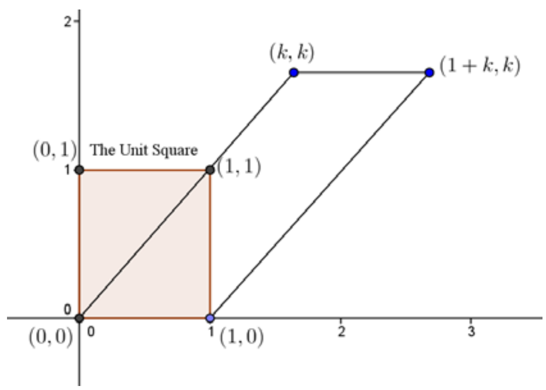
	c **N-CN.B.4** **N-CN.B.5**	Student makes little or no attempt to solve for a complex number.	Student attempts to find a complex number but lacks the proper steps in order to do so, resulting in an incorrect answer.	Student may find a correct answer but does not show any steps taken to solve the problem. <u>OR</u> Student has an answer that does not have a and b as positive real numbers.	Student correctly finds a complex number in the form $a + bi$, where a and b are positive real numbers, that satisfies the given equation and shows all steps such as finding the modulus and argument of i.
	d **N-CN.B.4** **N-CN.B.5**	Student does not give any explanation as to whether a complex number, w, exists for the given equation and conditions and answers incorrectly.	Student answers incorrectly but gives an explanation that has somewhat valid points but is lacking proper information.	Student answers correctly but does not give an accurate written and algebraic explanation such as stating the modulus and argument of z and w for both zero and nonzero cases.	Student answers correctly and provides correct reasoning as to why w is sure to exist, including stating the modulus and argument of z and w if they are nonzero.
	e **N-CN.B.4** **N-CN.B.5**	Student does not give any explanation as to whether a complex number, w, exists for the given equation and conditions and answers incorrectly.	Student answers incorrectly but gives an explanation that has somewhat valid points but is lacking proper information.	Student answers correctly but lacks proper reasoning to support the answer.	Student answers correctly and provides correct reasoning as to why w is sure to exist, including an algebraic solution.
4	**a** **N-VM.C.8** **N-VM.C.10**	Student makes little to no attempt to find matrix.	Student sets up a matrix equation but does not use the correct matrices in order to solve the problem.	Student correctly sets up the matrix equation but, due to errors in calculations, fails to find the correct matrix.	Student correctly sets up and solves the matrix equation leading to the correct matrix.
	b **N-VM.C.8** **N-VM.C.10**	Student makes little to no attempt to find matrix.	Student sets up a matrix equation but does not use the correct matrices in order to solve the problem.	Student correctly sets up the matrix equation but, due to errors in calculations, fails to find the correct matrix.	Student correctly sets up and solves the matrix equation leading to the correct matrix.
	c **N-VM.C.8** **N-VM.C.10**	Student makes little to no attempt to find matrix.	Student sets up a matrix equation but does not use the identity matrix in order to solve the problem.	Student identifies the identity matrix as the answer but writes the matrix incorrectly.	Student identifies the identity matrix as the answer and writes it correctly.
	d **N-VM.C.8** **N-VM.C.10**	Student makes little to no attempt to find matrix.	Student sets up a matrix equation but does not use the correct matrices in order to solve the problem.	Student correctly sets up the matrix equation but, due to errors in calculations, fails to find the correct matrix.	Student correctly sets up and solves the matrix equation leading to the correct matrix.

	e **N-VM.C.8** **N-VM.C.10**	Student makes little to no attempt to find matrix.	Student sets up a matrix equation but does not use the correct matrices in order to answer the question.	Student correctly sets up one or both matrix equations but, due to errors in calculations, fails to arrive at the correct answer.	Student correctly sets up and solves both matrix equations leading to the correct answer.
5	**a** **N-VM.C.10** **N-VM.C.11** **N-VM.C.12**	Student makes little to no attempt to solve for the location of the rocket at either time given.	Student sets up a matrix equation but does not use the correct matrices in order to solve the problem.	Student correctly sets up the matrix equation but, due to errors in calculations, fails to reach a correct final answer for the location of the rocket at both times.	Student correctly solves for the location of the rocket at both times given, using the correct matrix equation.
	b **N-VM.C.10** **N-VM.C.11** **N-VM.C.12**	Student makes little to no attempt to find the location of the rocket at the given time for either set of instructions and gives no explanation.	Student sets up matrix equations to solve for the location of the rocket but fails to properly solve the equations and produce an accurate explanation.	Student correctly finds the location of the rocket for one set of instructions but fails to verify that the location of the rocket for the other set of instructions is consistent with the first.	Student correctly gives the location of the rocket for the given time for both sets of instructions and correctly makes the correlations between the two.
	c **N-VM.C.10** **N-VM.C.11** **N-VM.C.12**	Student makes little to no attempt to solve for the area.	Student attempts to find the area of the region enclosed by the path of the rocket but does not make the correct conclusion that it travels in a semicircle.	Student correctly finds that the path traversed is a semicircle but has minor errors in calculations that prevent the correct area from being found.	Student correctly finds the area of the enclosed path of the rocket including finding the radius of the traversed path.
	d **N-VM.C.10** **N-VM.C.11** **N-VM.C.12**	Student makes little to no attempt to adjust the matrix five places farther right.	Student sets up matrix/matrices for one or both sets of instructions but incorrectly translates the points 5 units to the right.	Student correctly sets up the shifted matrix for one set of instructions but fails to correctly set up the shifted matrices for both sets of instructions.	Student correctly sets up the matrices for both sets of instructions that results in a shift of the rocket five places to the right.

Name ______________________________ Date ______________

1. Consider the transformation on the plane given by the 2×2 matrix $\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}$ for a fixed positive number $k > 1$.

   a. Draw a sketch of the image of the unit square under this transformation (the unit square has vertices $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$). Be sure to label all four vertices of the image figure.



To find the coordinates of the image, multiply the vertices of the unit square by the matrix.

$$\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} k \\ k \end{pmatrix}$$

$$\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 + k \\ k \end{pmatrix}$$

$$\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The image is a parallelogram with base = 1 and height = k.

b. What is the area of the image parallelogram?

To find the area of the image figure, multiply the area of the unit square by the absolute value of $\begin{bmatrix} 1 & k \\ 0 & k \end{bmatrix}$.

$$\begin{vmatrix} 1 & k \\ 0 & k \end{vmatrix} = (1 \times k) - (0 \times k) = k$$

$$\text{Area} = 1 \times |k| = k \text{ since } k > 0$$

c. Find the coordinates of a point $\begin{pmatrix} x \\ y \end{pmatrix}$ whose image under the transformation is $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Solve the equation to find the coordinates of $\begin{pmatrix} x \\ y \end{pmatrix}$.

$$\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Converting the matrix equation to a system of linear equations gives us

$$x + ky = 2$$
$$ky = 3$$

Solve this system.

$$y = \frac{3}{k}$$
$$x + k\left(\frac{3}{k}\right) = 2$$
$$x + 3 = 2$$
$$x = -1$$

The point is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ \frac{3}{k} \end{pmatrix}$.

d. The transformation $\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}$ is applied once to the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then once to the image point, then once to the image of the image point, and then once to the image of the image of the image point, and so on. What are the coordinates of tenfold image of the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, that is, the image of the point after the transformation has been applied 10 times?

Multiply to apply the transformation once: $\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+k \\ k \end{pmatrix}$

Multiply again by the 2×2 *matrix:* $\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}\begin{pmatrix} 1+k \\ k \end{pmatrix} = \begin{pmatrix} 1+k+k^2 \\ k^2 \end{pmatrix}$

Multiply again by the 2×2 *matrix:* $\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}\begin{pmatrix} 1+k+k^2 \\ k^2 \end{pmatrix} = \begin{pmatrix} 1+k+k^2+k^3 \\ k^3 \end{pmatrix}$

By observing the patterns, we can see that the result of n multiplications is a 2×1 *matrix whose top row is the previous row plus* k^n *and whose bottom row is* k^n*.*

The tenfold image would be $\begin{pmatrix} 1+k+k^2+k^3+\dots k^{10} \\ k^{10} \end{pmatrix}$.

2. Consider the transformation given by $\begin{pmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{pmatrix}$.

a. Describe the geometric effect of applying this transformation to a point $\begin{pmatrix} x \\ y \end{pmatrix}$ in the plane.

This transformation will rotate the point $\begin{pmatrix} x \\ y \end{pmatrix}$ *counterclockwise about the origin through an angle of 1 radian.*

b. Describe the geometric effect of applying this transformation to a point $\begin{pmatrix} x \\ y \end{pmatrix}$ in the plane twice: once to the point, and then once to its image.

This transformation will rotate the point $\begin{pmatrix} x \\ y \end{pmatrix}$ *counterclockwise about the origin an additional 1 radian for a total rotation of 2 radians.*

c. Use part (b) to prove $\cos(2) = \cos^2(1) - \sin^2(1)$ and $\sin(2) = 2\sin(1)\cos(1)$.

To prove this, multiply $\begin{pmatrix} x \\ y \end{pmatrix}$ by the transformation matrix:

$$\begin{pmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos(1) - y\sin(1) \\ x\sin(1) + y\cos(1) \end{pmatrix}$$

Then, multiply this answer by the transformation matrix:

$$\begin{pmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{pmatrix}\begin{pmatrix} x\cos(1) - y\sin(1) \\ x\sin(1) + y\cos(1) \end{pmatrix}$$

Apply matrix multiplication:

$$\begin{pmatrix} \cos(1)\big(x\cos(1) - y\sin(1)\big) - \sin(1)\big(x\sin(1) + y\cos(1)\big) \\ \sin(1)\big(x\cos(1) - y\sin(1)\big) + \cos(1)\big(x\sin(1) + y\cos(1)\big) \end{pmatrix}$$

Distribute:

$$\begin{pmatrix} x(\cos(1))^2 - y\cos(1)\sin(1) - x\sin(1)^2 - y\sin(1)\cos(1) \\ x\sin(1)\cos(1) - y\sin(1)^2 + x\cos(1)\sin(1) + y(\cos(1))^2 \end{pmatrix}$$

Rearrange and factor:

$$\begin{pmatrix} x((\cos(1))^2 - \sin(1)^2) - y(2\sin(1)\cos(1)) \\ x(2\sin(1)\cos(1)) + y(\cos(1)^2 - \sin(1)^2) \end{pmatrix}$$

This matrix is equal to the matrix resulting from the 2-radian rotation.

$$\begin{pmatrix} \cos(2) & -\sin(2) \\ \sin(2) & \cos(2) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos(2) - y\sin(2) \\ x\sin(2) + y\cos(2) \end{pmatrix}$$

When you equate the answers and compare the coefficients of x and y, you can see that

$\cos(2) = \cos(1)^2 - \sin(1)^2$ and $\sin(2) = 2\sin(1)\cos(1)$.

The matrices are equal because they represent the same transformation.

$$\begin{pmatrix} x((\cos(1))^2 - \sin(1)^2) - y(2\sin(1)\cos(1)) \\ x(2\sin(1)\cos(1)) + y(\cos(1)^2 - \sin(1)^2) \end{pmatrix} = \begin{pmatrix} x\cos(2) - y\sin(2) \\ x\sin(2) + y\cos(2) \end{pmatrix}$$

3.

a. Explain the geometric representation of multiplying by $1+i$.

$1+i$ has argument $\frac{\pi}{4}$ and modulus $\sqrt{2}$, so geometrically this represents a dilation with a scale factor of $\sqrt{2}$ and a counterclockwise rotation of $\frac{\pi}{4}$ about the origin.

b. Write $(1+i)^{10}$ as a complex number of the form $a+bi$ for real numbers a and b.

$1+i$ has argument $\frac{\pi}{4}$ and modulus $\sqrt{2}$, and so $(1+i)^{10}$ has argument $10\times\frac{\pi}{4}=\frac{\pi}{2}+2\pi$ and modulus $(\sqrt{2})^{10}=2^5=32$. Thus, $(1+i)^{10}=32i$.

c. Find a complex number $a+bi$, with a and b positive real numbers, such that $(a+bi)^3=i$.

i has argument $\frac{\pi}{2}$ and modulus 1. Thus, a complex number $a+bi$ of argument $\frac{\pi}{6}$ and modulus 1 will satisfy $(a+bi)^3=i$. We have $a+bi=\frac{\sqrt{3}}{2}+i\frac{1}{2}$.

d. If z is a complex number, is there sure to exist, for any positive integer n, a complex number w such that $w^n=z$? Explain your answer.

Yes. If $z=0$, then $w=0$ works. If, on the other hand, z is not zero and has argument θ and modulus m, then let w be the complex number with argument $\frac{\theta}{n}$ and modulus $m^{\frac{1}{n}}$:

$$w=m^{\frac{1}{n}}\left(\cos\left(\frac{\theta}{n}\right)+i\sin\left(\frac{\theta}{n}\right)\right).$$

e. If z is a complex number, is there sure to exist, for any negative integer n, a complex number w such that $w^n = z$? Explain your answer.

If $z = 0$, then there is no such complex number w. If $z \neq 0$, then $\frac{1}{w}$, with w as given in part (c), satisfies $\left(\frac{1}{w}\right)^{-n} = z$, showing that the answer to the question is yes in this case.

4. Let $P = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

a. Give an example of a 2×2 matrix A, not with all entries equal to zero, such that $PA = O$.

Notice that for any matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have $PA = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}$.

If we choose $A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, for example, then $PA = O$.

b. Give an example of a 2×2 matrix B with $PB \neq O$.

Following the discussion in part (a), we see that choosing $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ gives $PA = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, which is different from O.

c. Give an example of a 2×2 matrix C such that $CR = R$ for all 2×2 matrices R.

Choose $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The identity matrix has this property.

d. If a 2×2 matrix D has the property that $D + R = R$ for all 2×2 matrices R, must D be the zero matrix O? Explain.

Write $D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $R = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. Then, for $D + R = \begin{pmatrix} a+x & b+y \\ c+z & d+w \end{pmatrix}$ to equal $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ no matter the values of x, y, z, and w, we need:

$a + x = x$

$b + y = y$

$c + z = z$

$d + w = w$

to hold for all values x, y, z, and w. Thus, we need $a = 0$, $b = 0$, $c = 0$, and $d = 0$. That is, D must indeed be the zero matrix.

e. Let $E = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}$. Is there 2×2 matrix F so that $EF = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $FE = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$? If so, find one. If not, explain why no such matrix F can exist.

The determinant of E is $|2 \cdot 6 - 3 \cdot 4| = 0$ and so no inverse matrix like F can exist.

Alternatively:

Write $F = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then, $EF = \begin{pmatrix} 2a+4c & 2b+4d \\ 3a+6c & 3b+3d \end{pmatrix}$. For this to equal $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we need, at the very least:

$2a + 4c = 1$

$3a + 6c = 0$

The first of these equations gives $a + 2c = \frac{1}{2}$ and the second $a + 2c = 0$. There is no solution to this system of equations, and so there can be no matrix F with the desired property.

5. In programming a computer video game, Mavis coded the changing location of a space rocket as follows:

At a time t seconds between $t = 0$ seconds and $t = 2$ seconds, the location $\begin{pmatrix} x \\ y \end{pmatrix}$ of the rocket is given by:

$$\begin{pmatrix} \cos\left(\frac{\pi}{2}t\right) & -\sin\left(\frac{\pi}{2}t\right) \\ \sin\left(\frac{\pi}{2}t\right) & \cos\left(\frac{\pi}{2}t\right) \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

At a time of t seconds between $t = 2$ seconds and $t = 4$ seconds, the location of the rocket is given by $\begin{pmatrix} 3-t \\ 3-t \end{pmatrix}$.

a. What is the location of the rocket at time $t = 0$? What is its location at time $t = 4$?

At time $t = 0$, the location of the rocket is

$$\begin{pmatrix} \cos(0) & -\sin(0) \\ \sin(0) & \cos(0) \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

At time $t = 4$, the location of the rocket is

$$\begin{pmatrix} 3-4 \\ 3-4 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

the same as start.

b. Petrich is worried that Mavis may have made a mistake and the location of the rocket is unclear at time $t = 2$ seconds. Explain why there is no inconsistency in the location of the rocket at this time.

According to the first set of instructions, the location of the rocket at time $t = 2$ is

$$\begin{pmatrix} \cos(\pi) & -\sin(\pi) \\ \sin(\pi) & \cos(\pi) \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

According to the second set of instructions, its location at this time is

$$\begin{pmatrix} 3-2 \\ 3-2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

These are consistent.

c. What is the area of the region enclosed by the path of the rocket from time $t = 0$ to time $t = 4$?

The path traversed is a semicircle with a radius of $\sqrt{2}$. The area enclosed is $\frac{1}{2} \times 2\pi = \pi$ squared units.

d. Mavis later decided that the moving rocket should be shifted five places farther to the right. How should she adjust her formulations above to accomplish this translation?

Notice that:

$$\begin{pmatrix} \cos\left(\frac{\pi}{2}t\right) & -\sin\left(\frac{\pi}{2}t\right) \\ \sin\left(\frac{\pi}{2}t\right) & \cos\left(\frac{\pi}{2}t\right) \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -\cos\left(\frac{\pi}{2}t\right) + \sin\left(\frac{\pi}{2}t\right) \\ -\sin\left(\frac{\pi}{2}t\right) - \cos\left(\frac{\pi}{2}t\right) \end{pmatrix}$$

To translate these points 5 units to the right, use

$\begin{pmatrix} -\cos\left(\frac{\pi}{2}t\right) + \sin\left(\frac{\pi}{2}t\right) + 5 \\ -\sin\left(\frac{\pi}{2}t\right) - \cos\left(\frac{\pi}{2}t\right) \end{pmatrix}$ for $0 \le t \le 2$.

Also use

$\begin{pmatrix} 3 - t + 5 \\ 3 - t \end{pmatrix} = \begin{pmatrix} 8 - t \\ 3 - t \end{pmatrix}$ for $2 \le t \le 4$.