Lesson 1: Successive Differences in Polynomials

Classwork

Opening Exercise

John noticed patterns in the arrangement of numbers in the table below.

______	2.4		3.4		4.4		5.4		6.4
______	5.76		11.56		19.36		29.16		40.96
________		5.8		7.8		9.8		11.8	
__________			2		2		2		

Assuming that the pattern would continue, he used it to find the value of 7.4^2. Explain how he used the pattern to find 7.4^2, and then use the pattern to find 8.4^2.

How would you label each row of numbers in the table?

Discussion

Let the sequence $\{a_0, a_1, a_2, a_3, \ldots\}$ be generated by evaluating a polynomial expression at the values $0, 1, 2, 3, \ldots$. The numbers found by evaluating $a_1 - a_0$, $a_2 - a_1$, $a_3 - a_2, \ldots$ form a new sequence which we will call the first differences of the polynomial. The differences between successive terms of the first differences sequence are called the second differences, and so on.

Example 1

What is the sequence of first differences for the linear polynomial given by $ax + b$, where a and b are constant coefficients?

What is the sequence of second differences for $ax + b$?

Example 2

Find the first, second, and third differences of the polynomial $ax^2 + bx + c$ by filling in the blanks in the following table.

x	$ax^2 + bx + c$	First Differences	Second Differences	Third Differences
0	c			
1	$a + b + c$			
2	$4a + 2b + c$			
3	$9a + 3b + c$			
4	$16a + 4b + c$			
5	$25a + 5b + c$			

Example 3

Find the second, third, and fourth differences of the polynomial $ax^3 + bx^2 + cx + d$ by filling in the blanks in the following table.

x	$ax^3 + bx^2 + cx + d$	First Differences	Second Differences	Third Differences	Fourth Differences
0	d				
		$a+b+c$			
1	$a+b+c+d$				
		$7a+3b+c$			
2	$8a+4b+2c+d$				
		$19a+5b+c$			
3	$27a+9b+3c+d$				
		$37a+7b+c$			
4	$64a+16b+4c+d$				
		$61a+9b+c$			
5	$125a+25b+5c+d$				

Example 4

What type of relationship does the set of ordered pairs (x, y) satisfy? How do you know? Fill in the blanks in the table below to help you decide. (The first differences have already been computed for you.)

x	y	First Differences	Second Differences	Third Differences
0	2			
		-1		
1	1			
		5		
2	6			
		17		
3	23			
		35		
4	58			
		59		
5	117			

Find the equation of the form $y = ax^3 + bx^2 + cx + d$ that all ordered pairs (x, y) above satisfy. Give evidence that your equation is correct.

Relevant Vocabulary

Numerical Symbol: A *numerical symbol* is a symbol that represents a specific number. Examples: $1, 2, 3, 4, \pi, -3.2$.

Variable Symbol: A *variable symbol* is a symbol that is a placeholder for a number from a specified set of numbers. The set of numbers is called the *domain of the variable.* Examples: x, y, z.

Algebraic Expression: An *algebraic expression* is either

1. a numerical symbol or a variable symbol or
2. the result of placing previously generated algebraic expressions into the two blanks of one of the four operators ((__)+(__), (__)−(__), (__)×(__), (__)÷(__)) or into the base blank of an exponentiation with an exponent that is a rational number.

Following the definition above, $\big(\big((x) \times (x)\big) \times (x)\big) + \big((3) \times (x)\big)$ is an algebraic expression, but it is generally written more simply as $x^3 + 3x$.

Numerical Expression: A *numerical expression* is an algebraic expression that contains only numerical symbols (no variable symbols) that evaluates to a single number. Example: The numerical expression $\frac{(3 \cdot 2)^2}{12}$ evaluates to 3.

Monomial: A *monomial* is an algebraic expression generated using only the multiplication operator (__×__). The expressions x^3 and $3x$ are both monomials.

Binomial: A *binomial* is the sum of two monomials. The expression $x^3 + 3x$ is a binomial.

Polynomial Expression: A *polynomial expression* is a monomial or sum of two or more monomials.

Sequence: A *sequence* can be thought of as an ordered list of elements. The elements of the list are called the *terms of the sequence.*

Arithmetic Sequence: A sequence is called *arithmetic* if there is a real number d such that each term in the sequence is the sum of the previous term and d.

Problem Set

1. Create a table to find the second differences for the polynomial $36 - 16t^2$ for integer values of t from 0 to 5.

2. Create a table to find the third differences for the polynomial $s^3 - s^2 + s$ for integer values of s from -3 to 3.

3. Create a table of values for the polynomial x^2, using $n, n+1, n+2, n+3, n+4$ as values of x. Show that the second differences are all equal to 2.

4. Show that the set of ordered pairs (x, y) in the table below satisfies a quadratic relationship. (Hint: Find second differences.) Find the equation of the form $y = ax^2 + bx + c$ that all of the ordered pairs satisfy.

x	0	1	2	3	4	5
y	5	4	-1	-10	-23	-40

5. Show that the set of ordered pairs (x, y) in the table below satisfies a cubic relationship. (Hint: Find third differences.) Find the equation of the form $y = ax^3 + bx^2 + cx + d$ that all of the ordered pairs satisfy.

x	0	1	2	3	4	5
y	20	4	0	20	76	180

6. The distance d ft. required to stop a car traveling at $10v$ mph under dry asphalt conditions is given by the following table.

v	0	1	2	3	4	5
d	0	5	19.5	43.5	77	120

   a. What type of relationship is indicated by the set of ordered pairs?
   b. Assuming that the relationship continues to hold, find the distance required to stop the car when the speed reaches 60 mph, when $v = 6$.
   c. (Challenge) Find an equation that describes the relationship between the speed of the car v and its stopping distance d.

7. Use the polynomial expressions $5x^2 + x + 1$ and $2x + 3$ to answer the questions below.
   a. Create a table of second differences for the polynomial $5x^2 + x + 1$ for the integer values of x from 0 to 5.
   b. Justin claims that for $n \geq 2$, the n^{th} differences of the sum of a degree n polynomial and a linear polynomial are the same as the n^{th} differences of just the degree n polynomial. Find the second differences for the sum $(5x^2 + x + 1) + (2x + 3)$ of a degree 2 and a degree 1 polynomial and use the calculation to explain why Justin might be correct in general.
   c. Jason thinks he can generalize Justin's claim to the product of two polynomials. He claims that for $n \geq 2$, the $(n+1)^{th}$ differences of the product of a degree n polynomial and a linear polynomial are the same as the n^{th} differences of the degree n polynomial. Use what you know about second and third differences (from Examples 2 and 3) and the polynomial $(5x^2 + x + 1)(2x + 3)$ to show that Jason's generalization is incorrect.

Lesson 2: The Multiplication of Polynomials

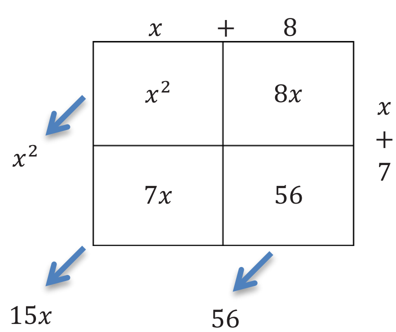
Classwork

Opening Exercise

Show that $28 \times 27 = (20 + 8)(20 + 7)$ using an area model. What do the numbers you placed inside the four rectangular regions you drew represent?

Example 1

Use tabular method to multiply $(x + 8)(x + 7)$ and combine like terms.



Exercises 1–2

1. Use the tabular method to multiply $(x^2 + 3x + 1)(x^2 - 5x + 2)$ and combine like terms.

2. Use the tabular method to multiply $(x^2 + 3x + 1)(x^2 - 2)$ and combine like terms.

Example 2

Multiply the polynomials $(x-1)(x^4+x^3+x^2+x+1)$ using a table. Generalize the pattern that emerges by writing down an identity for $(x-1)(x^n+x^{n-1}+\cdots+x^2+x+1)$ for n a positive integer.

	x	-1	
x^5	x^5	$-x^4$	x^4
$0x^4$	x^4	$-x^3$	x^3
$0x^3$	x^3	$-x^2$	x^2
$0x^2$	x^2	$-x$	x
$0x$	x	-1	1
	-1		

Exercises 3–4

3. Multiply $(x-y)(x^3+x^2y+xy^2+y^3)$ using the distributive property and combine like terms. How is this calculation similar to Example 2?

4. Multiply $(x^2-y^2)(x^2+y^2)$ using the distributive property and combine like terms. Generalize the pattern that emerges to write down an identity for $(x^n-y^n)(x^n+y^n)$ for positive integers n.

Relevant Vocabulary

Equivalent Polynomial Expressions: Two polynomial expressions in one variable are *equivalent* if, whenever a number is substituted into all instances of the variable symbol in both expressions, the numerical expressions created are equal.

Polynomial Identity: A *polynomial identity* is a statement that two polynomial expressions are equivalent. For example, $(x+3)^2 = x^2 + 6x + 9$ for any real number x is a polynomial identity.

Coefficient of a Monomial: The *coefficient of a monomial* is the value of the numerical expression found by substituting the number 1 into all the variable symbols in the monomial. The coefficient of $3x^2$ is 3, and the coefficient of the monomial $(3xyz) \cdot 4$ is 12.

Terms of a Polynomial: When a polynomial is expressed as a monomial or a sum of monomials, each monomial in the sum is called a *term* of the polynomial.

Like Terms of a Polynomial: Two terms of a polynomial that have the same variable symbols each raised to the same power are called *like terms*.

Standard Form of a Polynomial in One Variable: A polynomial expression with one variable symbol, x, is in *standard form* if it is expressed as

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where n is a non-negative integer, and $a_0, a_1, a_2 \ldots, a_n$ are constant coefficients with $a_n \neq 0$.

A polynomial expression in x that is in standard form is often just called a *polynomial in* x or a *polynomial*.

The *degree of the polynomial in standard form* is the highest degree of the terms in the polynomial, namely n. The term $a_n x^n$ is called the *leading term* and a_n (thought of as a specific number) is called the *leading coefficient.* The *constant term* is the value of the numerical expression found by substituting 0 into all the variable symbols of the polynomial, namely a_0.

Problem Set

1. Complete the following statements by filling in the blanks.
   a. $(a+b)(c+d+e) = ac + ad + ae + \_\_\_ + \_\_\_ + \_\_\_$
   b. $(r-s)^2 = (\_\_\_\_)^2 - (\_\_\_\_)rs + s^2$
   c. $(2x+3y)^2 = (2x)^2 + 2(2x)(3y) + (\_\_\_\_)^2$
   d. $(w-1)(1+w+w^2) = \_\_\_\_ - 1$
   e. $a^2 - 16 = (a + \_\_\_\_)(a - \_\_\_\_)$
   f. $(2x+5y)(2x-5y) = \_\_\_\_ - \_\_\_\_$
   g. $(2^{21}-1)(2^{21}+1) = \_\_\_\_ - 1$
   h. $[(x-y)-3][(x-y)+3] = (\_\_\_\_)^2 - 9$

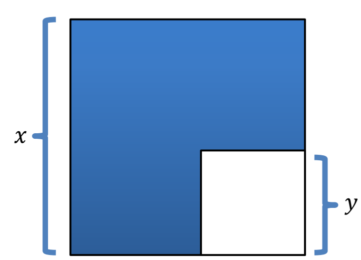
2. Use the tabular method to multiply and combine like terms.
   a. $(x^2 - 4x + 4)(x + 3)$
   b. $(11 - 15x - 7x^2)(25 - 16x^2)$
   c. $(3m^3 + m^2 - 2m - 5)(m^2 - 5m - 6)$
   d. $(x^2 - 3x + 9)(x^2 + 3x + 9.$

3. Multiply and combine like terms to write as the sum or difference of monomials.
   a. $2a(5 + 4a)$
   b. $x^3(x + 6) + 9$
   c. $\frac{1}{8}(96z + 24z^2)$
   d. $2^{23}(2^{84} - 2^{81})$
   e. $(x - 4)(x + 5)$
   f. $(10w - 1)(10w + 1)$
   g. $(3z^2 - 8)(3z^2 + 8)$
   h. $(-5w - 3)w^2$
   i. $8y^{1000}(y^{12200} + 0.125y)$
   j. $(2r + 1)(2r^2 + 1)$
   k. $(t - 1)(t + 1)(t^2 + 1)$
   l. $(w - 1)(w^5 + w^4 + w^3 + w^2 + w + 1)$
   m. $(x+2)(x+2)(x+2)$
   n. $n(n + 1)(n + 2)$
   o. $n(n + 1)(n + 2)(n + 3)$
   p. $n(n + 1)(n + 2)(n + 3)(n + 4)$
   q. $(x + 1)(x^3 - x^2 + x - 1)$
   r. $(x + 1)(x^5 - x^4 + x^3 - x^2 + x - 1)$
   s. $(x + 1)(x^7 - x^6 + x^5 - x^4 + x^3 - x^2 + x - 1)$
   t. $(m^3 - 2m + 1)(m^2 - m + 2)$

4. Polynomial expressions can be thought of as a generalization of place value.
   a. Multiply 214×112 using the standard paper-and-pencil algorithm.
   b. Multiply $(2x^2 + x + 4)(x^2 + x + 2)$ using the tabular method and combine like terms.
   c. Put $x = 10$ into your answer from part (b).
   d. Is the answer to part (c) equal to the answer from part (a)? Compare the digits you computed in the algorithm to the coefficients of the entries you computed in the table. How do the place-value units of the digits compare to the powers of the variables in the entries?

5. Jeremy says $(x - 9)(x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$ must equal $x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ because when $x = 10$, multiplying by $x - 9$ is the same as multiplying by 1.
   a. Multiply $(x - 9)(x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$.
   b. Put $x = 10$ into your answer.
   c. Is the answer to part (b) the same as the value of $x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ when $x = 10$?
   d. Was Jeremy right?

6. In the diagram, the side of the larger square is x units and the side of the smaller square is y units. The area of the shaded region is $(x^2 - y^2)$ square units. Show how the shaded area might be cut and rearranged to illustrate that the area is $(x - y)(x + y)$ square units.



Lesson 3: The Division of Polynomials

Opening Exercise

a. Multiply these polynomials using the tabular method.

$$(2x + 5)(x^2 + 5x + 1)$$

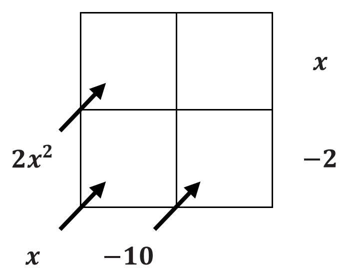
b. How can you use the expression in part (a) to quickly multiply 25×151?

Exploratory Challenge

1. Does $\frac{2x^3+15x^2+27x+5}{2x+5} = (x^2 + 5x + 1)$? Justify your answer.

2. Describe the process you used to determine your answer to Exercise 1.

3. Reverse the tabular method of multiplication to find the quotient: $\frac{2x^2+x-10}{x-2}$.



4. Test your conjectures. Create your own table and use the *reverse tabular method* to find the quotient.

$$\frac{x^4+4x^3+3x^2+4x+2}{x^2+1}$$

5. Test your conjectures. Use the *reverse tabular method* to find the quotient.

$$\frac{3x^5 - 2x^4 + 6x^3 - 4x^2 - 24x + 16}{x^2 + 4}$$

6. What is the quotient of $\frac{x^5-1}{x-1}$? Of $\frac{x^6-1}{x-1}$?

Problem Set

Use the reverse tabular method to solve these division problems.

1. $(2x^3 + x^2 - 16x + 15) \div (2x - 3)$

2. $(3x^5 + 12x^4 + 11x^3 + 2x^2 - 4x - 2) \div (3x^2 - 1)$

3. $\frac{x^3-4x^2+7x-28}{x^2+7}$

4. $\frac{x^4-2x^3-29x-12}{x^3+2x^2+8x+3}$

5. $\frac{6x^5+4x^4-6x^3+14x^2-8}{6x+4}$

6. $(x^3 - 8) \div (x - 2)$

7. $\frac{x^3+2x^2+2x+1}{x+1}$

8. $\frac{x^4+2x^3+2x^2+2x+1}{x+1}$

9. Use the results of Problems 7 and 8 to predict the quotient of $\frac{x^5+2x^4+2x^3+2x^2+2x+1}{x+1}$. Explain your prediction. Then check your prediction using the reverse tabular method.

10. Use the results of Exercise 5 in the Exploratory Challenge and Problems 7 through 9 above to predict the quotient of $\frac{x^4-2x^3+2x^2-2x+1}{x-1}$. Explain your prediction. Then check your prediction using the reverse tabular method.

11. Make and test a conjecture about the quotient of $\frac{x^6+x^5+2x^4+2x^3+2x^2+x+1}{x^2+1}$. Explain your reasoning.

12. Given the following quotients:

$$\frac{4x^2+8x+3}{2x+1} \text{ and } \frac{483}{21}$$

   a. How are these expressions related?
   b. Find each quotient.
   c. Explain the connection between the quotients.

Lesson 4: Comparing Methods—Long Division, Again?

Opening Exercises

1. Use the reverse tabular method to determine the quotient $\frac{2x^3+11x^2+7x+10}{x+5}$.

2. Use your work from Exercise 1 to write the polynomial $2x^3 + 11x^2 + 7x + 10$ in factored form, and then multiply the factors to check your work above.

Example 1

If $x = 10$, then the division $1573 \div 13$ can be represented using polynomial division.

$$x+3\overline{)x^3+5x^2+7x+3}$$

Example 2

Use the long division algorithm for polynomials to evaluate

$$\frac{2x^3-4x^2+2}{2x-2}.$$

Exercises 1–8

Use the long division algorithm to determine the quotient. For each problem, check your work by using the reverse tabular method.

1. $\frac{x^2+6x+9}{x+3}$

2. $(7x^3 - 8x^2 - 13x + 2) \div (7x - 1)$

3. $(x^3 - 27\ ) \div (x - 3)$

4. $\frac{2x^4+14x^3+x^2-21x-6}{2x^2-3}$

5. $\frac{5x^4-6x^2+1}{x^2-1}$

6. $\frac{x^6+4x^4-4x-1}{x^3-1}$

7. $(2x^7 + x^5 - 4x^3 + 14x^2 - 2x + 7) \div (2x^2 + 1)$

8. $\frac{x^6-64}{x+2}$

Lesson Summary

The long division algorithm to divide polynomials is analogous to the long division algorithm for integers. The long division algorithm to divide polynomials produces the same results as the reverse tabular method.

Problem Set

Use the long division algorithm to determine the quotient.

1. $\frac{2x^3-13x^2-x+3}{2x+1}$

2. $\frac{3x^3+4x^2+7x+22}{x+2}$

3. $\frac{x^4+6x^3-7x^2-24x+12}{x^2-4}$

4. $(12x^4 + 2x^3 + x - 3) \div (2x^2 + 1)$

5. $(2x^3 + 2x^2 + 2x) \div (x^2 + x + 1)$

6. Use long division to find the polynomial, p, that satisfies the equation below.
$$2x^4 - 3x^2 - 2 = (2x^2 + 1)(p(x))$$

7. Given $q(x) = 3x^3 - 4x^2 + 5x + k$.
   a. Determine the value of k so that $3x - 7$ is a factor of the polynomial q.
   b. What is the quotient when you divide the polynomial q by $3x - 7$?

8. In parts (a)–(b) and (d)–(e), use long division to evaluate each quotient. Then, answer the remaining questions.
   a. $\frac{x^2-9}{x+3}$
   b. $\frac{x^4-81}{x+3}$
   c. Is $x + 3$ a factor of $x^3 - 27$? Explain your answer using the long division algorithm.
   d. $\frac{x^3+27}{x+3}$
   e. $\frac{x^5+243}{x+3}$
   f. Is $x + 3$ a factor of $x^2 + 9$? Explain your answer using the long division algorithm.
   g. For which positive integers n is $x + 3$ a factor of $x^n + 3^n$? Explain your reasoning.
   h. If n is a positive integer, is $x + 3$ a factor of $x^n - 3^n$? Explain your reasoning.

Lesson 5: Putting It All Together

Classwork

Exercises 1–15: Polynomial Pass

Perform the indicated operation to write each polynomial in standard form.

1. $(x^2 - 3)(x^2 + 3x - 1)$

2. $(5x^2 - 3x - 7) - (x^2 + 2x - 5)$

3. $(x^3 - 8) \div (x - 2)$

4. $(x + 1)(x - 2)(x + 3)$

5. $(x + 1) - (x - 2) - (x + 3)$

6. $(x + 2)(2x^2 - 5x + 7)$

7. $\frac{x^3 - 2x^2 - 65x + 18}{x - 9}$

8. $(x^2 - 3x + 2) - (2 - x + 2x^2)$

9. $(x^2 - 3x + 2)(2 - x + 2x^2)$

10. $\frac{x^3 - x^2 - 5x - 3}{x - 3}$

11. $(x^2 + 7x - 12)(x^2 - 9x + 1)$

12. $(2x^3 - 6x^2 - 7x - 2) + (x^3 + x^2 + 6x - 12)$

13. $(x^3 - 8)(x^2 - 4x + 4)$

14. $(x^3 - 2x^2 - 5x + 6) \div (x + 2)$

15. $(x^3 + 2x^2 - 3x - 1) + (4 - x - x^3)$

Exercises 16–22

16. Review Exercises 1–15 and then select one exercise for each category and record the steps in the operation below as an example. Be sure to show all your work.

Addition Exercise	**Multiplication Exercise**
Subtraction Exercise	**Division Exercise**

For Exercises 17–20, re-write each polynomial in standard form by applying the operations in the appropriate order.

17. $\frac{(x^2+5x+20)+(x^2+6x-6)}{x+2}$

18. $(x^2 - 4)(x + 3) - (x^2 + 2x - 5)$

19. $\frac{(x-3)^3}{x^2-6x+9}$

20. $(x + 7)(2x - 3) - (x^3 - 2x^2 + x - 2) \div (x - 2)$

21. What would be the first and last terms of the polynomial if it was re-written in standard form? Answer these quickly without performing all of the indicated operations.
    a. $(2x^3 - x^2 - 9x + 7) + (11x^2 - 6x^3 + 2x - 9)$

    b. $(x - 3)(2x + 3)(x - 1)$

    c. $(2x - 3)(3x + 5) - (x + 1)(2x^2 - 6x + 3)$

    d. $(x + 5)(3x - 1) - (x - 4)^2$

22. What would the first and last terms of the polynomial be if it was re-written in standard form?
    a. $(n + 1)(n + 2)(n + 3) \dots (n + 9)(n + 10)$

    b. $(x - 2)^{10}$

    c. $\frac{(x-2)^{10}}{(x-2)}$

    d. $\frac{n(n+1)(2n+1)}{6}$

Problem Set

For Problems 1–7, rewrite each expression as a polynomial is standard form.

1. $(3x-4)^3$

2. $(2x^2-x^3-9x+1)-(x^3+7x-3x^2+1)$

3. $(x^2-5x+2)(x-3)$

4. $\frac{x^4-x^3-6x^2-9x+27}{x-3}$

5. $(x+3)(x-3)-(x+4)(x-4)$

6. $(x+3)^2-(x+4)^2$

7. $\frac{x^2-5x+6}{x-3}+\frac{x^3-1}{x-1}$

For Problems 8–9: Quick, what would be the first and last terms of the polynomial if it was written in standard form?

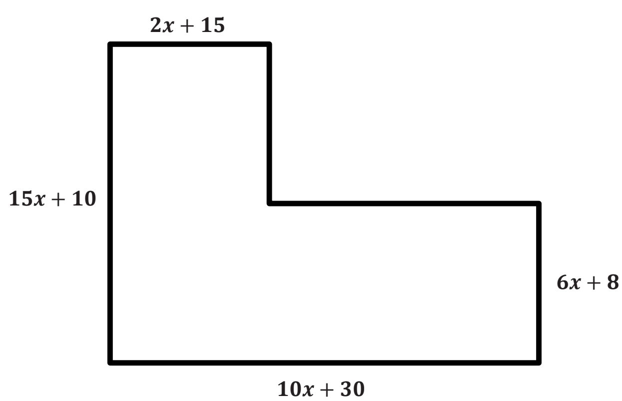
8. $2(x^2-5x+4)-(x+3)(x+2)$

9. $\frac{(x-2)^5}{x-2}$

10. The profit a business earns by selling x items is given by the polynomial function
$$p(x)=x(160-x)-(100x+500).$$
What is the last term in the standard form of this polynomial? What does it mean in this situation?

11. Explain why these two quotients are different. Compute each one. What do they have in common? Why?
$$\frac{(x-2)^4}{x-2} \text{ and } \frac{x^4-16}{x-2}$$

12. What are the area and perimeter of the figure? Assume a right angle at each vertex.



Lesson 6: Dividing by $x - a$ and by $x + a$

Classwork

Opening Exercise

Find the following quotients, and write the quotient in standard form.

a. $\frac{x^2-9}{x-3}$

b. $\frac{x^3-27}{x-3}$

c. $\frac{x^4-81}{x-3}$

Exercise 1

1. Use patterns to predict each quotient. Explain how you arrived at your prediction, and then test it by applying the reverse tabular method or long division.

   a. $\frac{x^2-144}{x-12}$

   b. $\frac{x^3-8}{x-2}$

   c. $\frac{x^3-125}{x-5}$

   d. $\frac{x^6-1}{x-1}$

Example 1

What is the quotient of $\frac{x^2-a^2}{x-a}$? Use the reverse tabular method or long division.

Exercises 2–4

2. Work with your group to find the following quotients.

   a. $\frac{x^3-a^3}{x-a}$

   b. $\frac{x^4-a^4}{x-a}$

3. Predict without performing division whether or not the divisor will divide into the dividend without a remainder for the following problems. If so, find the quotient. Then check your answer.

   a. $\frac{x^2-a^2}{x+a}$

   b. $\frac{x^3-a^3}{x+a}$

   c. $\frac{x^2+a^2}{x+a}$

   d. $\frac{x^3+a^3}{x+a}$

4. Find the quotient $\frac{x^n-1}{x-1}$ for $n = 2, 3, 4$ and 8

a. What patterns do you notice?

b. Use your work in this problem to write an expression equivalent to $\frac{x^n-1}{x-1}$ for any integer $n > 1$.

Lesson Summary

Based on the work in this lesson, we can conclude the following statements are true for all real values of x and a:

$$x^2 - a^2 = (x-a)(x+a)$$
$$x^3 - a^3 = (x-a)(x^2+ax+a^2)$$
$$x^3 + a^3 = (x+a)(x^2-ax+a^2),$$

and it seems that the following statement is also an identity for all real values of x and a:

$x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + x^{n-3} + \cdots + x^1 + 1)$, for integers $n > 1$.

Problem Set

1. Compute each quotient.

   a. $\frac{x^2-625}{x-25}$

   b. $\frac{x^3+1}{x+1}$

   c. $\frac{x^3-\frac{1}{8}}{x-\frac{1}{2}}$

   d. $\frac{x^2-0.01}{x-0.1}$

2. In the next exercises, you can use the same identities you applied in the previous problem. Fill in the blanks in the problems below to help you get started. Check your work by using the reverse tabular method or long division to make sure you are applying the identities correctly.

   a. $\frac{16x^2-121}{4x-11} = \frac{(\_\_)^2-(\_\_)^2}{4x-11} = (\_\_) + 11$

   b. $\frac{25x^2-49}{5x+7} = \frac{(\_\_)^2-(\_\_)^2}{5x+7} = (\_\_) - (\_\_) = \_\_\_\_\_\_\_\_\_\_\_\_$

   c. $\frac{8x^3-27}{2x-3} = \frac{(\_\_)^3-(\_\_)^3}{2x-3} = (\_\_)^2 + (\_\_)(\_\_) + (\_\_)^2 = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

3. Show how the patterns and relationships learned in this lesson could be applied to solve the following arithmetic problems by filling in the blanks.

   a. $\frac{625-81}{16} = \frac{(\_\_)^2-(9)^2}{25-(\_\_)} = (\_\_) + (\_\_) = 34$

   b. $\frac{1000-27}{7} = \frac{(\_\_)^3-(\_\_)^3}{(\_\_)-3} = (\_\_)^2 + (10)(\_\_) + (\_\_)^2 = \_\_\_\_\_\_\_$

   c. $\frac{100-9}{7} = \frac{(\_\_)^2-(\_\_)^2}{(\_\_)-3} = \_\_\_\_\_\_\_$

   d. $\frac{1000+64}{14} = \frac{(\_\_)^3+(\_\_)^3}{(\_\_)+(\_\_)} = (\_\_)^2 - (\_\_)(\_\_) + (\_\_)^2 = \_\_\_\_\_\_\_$

4. Apply the identities from this lesson to compute each quotient. Check your work using the reverse tabular method or long division.

   a. $\frac{16x^2-9}{4x+3}$

   b. $\frac{81x^2-25}{18x-10}$

   c. $\frac{27x^3-8}{3x-2}$

5. Extend the patterns and relationships you learned in this lesson to compute the following quotients. Explain your reasoning, and then check your answer by using long division or the tabular method.

   a. $\frac{8+x^3}{2+x}$

   b. $\frac{x^4-y^4}{x-y}$

   c. $\frac{27x^3+8y^3}{3x+2y}$

   d. $\frac{x^7-y^7}{x-y}$

Lesson 7: Mental Math

Classwork

Opening Exercise

a. How are these two equations related?

$$\frac{x^2 - 1}{x + 1} = x - 1 \quad \text{and} \quad x^2 - 1 = (x + 1)(x - 1)$$

b. Explain the relationship between the polynomial identities $x^2 - 1 = (x + 1)(x - 1)$ and $x^2 - a^2 = (x - a)(x + a)$.

Exercises 1–3

1. Compute the following products using the identity $x^2 - a^2 = (x - a)(x + a)$. Show your steps.

   a. $6 \cdot 8$

   b. $11 \cdot 19$

c. $23 \cdot 17$

d. $34 \cdot 26$

2. Find two additional factors of $2^{100} - 1$.

3. Show that $8^3 - 1$ is divisible by 7.

Lesson Summary

Based on the work in this lesson, we can convert differences of squares into products (and vice versa) using

$$x^2 - a^2 = (x - a)(x + a).$$

If x, a, and n are integers and $n > 1$, then numbers of the form $x^n - a^n$ are not prime because

$$x^n - a^n = (x - a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1}).$$

Problem Set

1. Using an appropriate polynomial identity, quickly compute the following products. Show each step. Be sure to state your values for x and a.
   a. $41 \cdot 19$
   b. $993 \cdot 1{,}007$
   c. $213 \cdot 187$
   d. $29 \cdot 51$
   e. $125 \cdot 75$

2. Give the general steps you take to determine x and a when asked to compute a product such as those in Problem 1.

3. Why is $17 \cdot 23$ easier to compute than $17 \cdot 22$?

4. Rewrite the following differences of squares as a product of two integers.
   a. $81 - 1$
   b. $400 - 121$

5. Quickly compute the following differences of squares.
   a. $64^2 - 14^2$
   b. $112^2 - 88^2$
   c. $785^2 - 215^2$

6. Is 323 prime? Use the fact that $18^2 = 324$ and an identity to support your answer.

7. The number $2^3 - 1$ is prime and so are $2^5 - 1$ and $2^7 - 1$. Does that mean $2^9 - 1$ is prime? Explain why or why not.

8. Show that $9{,}999{,}999{,}991$ is not prime without using a calculator or computer.

9. Show that $999{,}973$ is not prime without using a calculator or computer.

10. Find a value of b so that the expression $b^n - 1$ is always divisible by 5 for any positive integer n. Explain why your value of b works for any positive integer n.

11. Find a value of b so that the expression $b^n - 1$ is always divisible by 7 for any positive integer n. Explain why your value of b works for any positive integer n.

12. Find a value of b so that the expression $b^n - 1$ is divisible by both 7 and 9 for any positive integer n. Explain why your value of b works for any positive integer n.

Lesson 8: The Power of Algebra—Finding Primes

Classwork

Opening Exercise: When is $2^n - 1$ prime and when is it composite?

Complete the table to investigate which numbers of the form $2^n - 1$ are prime and which are composite.

Exponent n	Expression $2^n - 1$	Value	Prime or Composite? Justify your answer if composite.
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			
11			

What patterns do you notice in this table about which expressions are prime and which are composite?

Example 1: Proving a Conjecture

Conjecture: If m is a positive odd composite number, then $2^m - 1$ is a composite number.

Start with an identity: $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots x^1 + 1)$

In this case, $x = 2$, so the identity above becomes:

$$\begin{aligned} 2^m - 1 &= (2-1)(2^{m-1} + 2^{m-2} + \cdots + 2^1 + 1) \\ &= (2^{m-1} + 2^{m-2} + \cdots + 2^1 + 1), \end{aligned}$$

and it is not clear whether or not $2^m - 1$ is composite.

Rewrite the expression: Let $m = ab$ be a positive odd composite number. Then a and b must also be odd, or else the product ab would be even. The smallest such number m is 9, so we have $a \geq 3$ and $b \geq 3$.

Then we have

$$\begin{aligned} 2^m - 1 &= (2^a)^b - 1 \\ &= (2^a - 1)\underbrace{\left((2^a)^{b-1} + (2^a)^{b-2} + \cdots + (2^a)^1 + 1\right)}_{\text{Some number larger than 1}}. \end{aligned}$$

Since $a \geq 3$, we have $2^a \geq 8$; thus, $2^a - 1 \geq 7$. Since the other factor is also larger than 1, $2^m - 1$ is composite and we have proven our conjecture.

Exercises 1–3

For Exercises 1–3, find a factor of each expression using the method discussed in Example 1.

1. $2^{15} - 1$

2. $2^{99} - 1$

3. $2^{537} - 1$ (Hint: 537 is the product of two prime numbers that are both less than 50.)

Exercise 4: How quickly can a computer factor a very large number?

4. How long would it take a computer to factor some squares of very large prime numbers?

   The time in seconds required to factor an n-digit number of the form p^2, where p is a large prime, can roughly be approximated by $f(n) = 3.4 \times 10^{(n-13)/2}$. Some values of this function are listed in the table below.

p	p^2	Number of Digits	Time needed to factor the number (sec)
10,007	100,140,049	9	0.034
100,003	10,000,600,009	11	0.34
1,000,003	1,000,006,000,009	13	3.4
10,000,019	100,000,380,000,361	15	34
100,000,007	10,000,001,400,000,049	17	340
1000,000,007	1,000,000,014,000,000,049	19	3,400

Use the function given above to determine how long it would take this computer to factor a number that contains 32 digits.

Problem Set

1. Factor $4^{12} - 1$ in two different ways using the identity $x^n - a^n = (x - a)(x^n + ax^{n-1} + a^2x^{n-2} + \cdots + a^n)$ and the difference of squares identity.

2. Factor $2^{12} + 1$ using the identity $x^n + a^n = (x + a)(x^n - ax^{n-1} + a^2x^{n-2} - \cdots + a^n)$ for odd numbers n.

3. Is $10{,}000{,}000{,}001$ prime? Explain your reasoning.

4. Explain why $2^n - 1$ is never prime if n is a composite number.

5. Fermat numbers are of the form $2^n + 1$ where n is positive integer.
   a. Create a table of Fermat numbers for odd values of n up to 9.

n	$2^n + 1$
1	
3	
5	
7	
9	

   b. Explain why if n is odd, the Fermat number $2^n + 1$ will always be divisible by 3.
   c. Complete the table of values for even values of n up to 12.

n	$2^n + 1$
2	
4	
6	
8	
10	
12	

   d. Show that if n can be written in the form $2k$ where k is odd, then $2^n + 1$ is divisible by 5.
   e. Which even numbers are not divisible by an odd number? Make a conjecture about the only Fermat numbers that might be prime.

6. Complete this table to explore which numbers can be expressed as the difference of two perfect squares.

Number	Difference of Two Squares	Number	Difference of Two Squares
1	$1^2 - 0^2 = 1 - 0 = 1$	11	
2	Not possible	12	
3	$2^2 - 1^2 = 4 - 1 = 3$	13	
4	$2^2 - 0^2 = 4 - 0 = 4$	14	
5		15	
6		16	
7		17	
8		18	
9		19	
10		20	

a. For which odd numbers does it appear to be possible to write the number as the difference of two squares?

b. For which even numbers does it appear to be possible to write the number as the difference of two squares?

c. Suppose that n is an odd number that can be expressed as $n = a^2 - b^2$ for positive integers a and b. What do you notice about a and b?

d. Suppose that n is an even number that can be expressed as $n = a^2 - b^2$ for positive integers a and b. What do you notice about a and b?

7. Express the numbers from 21 to 30 as the difference of two squares, if possible.

8. Prove this conjecture: Every positive odd number m can be expressed as the difference of the squares of two consecutive numbers that sum to the original number m.

a. Let m be a positive odd number. Then for some integer n, $m = 2n + 1$. We will look at the consecutive integers n and $n + 1$. Show that $n + (n + 1) = m$.

b. What is the difference of squares of $n + 1$ and n?

c. What can you conclude from parts (a) and (b)?

9. Prove this conjecture: Every positive multiple of 4 can be expressed as the difference of squares of two numbers that differ by 2. Use the table below to organize your work for parts (a)–(c).
   a. Write each multiple of 4 in the table as a difference of squares.

n	$4n$	*Difference of squares* $a^2 - b^2$	a	b
1	4	$2^2 - 0^2$	2	*0*
2				
3				
4				
5				
n	$4n$	$(\quad)^2 - (\quad)^2$		

   b. What do you notice about the numbers a and b that are squared? How do they relate to the number n?
   c. Given a positive integer of the form $4n$, prove that there are integers a and b so that $4n = a^2 - b^2$ and that $a - b = 2$. (Hint: Refer to parts (a) and (b) for the relationship between n and a and b.)

10. The steps below prove that the only positive even numbers that can be written as a difference of square integers are the multiples of 4. That is, completing this exercise will prove that is it impossible to write a number of the form $4n - 2$ as a difference of square integers.
   a. Let m be a positive even integer that we can write as the difference of square integers $m = a^2 - b^2$. Then $m = (a + b)(a - b)$ for integers a and b. How do we know that either a and b are both even or a and b are both odd?
   b. Is $a + b$ even or odd? What about $a - b$? How do you know?
   c. Is 2 a factor of $a + b$? Is 2 a factor of $a - b$? Is 4 a factor of $(a + b)(a - b)$? Explain how you know.
   d. Is 4 a factor of any integer of the form $4n - 2$?
   e. What can you conclude from your work in parts (a)–(d)?

11. Explain why the prime number 17 can only be expressed as the difference of two squares in only one way, but the composite number 24 can be expressed as the difference of two squares in more than one way.

12. Explain why you cannot use the factors of 3 and 8 to rewrite 24 as the difference of two square integers.

Lesson 9: Radicals and Conjugates

Classwork

Opening Exercise

Which of these statements are true for all $a, b > 0$? Explain your conjecture.

i. $2(a + b) = 2a + 2b$

ii. $\frac{a+b}{2} = \frac{a}{2} + \frac{b}{2}$

iii. $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$

Example 1

Express $\sqrt{50} - \sqrt{18} + \sqrt{8}$ in simplest radical form and combine like terms.

Exercises 1–5

1. $\sqrt{\frac{1}{4}} + \sqrt{\frac{9}{4}} - \sqrt{45}$

2. $\sqrt{2}\left(\sqrt{3} - \sqrt{2}\right)$

3. $\sqrt{\frac{3}{8}}$

4. $\sqrt[3]{\frac{5}{32}}$

5. $\sqrt[3]{16x^5}$

Example 2

Multiply and combine like terms. Then explain what you notice about the two different results.

$\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}+\sqrt{2}\right)$

$\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right)$

Exercise 6

6. Find the product of the conjugate radicals.

$(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})$

$(7+\sqrt{2})(7-\sqrt{2})$

$(\sqrt{5}+2)(\sqrt{5}-2)$

Example 3

Write $\frac{\sqrt{3}}{5-2\sqrt{3}}$ in simplest radical form.

Lesson Summary

- For real numbers $a \geq 0$ and $b \geq 0$, where $b \neq 0$ when b is a denominator,
$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b} \text{ and } \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$
- For real numbers $a \geq 0$ and $b \geq 0$, where $b \neq 0$ when b is a denominator,
$$\sqrt[3]{ab} = \sqrt[3]{a} \cdot \sqrt[3]{b} \text{ and } \sqrt[3]{\frac{a}{b}} = \frac{\sqrt[3]{a}}{\sqrt[3]{b}}.$$
- Two binomials of the form $\sqrt{a} + \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$ are called conjugate radicals:
  $\sqrt{a} + \sqrt{b}$ is the conjugate of $\sqrt{a} - \sqrt{b}$, and
  $\sqrt{a} - \sqrt{b}$ is the conjugate of $\sqrt{a} + \sqrt{b}$.
  For example, the conjugate of $2 - \sqrt{3}$ is $2 + \sqrt{3}$.
- To express a numeric expression with a denominator of the form $\sqrt{a} + \sqrt{b}$ in simplest radical form, multiply the numerator and denominator by the conjugate $\sqrt{a} - \sqrt{b}$ and combine like terms.

Problem Set

1. Express each of the following as a rational number or in simplest radical form. Assume that the symbols a, b, and x represent positive numbers.
   a. $\sqrt{36}$
   b. $\sqrt{72}$
   c. $\sqrt{18}$
   d. $\sqrt{9x^3}$
   e. $\sqrt{27x^2}$
   f. $\sqrt[3]{16}$
   g. $\sqrt[3]{24a}$
   h. $\sqrt{9a^2 + 9b^2}$

2. Express each of the following in simplest radical form, combining terms where possible.
   a. $\sqrt{25} + \sqrt{45} - \sqrt{20}$
   b. $3\sqrt{3} - \sqrt{\frac{3}{4}} + \sqrt{\frac{1}{3}}$
   c. $\sqrt[3]{54} - \sqrt[3]{8} + 7\sqrt[3]{\frac{1}{4}}$
   d. $\sqrt[3]{\frac{5}{8}} + \sqrt[3]{40} - \sqrt[3]{\frac{8}{9}}$

3. Evaluate $\sqrt{x^2 - y^2}$ when $x = 33$ and $y = 15$.

4. Evaluate $\sqrt{x^2 + y^2}$ when $x = 20$ and $y = 10$.

5. Express each of the following as a rational expression or in simplest radical form. Assume that the symbols x and y represent positive numbers.
   a. $\sqrt{3}(\sqrt{7} - \sqrt{3})$
   b. $(3 + \sqrt{2})^2$
   c. $(2 + \sqrt{3})(2 - \sqrt{3})$
   d. $(2 + 2\sqrt{5})(2 - 2\sqrt{5})$
   e. $(\sqrt{7} - 3)(\sqrt{7} + 3)$
   f. $(3\sqrt{2} + \sqrt{7})(3\sqrt{2} - \sqrt{7})$
   g. $(x - \sqrt{3})(x + \sqrt{3})$
   h. $(2x\sqrt{2} + y)(2x\sqrt{2} - y)$

6. Simplify each of the following quotients as far as possible.
   a. $(\sqrt{21} - \sqrt{3}) \div \sqrt{3}$
   b. $(\sqrt{5} + 4) \div (\sqrt{5} + 1)$
   c. $(3 - \sqrt{2}) \div (3\sqrt{2} - 5)$
   d. $(2\sqrt{5} - \sqrt{3}) \div (3\sqrt{5} - 4\sqrt{2})$

7. If $x = 2 + \sqrt{3}$, show that $x + \frac{1}{x}$ has a rational value.

8. Evaluate $5x^2 - 10x$ when the value of x is $\frac{2-\sqrt{5}}{2}$.

9. Write the factors of $a^4 - b^4$. Use the result to obtain the factored form of $(\sqrt{3} + \sqrt{2})^4 - (\sqrt{3} - \sqrt{2})^4$.

10. The converse of the Pythagorean Theorem is also a theorem: If the square of one side of a triangle is equal to the sum of the squares of the other two sides, then the triangle is a right triangle.

    Use the converse of the Pythagorean Theorem to show that for $A, B, C > 0$, if $A + B = C$, then $\sqrt{A} + \sqrt{B} > \sqrt{C}$, so that $\sqrt{A} + \sqrt{B} > \sqrt{A + B}$.

Lesson 10: The Power of Algebra—Finding Pythagorean Triples

Classwork

Opening Exercise

Sam and Jill decide to explore a city. Both begin their walk from the same starting point.

- Sam walks 1 block north, 1 block east, 3 blocks north, and 3 blocks west.
- Jill walks 4 blocks south, 1 block west, 1 block north, and 4 blocks east.

If all city blocks are the same length, who is the farthest distance from the starting point?

Example 1

Prove that if $x > 1$, then a triangle with side lengths $x^2 - 1$, $2x$, and $x^2 + 1$ is a right triangle.

Example 2

Next we describe an easy way to find Pythagorean triples using the expressions from Example 1. Look at the multiplication table below for $\{1, 2, \dots, 9\}$. Notice that the square numbers $\{1, 4, 9, \dots, 81\}$ lie on the diagonal of this table.

a. What value of x is used to generate the Pythagorean triple $(15,8,17)$ by the formula $(x^2 - 1,\ 2x,\ x^2 + 1)$? How do the numbers $(1, 4, 4, 16)$ at the corners of the shaded square in the table relate to the values 15, 8, and 17?

×	1	2	3	4	5	6	7	8	9	10
1	**1**	2	3	**4**	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	**4**	8	12	**16**	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

b. Now you try one. Form a square on the multiplication table below whose left-top corner is the 1 (as in the example above) and whose bottom-right corner is a square number. Use the sums or differences of the numbers at the vertices of your square to form a Pythagorean triple. Check that the triple you generate is a Pythagorean triple.

×	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Let's generalize this square to any square in the multiplication table where two opposite vertices of the square are square numbers.

c. How can you use the sums or differences of the numbers at the vertices of the shaded square to get a triple $(16, 30, 34)$? Is this a Pythagorean triple?

×	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	**9**	12	**15**	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	**15**	20	**25**	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

d. Using x instead of 5 and y instead of 3 in your calculations in part (c), write down a formula for generating Pythagorean triples in terms of x and y.

Relevant Facts and Vocabulary

<u>**Pythagorean Theorem**</u>: If a right triangle has legs of length a and b units and hypotenuse of length c units, then $a^2 + b^2 = c^2$.

<u>**Converse to the Pythagorean Theorem**</u>**:** If the lengths a, b, c of the sides of a triangle are related by $a^2 + b^2 = c^2$, then the angle opposite the side of length c is a right angle.

<u>**Pythagorean Triple**</u>: A *Pythagorean triple* is a triplet of positive integers (a, b, c) such that $a^2 + b^2 = c^2$. The triplet $(3, 4, 5)$ is a Pythagorean triple but $(1, 1, \sqrt{2})$ is not, even though the numbers are side lengths of an isosceles right triangle.

Problem Set

1. Rewrite each expression as a sum or difference of terms.
   a. $(x-3)(x+3)$
   b. $(x^2-3)(x^2+3)$
   c. $(x^{15}+3)(x^{15}-3)$
   d. $(x-3)(x^2+9)(x+3)$
   e. $(x^2+y^2)(x^2-y^2)$
   f. $(x^2+y^2)^2$
   g. $(x-y)^2(x+y)^2$
   h. $(x-y)^2(x^2+y^2)^2(x+y)^2$

2. Tasha used a clever method to expand and simplify $(a+b+c)(a+b-c)$. She grouped the addends together like this $[(a+b)+c][(a+b)-c]$ and then expanded them to get the difference of two squares:
$$(a+b+c)(a+b-c)=[(a+b)+c][(a+b)-c]=(a+b)^2-c^2=a^2+2ab+b^2-c^2.$$
   a. Is Tasha's method correct? Explain why or why not.
   b. Use a version of her method to find $(a+b+c)(a-b-c)$.
   c. Use a version of her method to find $(a+b-c)(a-b+c)$.

3. Use the difference of two squares identity to factor each of the following expressions.
   a. x^2-81
   b. $(3x+y)^2-(2y)^2$
   c. $4-(x-1)^2$
   d. $(x+2)^2-(y+2)^2$

4. Show that the expression $(x+y)(x-y)-6x+9$ may be written as the difference of two squares, and then factor the expression.

5. Show that $(x+y)^2-(x-y)^2=4xy$ for all real numbers x and y.

6. Prove that a triangle with side lengths $2xy$, x^2-y^2, and x^2+y^2 with $x>y>0$ is a right triangle.

7. Complete the table below to find Pythagorean triples (the first row is done for you).

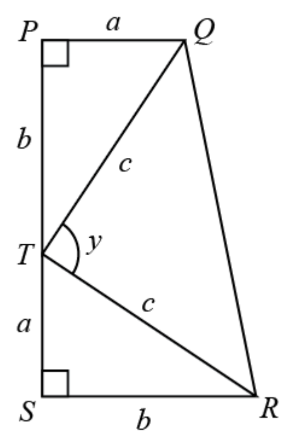
x	y	x^2-y^2	$2xy$	x^2+y^2	Check: Is it a Pythagorean Triple?
2	1	3	4	5	Yes: $3^2+4^2=25$, $5^2=25$
3	1				
3	2				
4	1				
4	2				
4	3				
5	1				

8. Answer the following parts about the triple $(9,12,15)$.
   a. Show that $(9, 12, 15)$ is a Pythagorean triple.
   b. Prove that neither $(9, 12, 15)$ nor $(12,9,15)$ can be found by choosing a pair of integers x and y with $x > y$ and computing $(x^2 - y^2, 2xy, x^2 + y^2)$.
   (Hint: What are the possible values of x and y if $2xy = 12$? What about if $2xy = 9$?)
   c. Wouldn't it be nice if all Pythagorean triples were generated by the expressions $x^2 - y^2$, $2xy$, $x^2 + y^2$? Research Pythagorean triples on the Internet to discover what is known to be true about generating all Pythagorean triples using these three expressions.

9. Follow the steps below to prove the identity $\left(a^2 + b^2\right)(x^2 + y^2) = (ax - by)^2 + (bx + ay)^2$.
   a. Multiply $\left(a^2 + b^2\right)(x^2 + y^2)$.
   b. Square both binomials in $(ax - by)^2 + (bx + ay)^2$ and collect like terms.
   c. Use your answers from part (a) and part (b) to prove the identity.

10. Many U.S. presidents took great delight in studying mathematics. For example, President James Garfield, while still a congressman, came up with a proof of the Pythagorean Theorem based upon the ideas presented below,

    In the diagram, two congruent right triangles with side lengths a, b, and hypotenuse c, are used to form a trapezoid $PQRS$ composed of three triangles.



   a. Explain why $\angle QTR$ is a right angle.
   b. What are the areas of ΔSTR, ΔPTQ, and ΔQTR in terms of a, b, and c?
   c. Using the formula for the area of a trapezoid, what is the total area of trapezoid $PQRS$ in terms of a and b?
   d. Set the sum of the areas of the three triangles from part (b) equal to the area of the trapezoid you found in part (c), and simplify the equation to derive a relationship between a, b, and c. Conclude that a right triangle with legs of length a and b and hypotenuse of length c must satisfy the relationship $a^2 + b^2 = c^2$.

Lesson 11: The Special Role of Zero in Factoring

Classwork

Opening Exercise

Find all solutions to the equation $(x^2 + 5x + 6)(x^2 - 3x - 4) = 0$.

Exercise 1

1. Find the solutions of $(x^2 - 9)(x^2 - 16) = 0$.

Example 1

Suppose we know that the polynomial equation $4x^3 - 12x^2 + 3x + 5 = 0$ has three real solutions and that one of the factors of $4x^3 - 12x^2 + 3x + 5$ is $(x - 1)$. How can we find all three solutions to the given equation?

Exercises 2–5

2. Find the zeros of the following polynomial functions, with their multiplicities.

   a. $f(x) = (x + 1)(x - 1)(x^2 + 1)$

   b. $g(x) = (x - 4)^3(x - 2)^8$

   c. $h(x) = (2x - 3)^5$

   d. $k(x) = (3x + 4)^{100}(x - 17)^4$

3. Find a polynomial function that has the following zeros and multiplicities. What is the degree of your polynomial?

Zero	Multiplicity
2	3
-4	1
6	6
-8	10

4. Is there more than one polynomial function that has the same zeros and multiplicities as the one you found in Exercise 3?

5. Can you find a rule that relates the multiplicities of the zeros to the degree of the polynomial function?

Relevant Vocabulary Terms

In the definitions below, the symbol $\mathbb{R}$ stands for the set of real numbers.

Function: A *function* is a correspondence between two sets, X and Y, in which each element of X is assigned to one and only one element of Y.

The set X in the definition above is called the *domain of the function.* The *range (or image)* of the function is the subset of Y, denoted $f(X)$, that is defined by the following property: y is an element of $f(X)$ if and only if there is an x in X such that $f(x) = y$.

If $f(x) = x^2$ where x can be any real number, then the domain is all real numbers (denoted $\mathbb{R}$), and the range is the set of non-negative real numbers.

Polynomial Function: Given a polynomial expression in one variable, a *polynomial function in one variable* is a function $f: \mathbb{R} \to \mathbb{R}$ such that for each real number x in the domain, $f(x)$ is the value found by substituting the number x into all instances of the variable symbol in the polynomial expression and evaluating.

It can be shown that if a function $f: \mathbb{R} \to \mathbb{R}$ is a polynomial function, then there is some non-negative integer n and collection of real numbers $a_0, a_1, a_2, \ldots, a_n$ with $a_n \neq 0$ such that the function satisfies the equation

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

for every real number x in the domain, which is called the *standard form of the polynomial function.* The function $f(x) = 3x^3 + 4x^2 + 4x + 7$, where x can be any real number, is an example of a function written in standard form.

Degree of a Polynomial Function: The *degree of a polynomial function* is the degree of the polynomial expression used to define the polynomial function.

The degree of $f(x) = 8x^3 + 4x^2 + 7x + 6$ is 3, but the degree of $g(x) = (x+1)^2 - (x-1)^2$ is 1 because when g is put into standard form, it is $g(x) = 4x$.

Constant Function: A *constant function* is a polynomial function of degree 0. A constant function is of the form $f(x) = c$, for a constant c.

Linear Function: A *linear function* is a polynomial function of degree 1. A linear function is of the form $f(x) = ax + b$, for constants a and b with $a \neq 0$.

Quadratic Function: A *quadratic function* is a polynomial function of degree 2. A quadratic function is in *standard form* if it is written in the form $f(x) = ax^2 + bx + c$, for constants a, b, c with $a \neq 0$ and any real number x.

Cubic Function: A *cubic function* is a polynomial function of degree 3. A cubic function is of the form $f(x) = ax^3 + bx^2 + cx + d$, for constants a, b, c, d with $a \neq 0$.

Zeros or Roots of a Function: A *zero* (or *root*) of a function $f: \mathbb{R} \to \mathbb{R}$ is a number x of the domain such that $f(x) = 0$. A zero of a function is an element in the solution set of the equation $f(x) = 0$.

Lesson Summary

Given any two polynomial functions p and q, the solution set of the equation $p(x)q(x) = 0$ can be quickly found by solving the two equations $p(x) = 0$ and $q(x) = 0$ and combining the solutions into one set.

The number a is zero of a polynomial function p with multiplicity m if the factored form of p contains $(x - a)^m$.

Problem Set

For Problems 1–4, find all solutions to the given equations.

1. $(x - 3)(x + 2) = 0$

2. $(x - 5)(x + 2)(x + 3) = 0$

3. $(2x - 4)(x + 5) = 0$

4. $(2x - 2)(3x + 1)(x - 1) = 0$

5. Find four solutions to the equation $(x^2 - 9)(x^4 - 16) = 0$.

6. Find the zeros with multiplicity for the function $p(x) = (x^3 - 8)(x^5 - 4x^3)$.

7. Find two different polynomial functions that have zeros at 1, 3, and 5 of multiplicity 1.

8. Find two different polynomial functions that have a zero at 2 of multiplicity 5 and a zero at -4 of multiplicity 3.

9. Find three solutions to the equation $(x^2 - 9)(x^3 - 8) = 0$.

10. Find two solutions to the equation $(x^3 - 64)(x^5 - 1) = 0$.

11. If p, q, r, s are non-zero numbers, find the solutions to the equation $(px + q)(rx + s) = 0$ in terms of p, q, r, s.

Use the identity $a^2 - b^2 = (a - b)(a + b)$ to solve the equations given in Problems 12–13.

12. $(3x - 2)^2 = (5x + 1)^2$

13. $(x + 7)^2 = (2x + 4)^2$

14. Consider the polynomial function $P(x) = x^3 + 2x^2 + 2x - 5$.
    a. Divide P by the divisor $(x - 1)$ and rewrite in the form $P(x) = (\text{divisor})(\text{quotient}) + \text{remainder}$.
    b. Evaluate $P(1)$.

15. Consider the polynomial function $Q(x) = x^6 - 3x^5 + 4x^3 - 12x^2 + x - 3$.
    a. Divide Q by the divisor $(x - 3)$ and rewrite in the form $Q(x) = (\text{divisor})(\text{quotient}) + \text{remainder}$.
    b. Evaluate $Q(3)$.

16. Consider the polynomial function $R(x) = x^4 + 2x^3 - 2x^2 - 3x + 2$.
    a. Divide R by the divisor $(x + 2)$ and rewrite in the form $R(x) = (\text{divisor})(\text{quotient}) + \text{remainder}$.
    b. Evaluate $R(-2)$.

17. Consider the polynomial function $S(x) = x^7 + x^6 - x^5 - x^4 + x^3 + x^2 - x - 1$.
    a. Divide S by the divisor $(x + 1)$ and rewrite in the form $S(x) = (\text{divisor})(\text{quotient}) + \text{remainder}$.
    b. Evaluate $S(-1)$.

18. Make a conjecture based on the results of Questions 14—17.

Lesson 12: Overcoming Obstacles in Factoring

Classwork

Example 1

Find all real solutions to the equation $(x^2 - 6x + 3)(2x^2 - 4x - 7) = 0$.

Exercise 1

Factor and find all real solutions to the equation $(x^2 - 2x - 4)(3x^2 + 8x - 3) = 0$.

Example 2

Find all solutions to $x^3 + 3x^2 - 9x - 27 = 0$ by factoring the equation.

Exercise 2

Find all real solutions to $x^3 - 5x^2 - 4x + 20 = 0$.

Exercise 3

Find all real solutions to $x^3 - 8x^2 - 2x + 16 = 0$.

Lesson Summary

In this lesson, we learned some techniques to use when faced with factoring polynomials and solving polynomial equations.

- If a fourth degree polynomial can be factored into two quadratic expressions, then each quadratic expression might be factorable either using the quadratic formula or by completing the square.
- Some third degree polynomials can be factored using the technique of factoring by grouping.

Problem Set

1. Solve each of the following equations by completing the square.
   a. $x^2 - 6x + 2 = 0$
   b. $x^2 - 4x = -1$
   c. $x^2 + x - \frac{3}{4} = 0$
   d. $3x^2 - 9x = -6$
   e. $(2x^2 - 5x + 2)(3x^2 - 4x + 1) = 0$
   f. $x^4 - 4x^2 + 2 = 0$

2. Solve each of the following equations using the quadratic formula.
   a. $x^2 - 5x - 3 = 0$
   b. $(6x^2 - 7x + 2)(x^2 - 5x + 5) = 0$
   c. $(3x^2 - 13x + 14)(x^2 - 4x + 1) = 0$

3. Not all of the expressions in the equations below can be factored using the techniques discussed so far in this course. First, determine if the expression can be factored with real coefficients. If so, factor the expression and find all real solutions to the equation.
   a. $x^2 - 5x - 24 = 0$
   b. $3x^2 + 5x - 2 = 0$
   c. $x^2 + 2x + 4 = 0$
   d. $x^3 + 3x^2 - 2x + 6 = 0$
   e. $x^3 + 3x^2 + 2x + 6 = 0$
   f. $2x^3 + x^2 - 6x - 3 = 0$
   g. $8x^3 - 12x^2 + 2x - 3 = 0$
   h. $6x^3 + 8x^2 + 15x + 20 = 0$
   i. $4x^3 + 2x^2 - 36x - 18 = 0$
   j. $x^2 - \frac{1}{2}x - \frac{15}{2} = 0$

4. Solve the following equations by bringing all terms to one side of the equation and factoring out the greatest common factor.

   a. $(x-2)(x-1)=(x-2)(x+1)$
   b. $(2x+3)(x-4)=(2x+3)(x+5)$
   c. $(x-1)(2x+3)=(x-1)(x+2)$
   d. $(x^2+1)(3x-7)=(x^2+1)(3x+2)$
   e. $(x+3)(2x^2+7)=(x+3)(x^2+8)$

5. Consider the expression x^4+1. Since x^2+1 does not factor with real number coefficients, we might expect that x^4+1 also does not factor with real number coefficients. In this exercise, we will investigate the possibility of factoring x^4+1.

   a. Simplify the expression $(x^2+1)^2-2x^2$.
   b. Factor $(x^2+1)^2-2x^2$ as a difference of squares.
   c. Is it possible to factor x^4+1 with real number coefficients? Explain

Lesson 13: Mastering Factoring

Classwork

Opening Exercises

Factor each of the following expressions. What similarities do you notice between the examples in the left column and those on the right?

a. $x^2 - 1$

b. $9x^2 - 1$

c. $x^2 + 8x + 15$

d. $4x^2 + 16x + 15$

e. $x^2 - y^2$

f. $x^4 - y^4$

Example 1

Write $9 - 16x^4$ as the product of two factors.

Example 2

Factor $4x^2y^4 - 25x^4z^6$.

Exercise 1

1. Factor the following expressions:

   a. $4x^2 + 4x - 63$

   b. $12y^2 - 24y - 15$

Exercises 2–4

Factor each of the following, and show that the factored form is equivalent to the original expression.

2. $a^3 + 27$

3. $x^3 - 64$

4. $2x^3 + 128$

Lesson Summary

In this lesson we learned additional strategies for factoring polynomials.

- The difference of squares identity $a^2 - b^2 = (a-b)(a+b)$ can be used to to factor more advanced binomials.
- Trinomials can often be factored by looking for structure and then applying our previous factoring methods.
- Sums and differences of cubes can be factored by the formulas

$$x^3 + a^3 = (x+a)(x^2 - ax - a^2)$$
$$x^3 - a^3 = (x-a)(x^2 + ax + a^2).$$

Problem Set

1. If possible, factor the following expressions using the techniques discussed in this lesson.

   a. $25x^2 - 25x - 14$
   b. $9x^2y^2 - 18xy + 8$
   c. $45y^2 + 15y - 10$
   d. $y^6 - y^3 - 6$
   e. $x^3 - 125$
   f. $2x^4 - 16x$
   g. $9x^2 - 25y^4z^6$
   h. $36x^6y^4z^2 - 25x^2z^{10}$
   i. $4x^2 + 9$
   j. $x^4 - 36$
   k. $1 + 27x^9$
   l. $x^3y^6 + 8z^3$

2. Consider the polynomial expression $y^4 + 4y^2 + 16$.

   a. Is $y^4 + 4y^2 + 16$ factorable using the methods we have seen so far?
   b. Factor $y^6 - 64$ first as a difference of cubes, then factor completely: $(y^2)^3 - 4^3$.
   c. Factor $y^6 - 64$ first as a difference of squares, then factor completely: $(y^3)^2 - 8^2$.
   d. Explain how your answers to parts (b) and (c) provide a factorization of $y^4 + 4y^2 + 16$.
   e. If a polynomial can be factored as either a difference of squares or a difference of cubes, which formula should you apply first, and why?

3. Create expressions that have a structure that allows them to be factored using the specified identity. Be creative and produce challenging problems!

   a. Difference of squares
   b. Difference of cubes
   c. Sum of cubes

Lesson 14: Graphing Factored Polynomials

Classwork

Opening Exercise

An engineer is designing a rollercoaster for younger children and has tried some functions to model the height of the rollercoaster during the first 300 yards. She came up with the following function to describe what she believes would make a fun start to the ride:

$$H(x) = -3x^4 + 21x^3 - 48x^2 + 36x,$$

where $H(x)$ is the height of the rollercoaster (in yards) when the rollercoaster is $100x$ yards from the beginning of the ride. Answer the following questions to help determine at which distances from the beginning of the ride the rollercoaster is at its lowest height.

a. Does this function describe a roller coaster that would be fun to ride? Explain.

b. Can you see any obvious x-values from the equation where the rollercoaster is at height 0?

c. Using a graphing utility, graph the function H on the interval $0 \leq x \leq 3$, and identify when the rollercoaster is 0 yards off the ground.

d. What do the x-values you found in part (c) mean in terms of distance from the beginning of the ride?

e. Why do roller coasters always start with the largest hill first?

f. Verify your answers to part (c) by factoring the polynomial function H.

g. How do you think the engineer came up with the function for this model?

h. What is wrong with this rollercoaster model at distance 0 yards and 300 yards? Why might this not initially bother the engineer when she is first designing the track?

Example 1

Graph each of the following polynomial functions. What are the function's zeros (counting multiplicities)? What are the solutions to $f(x) = 0$? What are the x-intercepts to the graph of the function? How does the degree of the polynomial function compare to the x-intercepts of the graph of the function?

a. $f(x) = x(x-1)(x+1)$

b. $f(x) = (x+3)(x+3)(x+3)(x+3)$

c. $f(x) = (x-1)(x-2)(x+3)(x+4)(x+4)$

d. $f(x) = (x^2+1)(x-2)(x-3)$

Example 2

Consider the function $f(x) = x^3 - 13x^2 + 44x - 32$.

a. Use the fact that $x - 4$ is a factor of f to factor this polynomial.

b. Find the x-intercepts for the graph of f.

c. At which x-values can the function change from being positive to negative or from negative to positive?

d. To sketch a graph of f, we need to consider whether the function is positive or negative on the four intervals $x < 1$, $1 < x < 4$, $4 < x < 8$, and $x > 8$. Why is that?

e. How can we tell if the function is positive or negative on an interval between x-intercepts?

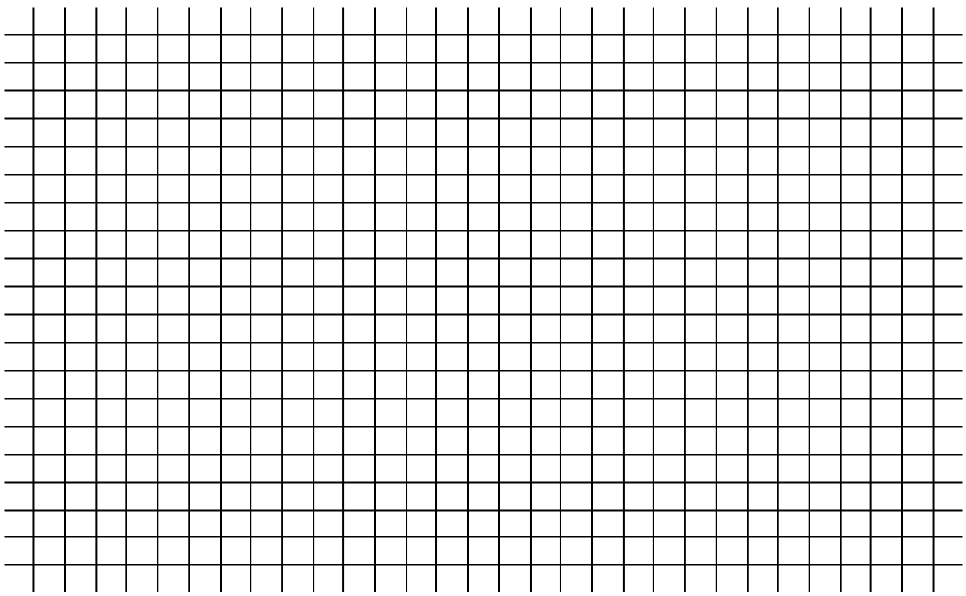
f. For $x < 1$, is the graph above or below the x-axis? How can you tell?

g. For $1 < x < 4$, is the graph above or below the x-axis? How can you tell?

h. For $4 < x < 8$, is the graph above or below the x-axis? How can you tell?

i. For $x > 8$, is the graph above or below the x-axis? How can you tell?

j. Use the information generated in parts (f)–(i) to sketch a graph of f.



k. Graph $y = f(x)$ on the interval from $[0,9]$ using a graphing utility, and compare your sketch with the graph generated by the graphing utility.

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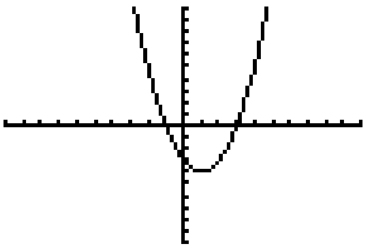
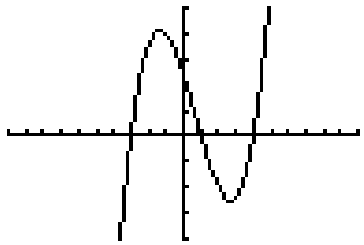
Discussion

For any particular polynomial, can we determine how many relative maxima or minima there are? Consider the following polynomial functions in factored form and their graphs.

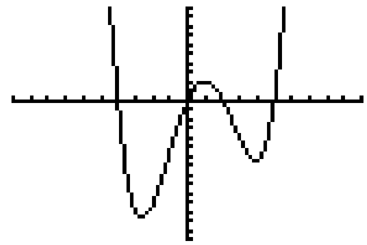
$f(x) = (x+1)(x-3)$



$g(x) = (x+3)(x-1)(x-4)$



$h(x) = (x)(x+4)(x-2)(x-5)$



Degree of each polynomial:

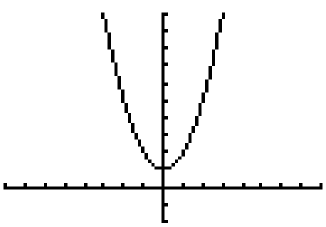
Number of x-intercepts in each graph:

Number of relative maxima or minima in each graph:

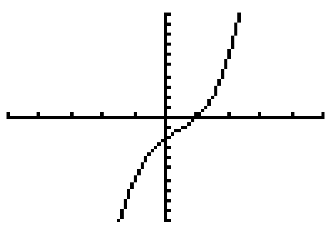
What observations can we make from this information?

Is this true for every polynomial? Consider the examples below.

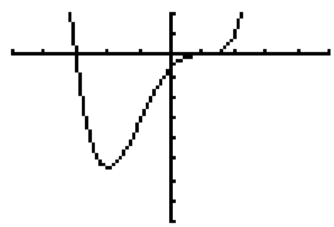
$\mathbf{r(x) = x^2 + 1}$



$\mathbf{s(x) = (x^2 + 2)(x - 1)}$



$\mathbf{t(x) = (x + 3)(x - 1)(x - 1)(x - 1)}$



Degree of each polynomial:

Number of x-intercepts in each graph:

Number of relative maximums or minimums in each graph:

What observations can we make from this information?

Relevant Vocabulary

<u>Increasing/Decreasing</u>: Given a function f whose domain and range are subsets of the real numbers and I is an interval contained within the domain, the function is called *increasing on the interval* I if

$$f(x_1) < f(x_2) \text{ whenever } x_1 < x_2 \text{ in } I.$$

It is called decreasing on the interval I if

$$f(x_1) > f(x_2) \text{ whenever } x_1 < x_2 \text{ in } I.$$

<u>Relative Maximum</u>: Let f be a function whose domain and range are subsets of the real numbers. The function has a *relative maximum at* c if there exists an open interval I of the domain that contains c such that

$$f(x) \leq f(c) \text{ for all } x \text{ in the interval } I.$$

If c is a relative maximum, then the value $f(c)$ is called the *relative maximum value*.

<u>Relative Minimum</u>: Let f be a function whose domain and range are subsets of the real numbers. The function has a *relative minimum at* c if there exists an open interval I of the domain that contains c such that

$$f(x) \geq f(c) \text{ for all } x \text{ in the interval } I.$$

If c is a relative minimum, then the value $f(c)$ is called the *relative minimum value*.

<u>Graph of f</u>: Given a function f whose domain D and the range are subsets of the real numbers, the graph of f is the set of ordered pairs in the Cartesian plane given by

$$\{(x, f(x)) \mid x \in D\}.$$

<u>Graph of $y = f(x)$</u>: Given a function f whose domain D and the range are subsets of the real numbers, the *graph of* $y = f(x)$ is the set of ordered pairs (x, y) in the Cartesian plane given by

$$\{(x, y) \mid x \in D \text{ and } y = f(x)\}.$$

Lesson Summary

A polynomial of degree n may have up to n x-intercepts and up to $n - 1$ relative maximum/minimum points.

A relative maximum is the x-value c that produces the highest point on a graph of f in a circle around $(c, f(c))$. That highest value $f(c)$ is a relative maximum value.

A relative minimum is the x-value d that produces the lowest point on a graph of f in a circle around $(d, f(d))$. That lowest value $f(d)$ is a relative minimum value.

Problem Set

1. For each function below, identify the largest possible number of x-intercepts and the largest possible number of relative maximum and minimum points based on the degree of the polynomial. Then use a calculator or graphing utility to graph the function and find the actual number of x-intercepts and relative maximum/minimum points.

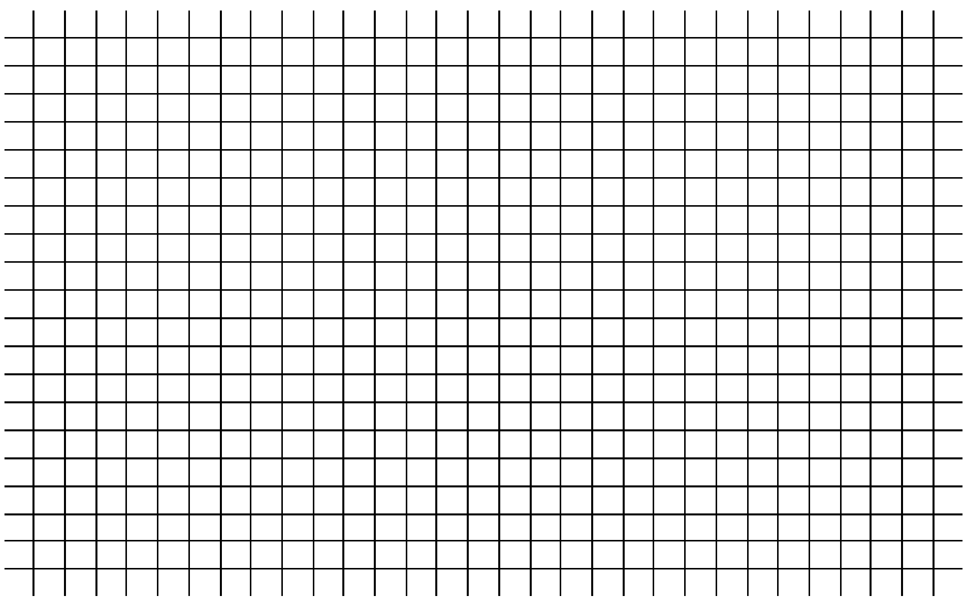
   a. $f(x) = 4x^3 - 2x + 1$

   b. $g(x) = x^7 - 4x^5 - x^3 + 4x$

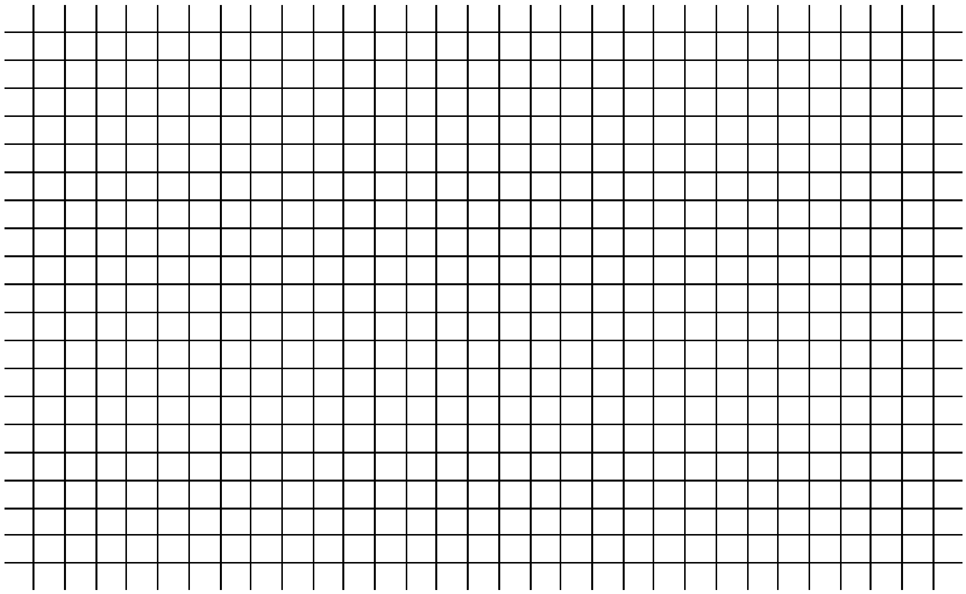
   c. $h(x) = x^4 + 4x^3 + 2x^2 - 4x + 2$

Function	Largest number of x-intercepts	Largest number of relative max/mins	Actual number of x-intercepts	Actual number of relative max/mins
a. $f(x)$				
b. $g(x)$				
c. $h(x)$				

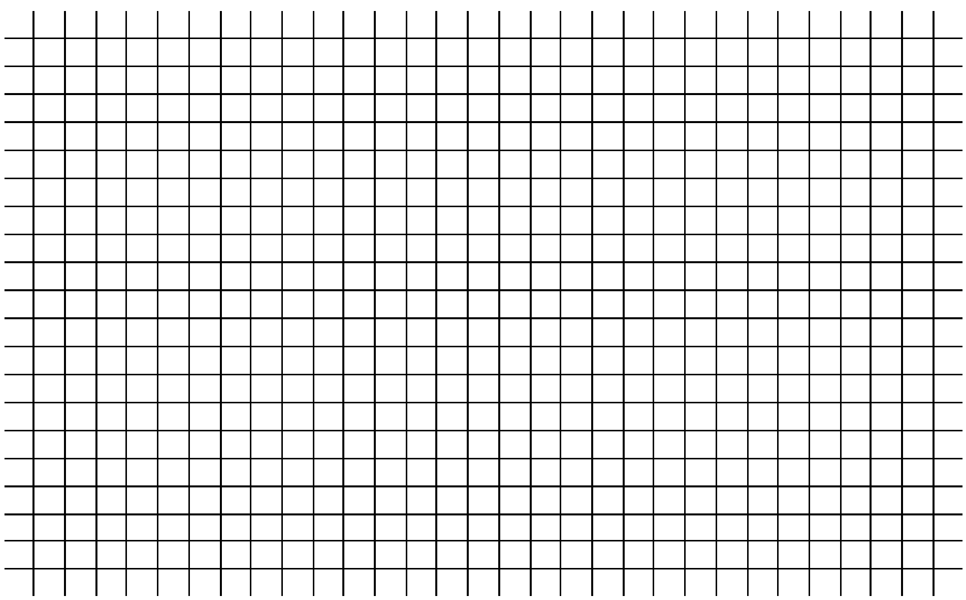
2. Sketch a graph of the function $f(x) = \frac{1}{2}(x+5)(x+1)(x-2)$ by finding the zeros and determining the sign of the values of the function between zeros.



3. Sketch a graph of the function $f(x) = -(x+2)(x-4)(x-6)$ by finding the zeros and determining the sign of the values of the function between zeros.



4. Sketch a graph of the function $f(x) = x^3 - 2x^2 - x + 2$ by finding the zeros and determining the sign of the values of the function between zeros.



5. Sketch a graph of the function $f(x) = x^4 - 4x^3 + 2x^2 + 4x - 3$ by determining the sign of the values of the function between the zeros -1, 1, and 3.

6. A function f has zeros at -1, 3, and 5. We know that $f(-2)$ and $f(2)$ are negative, while $f(4)$ and $f(6)$ are positive. Sketch a graph of f.

7. The function $h(t) = -16\mathrm{t}^2 + 33\mathrm{t} + 45$ represents the height of a ball tossed upward from the roof of a building 45 feet in the air after t seconds. Without graphing, determine when the ball will hit the ground.

Lesson 15: Structure in Graphs of Polynomial Functions

Classwork

Opening Exercise

Sketch the graph of $f(x) = x^2$. What will the graph of $g(x) = x^4$ look like? Sketch it on the same coordinate plane. What will the graph of $h(x) = x^6$ look like?

Example 1

Sketch the graph of $f(x) = x^3$. What will the graph of $g(x) = x^5$ look like? Sketch this on the same coordinate plane. What will the graph of $h(x) = x^7$ look like? Sketch this on the same coordinate plane.

Exercise 1

a. Consider the following function, $f(x) = 2x^4 + x^3 - x^2 + 5x + 3$, with a mixture of odd and even degree terms. Predict whether its end behavior will be like the functions in the Opening Exercise or more like the functions from Example 1. Graph the function f using a graphing utility to check your prediction.

b. Consider the following function, $f(x) = 2x^5 - x^4 - 2x^3 + 4x^2 + x + 3$, with a mixture of odd and even degree terms. Predict whether its end behavior will be like the functions in the Epening Exercise or more like the functions from Example 1. Graph the function f using a graphing utility to check your prediction.

c. Thinking back to our discussion of x-intercepts of graphs of polynomial functions from the previous lesson, sketch a graph of an even degree polynomial function that has no x-intercepts.

d. Similarly, can you sketch a graph of an odd degree polynomial function with no x-intercepts?

Exercise 2

The Center for Transportation Analysis (CTA) studies all aspects of transportation in the United States, from energy and environmental concerns to safety and security challenges. A 1997 study compiled the following data of the fuel economy in miles per gallon (mpg) of a car or light truck at various speeds measured in miles per hour (mph). The data is compiled in the table below.

Fuel Economy by Speed

Speed (mph)	Fuel Economy (mpg)
15	24.4
20	27.9
25	30.5
30	31.7
35	31.2
40	31.0
45	31.6
50	32.4
55	32.4
60	31.4
65	29.2
70	26.8
75	24.8

Source: Transportation Energy Data Book, Table 4.28. http://cta.ornl.gov/data/chapter4.shtml

a. Plot the data using a graphing utility. Which variable is the independent variable?

b. This data can be modeled by a polynomial function. Determine if the function that models the data would have an even or odd degree.

c. Is the leading coefficient of the polynomial that can be used to model this data positive or negative?.

d. List two possible reasons the data might have the shape that it does.

Relevant Vocabulary

Even Function: Let f be a function whose domain and range is a subset of the real numbers. The function f is called *even* if the equation $f(x) = f(-x)$ is true for every number x in the domain.

Even-degree polynomial functions are sometimes even functions, like $f(x) = x^{10}$, and sometimes not, like $g(x) = x^2 - x$.

Odd Function: Let f be a function whose domain and range is a subset of the real numbers. The function f is called *odd* if the equation $f(-x) = -f(x)$ is true for every number x in the domain.

Odd-degree polynomial functions are sometimes odd functions, like $f(x) = x^{11}$, and sometimes not, like $h(x) = x^3 - x^2$.

Problem Set

1. Graph the functions from the Opening Exercise simultaneously using a graphing utility and zoom in at the origin.
   a. At $x = 0.5$, order the values of the functions from least to greatest.
   b. At $x = 2.5$, order the values of the functions from least to greatest.
   c. Identify the x-value(s) where the order reverses. Write a brief sentence on why you think this switch occurs.

2. The National Agricultural Statistics Service (NASS) is an agency within the USDA that collects and analyzes data covering virtually every aspect of agriculture in the United States. The following table contains information on the amount (in tons) of the following vegetables produced in the U.S. from 1988–1994 for processing into canned, frozen, and packaged foods: lima beans, snap beans, beets, cabbage, sweet corn, cucumbers, green peas, spinach, and tomatoes.

Vegetable Production by Year

Year	Vegetable Production (tons)
1988	11,393,320
1989	14,450,860
1990	15,444,970
1991	16,151,030
1992	14,236,320
1993	14,904,750
1994	18,313,150

Source: NASS Statistics of Vegetables and Melons, 1995, Table 191.

http://www.nass.usda.gov/Publications/Ag_Statistics/1995-1996/agr95_4.pdf

   a. Plot the data using a graphing utility.
   b. Determine if the data displays the characteristics of an odd- or even-degree polynomial.
   c. List two possible reasons the data might have the shape that it does.

3. The U.S. Energy Information Administration (EIA) is responsible for collecting and analyzing information about energy production and use in the United States and for informing policy makers and the public about issues of energy, the economy, and the environment. The following table contains data from the EIA about natural gas consumption from 1950–2010, measured in millions of cubic feet.

U.S. Natural Gas Consumption by Year

Year	U.S. natural gas total consumption (millions of cubic feet)
1950	5.77
1955	8.69
1960	11.97
1965	15.28
1970	21.14
1975	19.54
1980	19.88
1985	17.28
1990	19.17
1995	22.21
2000	23.33
2005	22.01
2010	24.09

Source: U.S. Energy Information Administration. http://www.eia.gov/dnav/ng/hist/n9140us2a.htm

a. Plot the data using a graphing utility.

b. Determine if the data displays the characteristics of an odd- or even-degree polynomial function.

c. List two possible reasons the data might have the shape that it does.

4. We use the term *even function* when a function f satisfies the equation $f(-x) = f(x)$ for every number x in its domain. Consider the function $f(x) = -3x^2 + 7$. Note that the degree of the function is even, and each term is of an even degree (the constant term is degree 0.

a. Graph the function using a graphing utility.

b. Does this graph display any symmetry?

c. Evaluate $f(-x)$.

d. Is f an even function? Explain how you know.

5. We use the term *odd function* when a function f satisfies the equation $f(-x) = -f(x)$ for every number x in its domain. Consider the function $f(x) = 3x^3 - 4x$. The degree of this function is odd, and each term is of an odd degree.

a. Graph the function using a graphing utility.

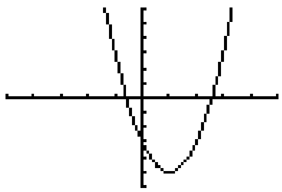
b. Does this graph display any symmetry?

c. Evaluate $f(-x)$.

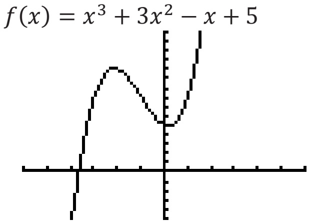
d. Is f an odd function? Explain how you know.

6. We have talked about x-intercepts of the graph of a function in both this lesson and the previous one. The x-intercepts correspond to the zeros of the function. Consider the following examples of polynomial functions and their graphs to determine an easy way to find the y-intercept of the graph of a polynomial function

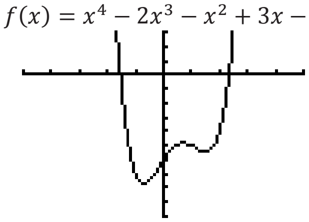
$f(x) = 2x^2 - 4x - 3$



$f(x) = x^3 + 3x^2 - x + 5$



$f(x) = x^4 - 2x^3 - x^2 + 3x - 6$



Lesson 16: Modeling with Polynomials—An Introduction

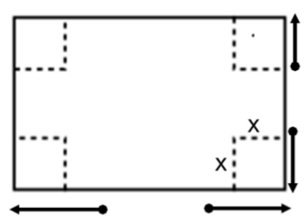
Classwork

Mathematical Modeling Exercise

You will be assigned to a group, which will create a box from a piece of construction paper. Each group will record its box's measurements and use said measurement values to calculate and record the volume of its box. Each group will contribute to the following class table on the board.

Group	Length	Width	Height	Volume
1				
2				
3				
4				

Using the given construction paper, cut out congruent squares from each corner and fold the sides in order to create an open-topped box as shown on the figure below.



1. Measure the length, width, and height of the box to the nearest tenth of a centimeter.

2. Calculate the volume.

3. Have a group member record the values on the table on the board.

4. Create a scatterplot of volume versus height using technology.

5. What type of polynomial function could we use to model this data?

6. Use the regression feature to find a function to model the data. Does a quadratic or a cubic regression provide a better fit to the data?

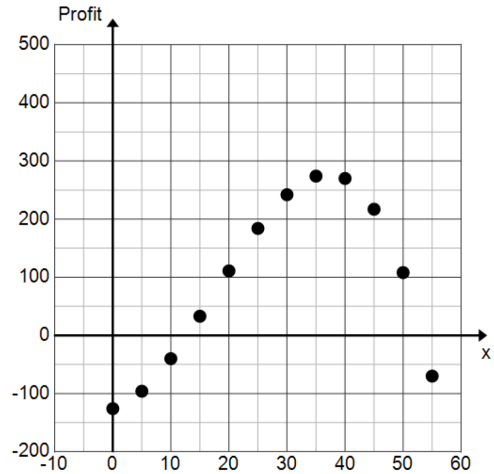
7. Find the maximum volume of the box.

8. What size square should be cut from each corner in order to maximize the volume?

9. What are the dimensions of the box of maximum volume?

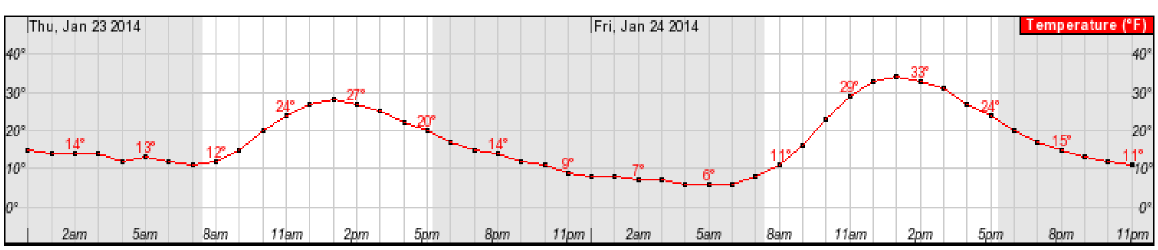
Problem Set

1. For a fundraiser, members of the math club decide to make and sell "Pythagoras may have been Fermat's first problem but not his last" t-shirts. They are trying to decide how many t-shirts to make and sell at a fixed price. They surveyed the level of interest of students around school and made a scatterplot of number of t-shirts sold (x) versus profit shown below.



   a. Identify the y-intercept. Interpret its meaning within the context of this problem.
   b. If we model this data with a function, what point on the graph of that function represents the number of t-shirts they need to sell in order to break even? Why?
   c. What is the smallest number of t-shirts they can sell and still make a profit?
   d. How many t-shirts should they sell in order to maximize the profit?
   e. What is the maximum profit?
   f. What factors would affect the profit?
   g. What would cause the profit to start decreasing?

2. The following graph shows the temperature in Aspen, Colorado during a 48-hour period beginning at midnight on Thursday, January 21, 2014. (Source: National Weather Service)



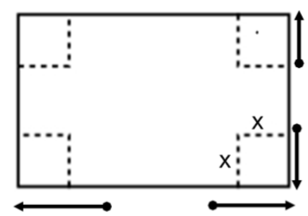
   a. We can model the data shown with a polynomial function. What degree polynomial would be a reasonable choice?
   b. Let T be the function that represents the temperature, in degrees Fahrenheit, as a function of time t, in hours. If we let $t = 0$ correspond to midnight on Thursday, interpret the meaning of $T(5)$. What is $T(5)$?
   c. What are the relative maximum points? Interpret their meanings.

Lesson 17: Modeling with Polynomials—An Introduction

Classwork

Opening Exercise

In Lesson 16, we created an open-topped box by cutting congruent squares from each corner of a piece of construction paper.



a. Express the dimensions of the box in terms of x.

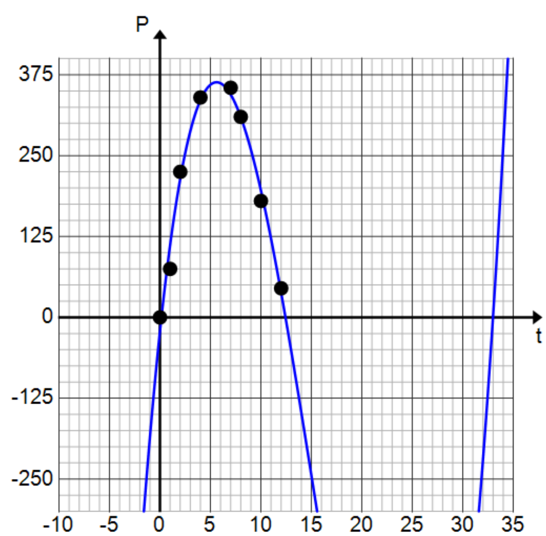
b. Write a formula for the volume of the box as a function of x. Give the answer in standard form.

Mathematical Modeling Exercises 1–13

The owners of Dizzy Lizzy's, an amusement park, are studying the wait time at their most popular roller coaster. The table below shows the number of people standing in line for the roller coaster t hours after Dizzy Lizzy's opens.

t (hours)	0	1	2	4	7	8	10	12
P (people in line)	0	75	225	345	355	310	180	45

Jaylon made a scatterplot and decided that a cubic function should be used to model the data. His scatterplot and curve are shown below.



1. Do you agree that a cubic polynomial function is a good model for this data? Explain.

2. What information would Dizzy Lizzy's be interested in learning about from this graph? How could they determine the answer?

3. Estimate the time at which the line is the longest. Explain how you know.

4. Estimate the number of people in line at that time. Explain how you know.

5. Estimate the t-intercepts of the function used to model this data.

6. Use the t-intercepts to write a formula for the function of the number of people in line, f, after t hours.

7. Use the maximum point to find the leading coefficient. Explain your reasoning.

8. What would be a reasonable domain for your function f? Why?

9. Use your function f to calculate the number of people in line 10 hours after the park opens.

10. Comparing the value calculated above to the actual value in the table, is your function f an accurate model for the data? Explain.

11. Use the regression feature of a graphing calculator to find a cubic function to model the data.

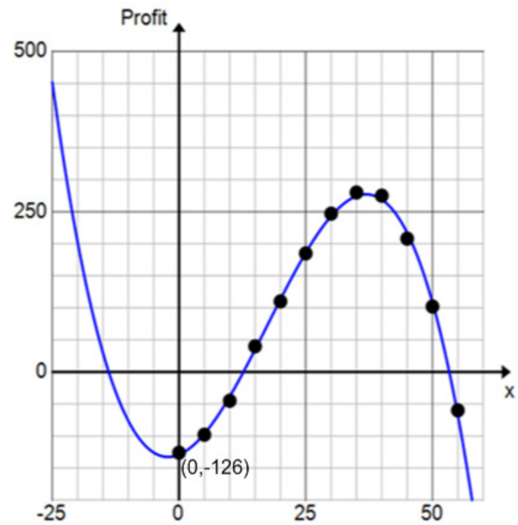
12. Graph your function and the one generated by the graphing calculator and use the graphing calculator to complete the table. Round your answers to the nearest integer.

t (hours)	0	1	2	4	7	8	10	12
P (people in line)	0	75	225	345	355	310	180	45
f (your equation)								
g (regression eqn.)								

13. Based on the results from the table, which model was more accurate at $t = 2$ hours? $t = 10$ hours?

Problem Set

1. Recall the math club fundraiser from yesterday's problem set. The club members would like to find a function to model their data, so Kylie draws a curve through the data points as shown below.



   a. What type of function does it appear that she has drawn?
   b. The function that models the profit in terms of the number of t-shirts made has the form $P(x) = c(x^3 - 53x^2 - 236x + 9828)$. Use the vertical intercept labeled on the graph to find the value of the leading coefficient c.
   c. From the graph, estimate the profit if the math club sells 30 t-shirts.
   d. Use your function to estimate the profit if the math club sells 30 t-shirts.
   e. Which estimate do you think is more reliable? Why?

2. A box is to be constructed so that it has a square base and no top.
   a. Draw and label the sides of the box. Label the sides of the base as x and the height of the box as h.
   b. The surface area is 108 cm^2. Write a formula for the surface area S and then solve for h.
   c. Write a formula for the function of the volume of the box in terms of x.
   d. Use a graphing utility to find the maximum volume of the box.
   e. What dimensions should the box be in order to maximize its volume?

Lesson 18: Overcoming a Second Obstacle in Factoring—What If There Is a Remainder?

Classwork

Opening Exercise

Write the rational number $\frac{13}{4}$ as a mixed number.

Example 1

a. Find the quotient by factoring the numerator.

$$\frac{x^2+3x+2}{x+2}$$

b. Find the quotient.

$$\frac{x^2+3x+5}{x+2}$$

Example 2

a. Find the quotient by factoring the numerator.

$$\frac{x^3-8}{x-2}$$

b. Find the quotient.

$$\frac{x^3-4}{x-2}$$

Exercises 1–10

Find each quotient by inspection.

1. $\dfrac{x+4}{x+1}$

2. $\dfrac{2x-7}{x-3}$

3. $\dfrac{x^2-21}{x+4}$

Find each quotient by using the reverse tabular method.

4. $\dfrac{x^2+4x+10}{x-8}$

5. $\dfrac{x^3-x^2+3x-1}{x+3}$

6. $\dfrac{x^2-2x-19}{x-1}$

Find each quotient by using long division.

7. $\dfrac{x^2-x-25}{x+6}$

8. $\dfrac{x^4-8x^2+12}{x+2}$

9. $\dfrac{4x^3+5x-8}{2x-5}$

Rewrite the numerator in the form $(x-h)^2+k$ by completing the square. Then find the quotient.

10. $\dfrac{x^2+4x-9}{x+2}$

Mental Math

$\dfrac{x^2-9}{x+3}$	$\dfrac{x^2-4x+3}{x-1}$	$\dfrac{x^2-16}{x+4}$	$\dfrac{x^2-3x-4}{x+1}$
$\dfrac{x^3-3x^2}{x-3}$	$\dfrac{x^4-x^2}{x^2-1}$	$\dfrac{x^2+x-6}{x+3}$	$\dfrac{x^2-4}{x+2}$
$\dfrac{x^2-8x+12}{x-2}$	$\dfrac{x^2-36}{x+6}$	$\dfrac{x^2+6x+8}{x+4}$	$\dfrac{x^2-4}{x-2}$
$\dfrac{x^2-x-20}{x+4}$	$\dfrac{x^2-25}{x+5}$	$\dfrac{x^2-2x+1}{x-1}$	$\dfrac{x^2-3x+2}{x-2}$
$\dfrac{x^2+4x-5}{x-1}$	$\dfrac{x^2-25}{x-5}$	$\dfrac{x^2-10x}{x}$	$\dfrac{x^2-12x+20}{x-2}$
$\dfrac{x^2+5x+4}{x+4}$	$\dfrac{x^2-1}{x-1}$	$\dfrac{x^2+16x+64}{x+8}$	$\dfrac{x^2+9x+8}{x+1}$

Problem Set

1. For each pair of problems, find the first quotient by factoring the numerator. Then, find the second quotient by using the first quotient.

   a. $\dfrac{3x-6}{x-2}$ $\qquad$ $\dfrac{3x-9}{x-2}$

   b. $\dfrac{x^2-5x-14}{x-7}$ $\qquad$ $\dfrac{x^2-5x+2}{x-7}$

   c. $\dfrac{x^3+1}{x+1}$ $\qquad$ $\dfrac{x^3}{x+1}$

   d. $\dfrac{x^2-13x+36}{x-4}$ $\qquad$ $\dfrac{x^2-13x+30}{x-4}$

Find each quotient by using the reverse tabular method.

2. $\dfrac{x^3-9x^2+5x+2}{x-1}$

3. $\dfrac{x^2+x+10}{x+12}$

4. $\dfrac{2x+6}{x-8}$

5. $\dfrac{x^2+8}{x+3}$

Find each quotient by using long division.

6. $\dfrac{x^4-9x^2+10x}{x+2}$

7. $\dfrac{x^5-35}{x-2}$

8. $\dfrac{x^2}{x-6}$

9. $\dfrac{x^3+2x^2+8x+1}{x+5}$

10. $\dfrac{x^3+2x+11}{x-1}$

11. $\dfrac{x^4+3x^3-2x^2+6x-15}{x}$

12. Rewrite the numerator in the form $(x-h)^2+k$ by completing the square. Then, find the quotient.

$$\frac{x^2-6x-10}{x-3}$$

Lesson 19: The Remainder Theorem

Classwork

Exercises 1–3

1. Consider the polynomial function $f(x) = 3x^2 + 8x - 4$.

   a. Divide f by $x - 2$.

   b. Find $f(2)$.

2. Consider the polynomial function $g(x) = x^3 - 3x^2 + 6x + 8$.

   a. Divide g by $x + 1$.

   b. Find $g(-1)$.

3. Consider the polynomial function $h(x) = x^3 + 2x - 3$.

   a. Divide h by $x - 3$.

   b. Find $h(3)$.

Exercises 4–6

4. Consider the polynomial $P(x) = x^3 + kx^2 + x + 6$.
   a. Find the value of k so that $x + 1$ is a factor of P.

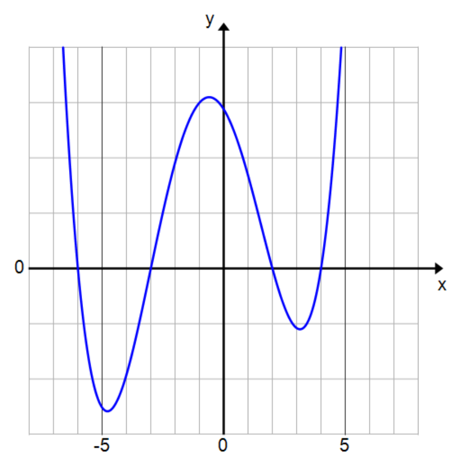
   b. Find the other two factors of P for the value of k found in part (a).

5. Consider the polynomial $P(x) = x^4 + 3x^3 - 28x^2 - 36x + 144$.
   a. Is 1 a zero of the polynomial P?

   b. Is $x + 3$ one of the factors of P?

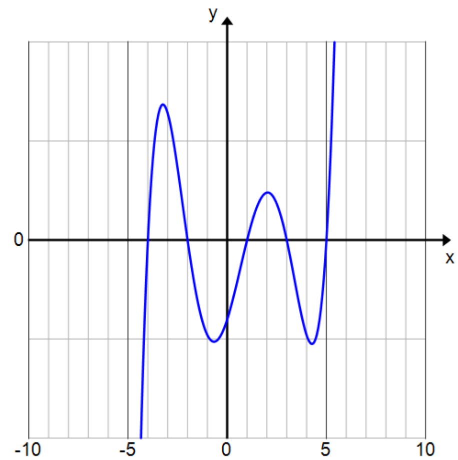
   c. The graph of P is shown to the right. What are the zeros of P?

   d. Write the equation of P in factored form.



6. Consider the graph of a degree 5 polynomial shown to the right, with x-intercepts -4, -2, 1, 3, and 5.

   a. Write a formula for a possible polynomial function that the graph represents using c as constant factor.



   b. Suppose the y-intercept is -4. Adjust your function to fit the y-intercept by finding the constant factor c.

Lesson Summary

Remainder Theorem:

Let P be a polynomial function in x, and let a be any real number. Then there exists a unique polynomial function q such that the equation

$$P(x) = q(x)(x-a) + P(a)$$

is true for all x. That is, when a polynomial is divided by $(x-a)$, the remainder is the value of the polynomial evaluated at a.

Factor Theorem:

Let P be a polynomial function in x, and let a be any real number. If a is a zero of P, then $(x-a)$ is a factor of P.

Example: If $P(x) = x^2 - 3$ and $a = 4$, then $P(x) = (x+4)(x-4) + 13$ where $q(x) = x + 4$ and $P(4) = 13$.

Example: If $P(x) = x^3 - 5x^2 + 3x + 9$, then $P(3) = 27 - 45 + 9 + 9 = 0$, so $(x-3)$ is a factor of P.

Problem Set

1. Use the Remainder Theorem to find the remainder for each of the following divisions.
   a. $(x^2 + 3x + 1) \div (x + 2)$
   b. $(x^3 - 6x^2 - 7x + 9) \div (x - 3)$
   c. $(32x^4 + 24x^3 - 12x^2 + 2x + 1) \div (x + 1)$
   d. $(32x^4 + 24x^3 - 12x^2 + 2x + 1) \div (2x - 1)$
   e. Hint for part (d): Can you rewrite the division expression so that the divisor is in the form $(x - c)$ for some constant c?

2. Consider the polynomial $P(x) = x^3 + 6x^2 - 8x - 1$. Find $P(4)$ in two ways.

3. Consider the polynomial function $P(x) = 2x^4 + 3x^2 + 12$.
   a. Divide P by $x + 2$ and rewrite P in the form $(\text{divisor})(\text{quotient}) + \text{remainder}$.
   b. Find $P(-2)$.

4. Consider the polynomial function $P(x) = x^3 + 42$.
   a. Divide P by $x - 4$ and rewrite P in the form $(\text{divisor})(\text{quotient}) + \text{remainder}$.
   b. Find $P(4)$.

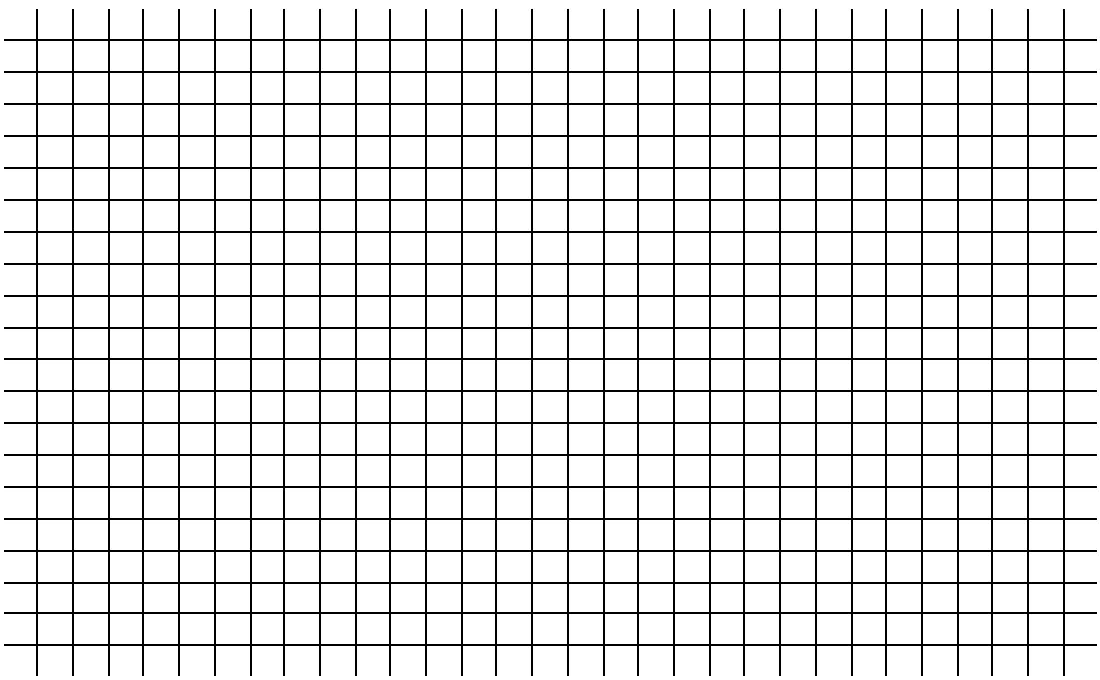
5. Explain why for a polynomial function P, $P(a)$ is equal to the remainder of the quotient of P and $x - a$.

6. Is $x - 5$ a factor of the function $f(x) = x^3 + x^2 - 27x - 15$? Show work supporting your answer.

7. Is $x + 1$ a factor of the function $f(x) = 2x^5 - 4x^4 + 9x^3 - x + 13$? Show work supporting your answer.

8. A polynomial function p has zeros of $2, 2, -3, -3, -3$, and 4. Find a possible formula for p and state its degree. Why is the degree of the polynomial not 3?

9. Consider the polynomial function $P(x) = x^3 - 8x^2 - 29x + 180$.
   a. Verify that $P(9) = 0$. Since $P(9) = 0$, what must one of the factors of P be?
   b. Find the remaining two factors of P.
   c. State the zeros of P.
   d. Sketch the graph of P.



10. Consider the polynomial function $P(x) = 2x^3 + 3x^2 - 2x - 3$.

    a. Verify that $P(-1) = 0$. Since $P(-1) = 0$, what must one of the factors of P be?

    b. Find the remaining two factors of P.

    c. State the zeros of P.

    d. Sketch the graph of P.

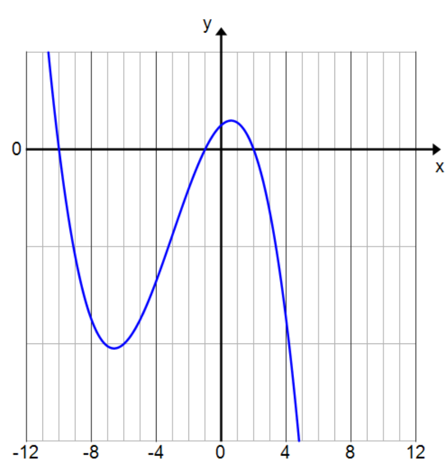
11. The graph to the right is of a third degree polynomial function f.

    a. State the zeros of f.

    b. Write a formula for f in factored form using c for the constant factor.

    c. Use the fact that $f(-4) = -54$ to find the constant factor.

    d. Verify your equation by using the fact that $f(1) = 11$.



12. Find the value of k so that $(x^3 - kx^2 + 2) \div (x - 1)$ has remainder 8.

13. Find the value k so that $(kx^3 + x - k) \div (x + 2)$ has remainder 16.

14. Show that $x^{51} - 21x + 20$ is divisible by $x - 1$.

15. Show that $x + 1$ is a factor of $19x^{42} + 18x - 1$.

Write a polynomial function that meets the stated conditions.

16. The zeros are -2 and 1.

17. The zeros are -1, 2, and 7.

18. The zeros are $-\frac{1}{2}$ and $\frac{3}{4}$.

19. The zeros are $-\frac{2}{3}$ and 5, and the constant term of the polynomial is -10.

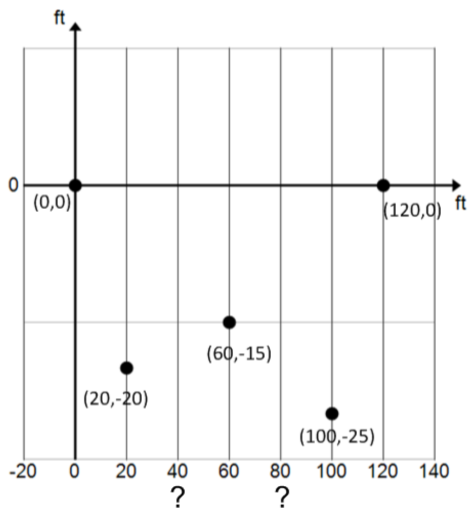
20. The zeros are 2 and $-\frac{3}{2}$, the polynomial has degree 3 and there are no other zeros.

Lesson 20: Modeling Riverbeds with Polynomials

Classwork

Discussion

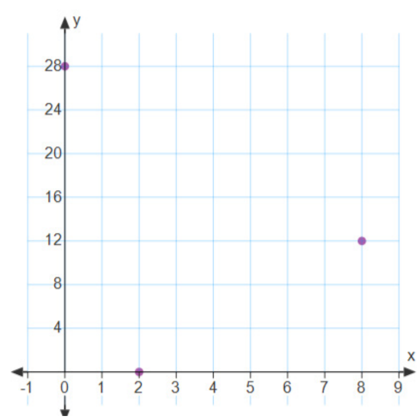
The Environmental Protection Agency (EPA) is studying the flow of a river in order to establish flood zones. The EPA hired a surveying company to determine the flow rate of the river, measured as volume of water per minute. The firm set up a coordinate system and found the depths of the river at five locations as shown on the graph below. After studying the data, the firm decided to model the riverbed with a polynomial function and divide the area into six regions that are either trapezoidal or triangular so that the overall area can be easily estimated. The firm will need to approximate the depth of the river at two more data points in order to do this.



Draw the four trapezoids and two triangles that will be used to estimate the area.

Example 1

Find a polynomial P such that $P(0) = 28$, $P(2) = 0$, and $P(8) = 12$.



Example 2

Find a degree 3 polynomial P such that $P(-1) = -3$, $P(0) = -2$, $P(1) = -1$, and $P(2) = 6$.

Value	Substitute point	Remainder Theorem for $a(x)$, $b(x)$, and $c(x)$	Substitute into $P(x)$
$P(-1) = -3$			
$P(0) = -2$			
$P(1) = -1$			
$P(2) = 6$			

Lesson Summary

A linear polynomial is determined by 2 points on its graph.

A degree 2 polynomial is determined by 3 points on its graph.

A degree 3 polynomial is determined by 4 points on its graph.

A degree 4 polynomial is determined by 5 points on its graph.

The Remainder Theorem can be used to find a polynomial P whose graph will pass through a given set of points.

Problem Set

1. Suppose a polynomial P is such that $P(2) = 5$ and $P(3) = 12$.
   a. What is the largest degree polynomial that can be uniquely determined given the information?
   b. Is this the only polynomial that satisfies $P(2) = 5$ and $P(3) = 12$?
   c. Use the Remainder Theorem to find the polynomial P of least degree that satisfies the two points given.
   d. Verify that your equation is correct by demonstrating that it satisfies the given points.

2. Write a quadratic equation P such that $P(0) = -10$, $P(5) = 0$, and $P(7) = 18$ using the specified method.
   a. Setting up a system of equations
   b. Using the Remainder Theorem

3. Find a degree three polynomial function P such that $P(-1) = 0$, $P(0) = 2$, $P(2) = 12$, and $P(3) = 32$. Use the table below to organize your work. Write your answer in standard form, and verify by showing that each point satisfies the equation.

Value	Substitute point	Remainder Theorem for $a(x)$, $b(x)$, and $c(x)$	Substitute into $P(x)$
$P(-1) = 0$			
$P(0) = 2$			
$P(2) = 12$			
$P(3) = 32$			

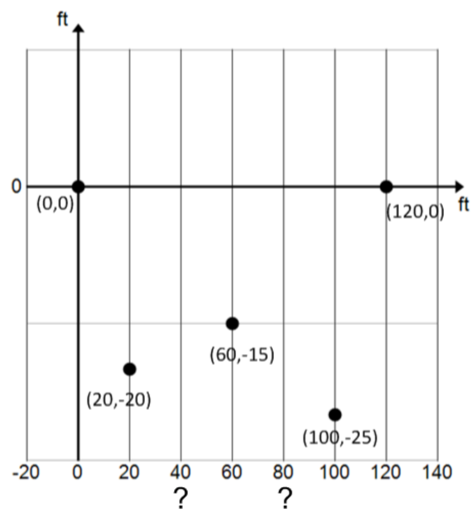
4. The method used in Problem 3 is based on the Lagrange Interpolation method. Research Joseph-Louis Lagrange and write a paragraph about his mathematical work.

Lesson 21: Modeling Riverbeds with Polynomials

Classwork

Mathematical Modeling Exercise

The Environmental Protection Agency (EPA) is studying the flow of a river in order to establish flood zones. The EPA hired a surveying company to determine the flow rate of the river, measured as volume of water per minute. The firm set up a coordinate system and found the depths of the river at five locations as shown on the graph below. After studying the data, the firm decided to model the riverbed with a polynomial function and divide the area into six regions that are either trapezoidal or triangular so that the overall area can be easily estimated. The firm will need to approximate the depth of the river at two more data points in order to do this.



1. Find a polynomial P that fits the five given data points.

2. Use the polynomial to estimate the depth of the river at $x = 40$ and $x = 80$.

3. Estimate the area of the cross section.

4. Suppose that the river flow speed was measured to be an average speed of $176 \frac{\text{ft}}{\text{min}}$ at the cross section. What is the volumetric flow of the water (the volume of water per minute)?

5. Convert the flow to gallons per minute. [Note: 1 cubic foot ≈ 7.48052 gallons.]

Problem Set

1. As the leader of the surveying team, write a short report to the EPA on your findings from the in-class exercises. Be sure to include data and calculations.

2. Suppose that depths of the riverbed were measured for a different cross-section of the river.

   a. Use Wolfram Alpha to find the interpolating polynomial Q with values:

      $Q(0) = 0$, $Q(16.5) = -27.4$, $Q(44.4) = -19.6$, $Q(77.6) = -25.1$,
      $Q(123.3) = -15.0$, $Q(131.1) = -15.1$, $Q(150) = 0$.

   b. Sketch the cross-section of the river and estimate its area.

   c. Suppose that the speed of the water was measured at $124 \frac{\text{ft}}{\text{min}}$. What is the approximate volumetric flow in this section of the river, measured in gallons per minute?

Lesson 22: Equivalent Rational Expressions

Classwork

Opening Exercise

On your own or with a partner, write two fractions that are equivalent to $\frac{1}{3}$, and use the slips of paper to create visual models to justify your response.

Example 1

Consider the following rational expression: $\frac{2(a-1)-2}{6(a-1)-3a}$. Turn to your neighbor and discuss the following: For what values of a is the expression undefined?

Exercise 1

Reduce the following rational expressions to lowest terms, and identify the values of the variable(s) that must be excluded to prevent division by zero.

a. $\frac{2(x+1)+2}{(2x+3)(x+1)-1}$

b. $\frac{x^2-x-6}{5x^2+10x}$

c. $\frac{3-x}{x^2-9}$

d. $\frac{3x-3y}{y^2-2xy+x^2}$

Lesson Summary

- If a, b, and n are integers with $n \neq 0$ and $b \neq 0$, then
$$\frac{na}{nb} = \frac{a}{b}.$$
- The rule for rational expressions is the same as the rule for integers but requires the domain of the rational expression to be restricted (i.e., no value of the variable that can make the denominator of the original rational expression zero is allowed).

Problem Set

1. Find an equivalent rational expression in lowest terms, and identify the value(s) of the variable that must be excluded to prevent division by zero.

   a. $\frac{16n}{20n}$
   b. $\frac{x^3y}{y^4x}$
   c. $\frac{2n-8n^2}{4n}$
   d. $\frac{db+dc}{db}$
   e. $\frac{x^2-9b^2}{x^2-2xb-3b^2}$
   f. $\frac{3n^2-5n-2}{2n-4}$
   g. $\frac{3x-2y}{9x^2-4y^2}$
   h. $\frac{4a^2-12ab}{a^2-6ab+9b^2}$
   i. $\frac{y-x}{x-y}$
   j. $\frac{a^2-b^2}{b+a}$
   k. $\frac{4x-2y}{3y-6x}$
   l. $\frac{9-x^2}{(x-3)^3}$
   m. $\frac{x^2-5x+6}{8-2x-x^2}$
   n. $\frac{a-b}{xa-xb-a+b}$
   o. $\frac{(x+y)^2-9a^2}{2x+2y-6a}$
   p. $\frac{8x^3-y^3}{4x^2-y^2}$

2. Write a rational expression with denominator $6b$ that is equivalent to

   a. $\frac{a}{b}$.
   b. one-half of $\frac{a}{b}$.
   c. $\frac{1}{3}$.

3. Remember that algebra is just another way to perform arithmetic, but with variables replacing numbers.

   a. Simplify the following rational expression: $\frac{\left(x^2y\right)^2(xy)^3z^2}{(xy^2)^2yz}$.
   b. Simplify the following rational expression without using a calculator: $\frac{12^2\cdot 6^3\cdot 5^2}{18^2\cdot 15}$.
   c. How are the calculations in parts (a) and (b) similar? How are they different? Which expression was easier to simplify?

Lesson 23: Comparing Rational Expressions

Classwork

Opening Exercise

Use the slips of paper you have been given to create visual arguments for which fraction is larger.

Exercises 1–5

We will start by working with positive integers. Let m and n be positive integers. Work through the following exercises with a partner.

1. Fill out the following table.

n	$\frac{1}{n}$
1	
2	
3	
4	
5	
6	

2. Do you expect $\frac{1}{n}$ to be bigger or smaller than $\frac{1}{n+1}$? Do you expect $\frac{1}{n}$ to be bigger or smaller than $\frac{1}{n+2}$? Explain why.

3. Compare the rational expressions $\frac{1}{n}$, $\frac{1}{n+1}$, and $\frac{1}{n+2}$ for $n = 1$, 2, and 3. Do your results support your conjecture from Exercise 2? Revise your conjecture if necessary.

4. From your work in Exercises 1 and 2, generalize how $\frac{1}{n}$ compares to $\frac{1}{n+m}$, where m and n are positive integers.

5. Will your conjecture change or stay the same if the numerator is 2 instead of 1? Make a conjecture about what happens when the numerator is held constant, but the denominator increases for positive numbers.

Example 1

x	$\frac{x+1}{x}$	$\frac{x+2}{x+1}$
0.5		
1		
1.5		
2		
5		
10		
100		

Lesson Summary

To compare two rational expressions, find equivalent rational expression with the same denominator. Then we can compare the numerators for values of the variable that do not cause the rational expression to change from positive to negative or vice versa.

We may also use numerical and graphical analysis to help understand the relative sizes of expressions.

Problem Set

1. For parts (a)–(d), rewrite each rational expression as an equivalent rational expression so that each expression has a common denominator.
   a. $\frac{3}{5}, \frac{9}{10}, \frac{7}{15}, \frac{7}{21}$
   b. $\frac{m}{sd}, \frac{s}{dm}, \frac{d}{ms}$
   c. $\frac{1}{(2-x)^2}, \frac{3}{(2x-5)(x-2)}$
   d. $\frac{3}{x-x^2}, \frac{5}{x}, \frac{2x+2}{2x^2-2}$

2. If x is a positive number, for which values of x is $x < \frac{1}{x}$?

3. Can we determine if $\frac{y}{y-1} > \frac{y+1}{y}$ for all values $y > 1$? Provide evidence to support your answer.

4. For positive x, determine when the following rational expressions have negative denominators:
   a. $\frac{3}{5}$
   b. $\frac{x}{5-2x}$
   c. $\frac{x+3}{x^2+4x+8}$
   d. $\frac{3x^2}{(x-5)(x+3)(2x+3)}$

5. Consider the rational expressions $\frac{x}{x-2}$ and $\frac{x}{x-4}$.

   a. Evaluate each expression for $x = 6$.

   b. Evaluate each expression for $x = 3$.

   c. Can you conclude that $\frac{x}{x-2} < \frac{x}{x-4}$ for all positive values of x? Explain how you know.

   d. EXTENSION: Raphael claims that the calculation below shows that $\frac{x}{x-2} < \frac{x}{x-4}$ for all values of x, where $x \neq 2$ and $x \neq 4$. Where is the error in the calculation?

   Starting with the rational expressions $\frac{x}{x-2}$ and $\frac{x}{x-4}$, we need to first find equivalent rational expressions with a common denominator. The common denominator we will use is $(x-4)(x-2)$. We then have

$$\frac{x}{x-2} = \frac{x(x-4)}{(x-4)(x-2)}$$
$$\frac{x}{x-4} = \frac{x(x-2)}{(x-4)(x-2)}.$$

   Since $x^2 - 4x < x^2 - 2x$ for $x > 0$, we can divide each expression by $(x-4)(x-2)$. We then have $\frac{x(x-4)}{(x-4)(x-2)} < \frac{x(x-2)}{(x-4)(x-2)}$, and we can conclude that $\frac{x}{x-2} < \frac{x}{x-4}$ for all positive values of x.

6. Consider the populations of two cities within the same state where the large city's population is P and the small city's population is Q. For each of the following pairs, state which of the expressions has a larger value. Explain your reasoning in the context of the populations.

   a. $P + Q$ and P

   b. $\frac{P}{P+Q}$ and $\frac{Q}{P+Q}$

   c. $2Q$ and $P + Q$

   d. $\frac{P}{Q}$ and $\frac{Q}{P}$

   e. $\frac{P}{P+Q}$ and $\frac{1}{2}$

   f. $\frac{P+Q}{P}$ and $P - Q$

   g. $\frac{P+Q}{2}$ and $\frac{P+Q}{Q}$

   h. $\frac{1}{P}$ and $\frac{1}{Q}$

Lesson 24: Multiplying and Dividing Rational Expressions

Classwork

If a, b, c, and d are rational expressions with $b \neq 0$, $d \neq 0$, then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

Example 1

Make a conjecture about the product $\frac{x^3}{4y} \cdot \frac{y^2}{x}$. What will it be? Explain your conjecture and give evidence that it is correct.

Example 2

Find the following product:

$$\left(\frac{3x-6}{2x+6}\right) \cdot \left(\frac{5x+15}{4x+8}\right).$$

Exercises 1–3

1. Summarize what you have learned so far with your neighbor.

2. Find the following product and reduce to lowest terms: $\left(\frac{2x+6}{x^2+x-6}\right) \cdot \left(\frac{x^2-4}{2x}\right)$.

3. Find the following product and reduce to lowest terms: $\left(\frac{4n-12}{3m+6}\right)^{-2} \cdot \left(\frac{n^2-2n-3}{m^2+4m+4}\right)$.

If a, b, c, and d are rational expressions with $b \neq 0$, $c \neq 0$, and $d \neq 0$, then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}.$$

Example 3

Find the quotient and reduce to lowest terms: $\frac{x^2-4}{3x} \div \frac{x-2}{2x}$.

Exercises 4–5

4. Find the quotient and reduce to lowest terms: $\frac{x^2-5x+6}{x+4} \div \frac{x^2-9}{x^2+5x+4}$.

5. Simplify the rational expression.

$$\frac{\left(\frac{x+2}{x^2-2x-3}\right)}{\left(\frac{x^2-x-6}{x^2+6x+5}\right)}$$

Lesson Summary

In this lesson we extended multiplication and division of rational numbers to multiplication and division of rational expressions.

- To multiply two rational expressions, multiply the numerators together and multiply the denominators together, and then reduce to lowest terms.
- To divide one rational expression by another, multiply the first by the multiplicative inverse of the second, and reduce to lowest terms.
- To simplify a complex fraction, apply the process for dividing one rational expression by another.

Problem Set

1. Complete the following operations:

 a. Multiply $\frac{1}{3}(x-2)$ by 9.
 b. Divide $\frac{1}{4}(x-8)$ by $\frac{1}{12}$.
 c. Multiply $\frac{1}{4}\left(\frac{1}{3}x+2\right)$ by 12.
 d. Divide $\frac{1}{3}\left(\frac{2}{5}x-\frac{1}{5}\right)$ by $\frac{1}{15}$.
 e. Multiply $\frac{2}{3}\left(2x+\frac{2}{3}\right)$ by $\frac{9}{4}$.
 f. Multiply $0.03(4-x)$ by 100.

2. Simplify each of the following expressions.

 a. $\left(\frac{a^3b^2}{c^2d^2}\cdot\frac{c}{ab}\right)\div\frac{a}{c^2d^3}$
 b. $\frac{a^2+6a+9}{a^2-9}\cdot\frac{3a-9}{a+3}$
 c. $\frac{6x}{4x-16}\div\frac{4x}{x^2-16}$
 d. $\frac{3x^2-6x}{3x+1}\cdot\frac{x+3x^2}{x^2-4x+4}$
 e. $\frac{2x^2-10x+12}{x^2-4}\cdot\frac{2+x}{3-x}$
 f. $\frac{a-2b}{a+2b}\div(4b^2-a^2)$
 g. $\frac{d+c}{c^2+d^2}\div\frac{c^2-d^2}{d^2-dc}$
 h. $\frac{12a^2-7ab+b^2}{9a^2-b^2}\div\frac{16a^2-b^2}{3ab+b^2}$
 i. $\left(\frac{x-3}{x^2-4}\right)^{-1}\cdot\left(\frac{x^2-x-6}{x-2}\right)$
 j. $\left(\frac{x-2}{x^2+1}\right)^{-3}\div\left(\frac{x^2-4x+4}{x^2-2x-3}\right)$
 k. $\frac{6x^2-11x-10}{6x^2-5x-6}\cdot\frac{6-4x}{25-20x+4x^2}$
 l. $\frac{3x^3-3a^2x}{x^2-2ax+a^2}\cdot\frac{a-x}{a^3x+a^2x^2}$

3. Simplify the following complex rational expressions.

   a. $\dfrac{\left(\frac{4a}{6b^2}\right)}{\left(\frac{20a^3}{12b}\right)}$

   b. $\dfrac{\left(\frac{x-2}{x^2-1}\right)}{\left(\frac{x^2-4}{x-6}\right)}$

   c. $\dfrac{\left(\frac{x^2+2x-3}{x^2+3x-4}\right)}{\left(\frac{x^2+x-6}{x+4}\right)}$

4. Suppose that $x = \frac{t^2+3t-4}{3t^2-3}$ and $y = \frac{t^2+2t-8}{2t^2-2t-4}$, for $t \neq 1$, $t \neq -1$, $t \neq 2$, and $t \neq -4$. Show that the value of x^2y^{-2} does not depend on the value of t.

5. Determine which of the following numbers is larger without using a calculator, $\frac{15^{16}}{16^{15}}$ or $\frac{20^{24}}{24^{20}}$.

   (Hint: We can compare two positive quantities a and b by computing the quotient $\frac{a}{b}$. If $\frac{a}{b} > 1$, then $a > b$. Likewise, if $0 < \frac{a}{b} < 1$, then $a < b$.)

6. One of two numbers can be represented by the rational expression $\frac{x-2}{x}$, where $x \neq 0$ and $x \neq 2$.

   a. Find a representation of the second number if the product of the two numbers is 1.
   b. Find a representation of the second number if the product of the two numbers is 0.

Lesson 25: Adding and Subtracting Rational Expressions

Classwork

Exercises 1–4

1. Calculate the following sum: $\frac{3}{10} + \frac{6}{10}$.

2. $\frac{3}{20} - \frac{4}{15}$

3. $\frac{\pi}{4} + \frac{\sqrt{2}}{5}$

4. $\frac{a}{m} + \frac{b}{2m} - \frac{c}{m}$

Example 1

Perform the indicated operations below and simplify.

a. $\frac{a+b}{4}+\frac{2a-b}{5}$

b. $\frac{4}{3x}-\frac{3}{5x^2}$

c. $\frac{3}{2x^2+2x}+\frac{5}{x^2-3x-4}$

Exercises 5–8

Perform the indicated operations for each problem below.

5. $\frac{5}{x-2}+\frac{3x}{4x-8}$

6. $\frac{7m}{m-3}+\frac{5m}{3-m}$

7. $\frac{b^2}{b^2-2bc+c^2}-\frac{b}{b-c}$

8. $\frac{x}{x^2-1}-\frac{2x}{x^2+x-2}$

Example 2

Simplify the following expression.

$$\frac{\frac{b^2+b-1}{2b-1}-1}{4-\frac{8}{(b+1)}}$$

Lesson Summary

In this lesson, we extended addition and subtraction of rational numbers to addition and subtraction of rational expressions. The process for adding or subtracting rational expressions can be summarized as follows:

- Find a common multiple of the denominators to use as a common denominator.
- Find equivalent rational expressions for each expression using the common denominator.
- Add or subtract the numerators as indicated and simplify if needed.

Problem Set

1. Write each sum or difference as a single rational expression.

   a. $\frac{7}{8} - \frac{\sqrt{3}}{5}$

   b. $\frac{\sqrt{5}}{10} + \frac{\sqrt{2}}{6} + 2$

   c. $\frac{4}{x} + \frac{3}{2x}$

2. Write as a single rational expression.

   a. $\frac{1}{x} - \frac{1}{x-1}$

   b. $\frac{3x}{2y} - \frac{5x}{6y} + \frac{x}{3y}$

   c. $\frac{a-b}{a^2} + \frac{1}{a}$

   d. $\frac{1}{p-2} - \frac{1}{p+2}$

   e. $\frac{1}{p-2} + \frac{1}{2-p}$

   f. $\frac{1}{b+1} - \frac{b}{1+b}$

   g. $1 - \frac{1}{1+p}$

   h. $\frac{p+q}{p-q} - 2$

   i. $\frac{r}{s-r} + \frac{s}{r+s}$

   j. $\frac{3}{x-4} + \frac{2}{4-x}$

   k. $\frac{3n}{n-2} + \frac{3}{2-n}$

   l. $\frac{8x}{3y-2x} + \frac{12y}{2x-3y}$

   m. $\frac{1}{2m-4n} - \frac{1}{2m+4n} - \frac{m}{m^2-4n^2}$

   n. $\frac{1}{(2a-b)(a-c)} + \frac{1}{(b-c)(b-2a)}$

   o. $\frac{b^2+1}{b^2-4} + \frac{1}{b+2} + \frac{1}{b-2}$

3. Simplify the following expressions.

   a. $\dfrac{\frac{1}{a} - \frac{1}{2a}}{\frac{4}{a}}$

   b. $\dfrac{\frac{5x}{2} + 1}{\frac{5x}{4} - \frac{1}{5x}}$

   c. $\dfrac{1 + \frac{4x+3}{x^2+1}}{1 - \frac{x+7}{x^2+1}}$

EXTENSION:

4. Suppose that $x \neq 0$ and $y \neq 0$. We know from our work in this section that $\frac{1}{x} \cdot \frac{1}{y}$ is equivalent to $\frac{1}{xy}$. Is it also true that $\frac{1}{x} + \frac{1}{y}$ is equivalent to $\frac{1}{x+y}$? Provide evidence to support your answer.

5. Suppose that $x = \frac{2t}{1+t^2}$ and $y = \frac{1-t^2}{1+t^2}$. Show that the value of $x^2 + y^2$ does not depend on the value of t.

6. Show that for any real numbers a and b, and any integers x and y so that $x \neq 0$, $y \neq 0$, $x \neq y$, and $x \neq -y$,

$$\left(\frac{y}{x} - \frac{x}{y}\right)\left(\frac{ax+by}{x+y} - \frac{ax-by}{x-y}\right) = 2(a-b).$$

7. Suppose that n is a positive integer.

   a. Simplify the expression $\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right)$.

   b. Simplify the expression $\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right)\left(1+\frac{1}{n+2}\right)$.

   c. Simplify the expression $\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right)\left(1+\frac{1}{n+2}\right)\left(1+\frac{1}{n+3}\right)$.

   d. If this pattern continues, what is the product of n of these factors?

Lesson 26: Solving Rational Equations

Classwork

Exercises 1–2

Solve the following equations for x, and give evidence that your solutions are correct.

1. $\frac{x}{2}+\frac{1}{3}=\frac{5}{6}$

2. $\frac{2x}{9}+\frac{5}{9}=\frac{8}{9}$.

Example 1

Solve the following equation: $\frac{x+3}{12}=\frac{5}{6}$.

Exercises 3–7

3. Solve the following equation: $\frac{3}{x} = \frac{8}{x-2}$.

4. Solve the following equation for a: $\frac{1}{a+2} + \frac{1}{a-2} = \frac{4}{a^2-4}$.

5. Solve the following equation. Remember to check for extraneous solutions.

$$\frac{4}{3x} + \frac{5}{4} = \frac{3}{x}$$

6. Solve the following equation. Remember to check for extraneous solutions.

$$\frac{7}{b+3}+\frac{5}{b-3}=\frac{10b-2}{b^2-9}$$

7. Solve the following equation. Remember to check for extraneous solutions.

$$\frac{1}{x-6}+\frac{x}{x-2}=\frac{4}{x^2-8x+12}$$

Lesson Summary

In this lesson, we applied what we have learned in the past two lessons about addition, subtraction, multiplication, and division of rational expressions to solve rational equations. An extraneous solution is a solution to a transformed equation that is not a solution to the original equation. For rational functions, extraneous solutions come from the excluded values of the variable.

Rational equations can be solved one of two ways:

1. Write each side of the equation as an equivalent rational expression with the same denominator and equate the numerators. Solve the resulting polynomial equation, and check for extraneous solutions.
2. Multiply both sides of the equation by an expression that is the common denominator of all terms in the equation. Solve the resulting polynomial equation, and check for extraneous solutions.

Problem Set

1. Solve the following equations and check for extraneous solutions.

a. $\frac{x-8}{x-4} = 2$

b. $\frac{4x-8}{x-2} = 4$

c. $\frac{x-4}{x-3} = 1$

d. $\frac{4x-8}{x-2} = 3$

e. $\frac{1}{2a} - \frac{2}{2a-3} = 0$

f. $\frac{3}{2x+1} = \frac{5}{4x+3}$

g. $\frac{4}{x-5} - \frac{2}{5+x} = \frac{2}{x}$

h. $\frac{y+2}{3y-2} + \frac{y}{y-1} = \frac{2}{3}$

i. $\frac{3}{x+1} - \frac{2}{1-x} = 1$

j. $\frac{4}{x-1} + \frac{3}{x} - 3 = 0$

k. $\frac{x+1}{x+3} - \frac{x-5}{x+2} = \frac{17}{6}$

l. $\frac{x+7}{4} - \frac{x+1}{2} = \frac{5-x}{3x-14}$

m. $\frac{b^2-b-6}{b^2} - \frac{2b+12}{b} = \frac{b-39}{2b}$

n. $\frac{1}{p(p-4)} + 1 = \frac{p-6}{p}$

o. $\frac{1}{h+3} = \frac{h+4}{h-2} + \frac{6}{h-2}$

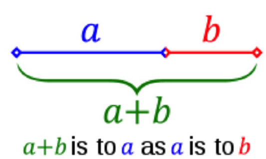
p. $\frac{m+5}{m^2+m} = \frac{1}{m^2+m} - \frac{m-6}{m+1}$

2. Create and solve a rational equation that has 0 as an extraneous solution.

3. Create and solve a rational equation that has 2 as an extraneous solution.

EXTENSION:

4. Two lengths a and b, where $a > b$, are in *golden ratio* if the ratio of $a + b$ is to a is the same as a is to b. Symbolically, this is expressed as $\frac{a}{b} = \frac{a+b}{a}$. We denote this common ratio by the Greek letter *phi (pronounced "fee")* with symbol φ, so that if a and b are in common ratio, then $\varphi = \frac{a}{b} = \frac{a+b}{a}$. By setting $b = 1$, we find that $\varphi = a$ and φ is the positive number that satisfies the equation $\varphi = \frac{\varphi+1}{\varphi}$. Solve this equation to find the numerical value for φ.



5. Remember that if we use x to represent an integer, then the next integer can be represented by $x + 1$.

   a. Does there exist a pair of consecutive integers whose reciprocals sum to $\frac{5}{6}$? Explain how you know.

   b. Does there exist a pair of consecutive integers whose reciprocals sum to $\frac{3}{4}$? Explain how you know.

   c. Does there exist a pair of consecutive even integers whose reciprocals sum to $\frac{3}{4}$? Explain how you know.

   d. Does there exist a pair of consecutive even integers whose reciprocals sum to $\frac{5}{6}$? Explain how you know.

Lesson 27: Word Problems Leading to Rational Equations

Classwork

Exercise 1

1. Anne and Maria play tennis almost about every weekend. So far, Anne has won 12 out of 20 matches.
   a. How many matches will Anne have to win in a row to improve her winning percentage to 75%?

   b. How many matches will Anne have to win in a row to improve her winning percentage to 90%?

   c. Can Anne reach a winning percentage of 100%?

   d. After Anne has reached a winning percentage of 90% by winning consecutive matches as in part (b), how many matches can she now lose in a row to have a winning percentage of 50%?

Example 1

Working together, it takes Sam, Jenna, and Francisco two hours to paint one room. When Sam is working alone, he can paint one room in 6 hours. When Jenna works alone, she can paint one room in 4 hours. Determine how long it would take Francisco to paint one room on his own.

Exercises 2–4

2. Melissa walks 3 miles to the house of a friend and returns home on a bike. She averages 4 miles per hour faster when cycling than when walking, and the total time for both trips is two hours. Find her walking speed.

3. You have 10 liters of a juice blend that is 60% juice.
   a. How many liters of pure juice need to be added in order to make a blend that is 75% juice?

b. How many liters of pure juice need to be added in order to make a blend that is 90% juice?

c. Write a rational equation that relates the desired percentage p to the amount A of pure juice that needs to be added to make a blend that is $p\%$ juice, where $0 < p < 100$. What is a reasonable restriction on the set of possible values of p? Explain your answer.

d. Suppose that you have added 15 liters of juice to the original 10 liters. What is the percentage of juice in this blend?

e. Solve your equation in part (c) for the amount A. Are there any excluded values of the variable p? Does this make sense in the context of the problem?

4. You have a solution containing 10% acid and a solution containing 30% acid.
   a. How much of the 30% solution must you add to 1 liter of the 10% solution to create a mixture that is 22% acid?

   b. Write a rational equation that relates the desired percentage p to the amount A of 30% acid solution that needs to be added to 1 liter of 10% acid solution to make a blend that is $p\%$ acid, where $0 < p < 100$. What is a reasonable restriction on the set of possible values of p? Explain your answer.

   c. Solve your equation in part (b) for A. Are there any excluded values of p? Does this make sense in the context of the problem?

   d. If you have added some 30% acid solution to 1 liter of 10% acid solution to make a 26% acid solution, how much of the stronger acid did you add?

Lesson Summary

In this lesson, we developed the students' problem solving skills by asking them to carefully read a problem, rephrase it in a form comfortable for their own understanding, and convert fact sentences about unknown quantities into algebraic equations. Specifically, they used rational equations to model and solve some application problems and further developed their skills in working with rational expressions.

Problem Set

1. If 2 inlet pipes can fill a pool in one hour and 30 minutes, and one pipe can fill the pool in two hours and 30 minutes on its own, how long would the other pipe take to fill the pool on its own?

2. If one inlet pipe can fill the pool in 2 hours with the outlet drain closed, and the same inlet pipe can fill the pool in 2.5 hours with the drain open, how long does it take the drain to empty the pool if there is no water entering the pool?

3. It takes 36 minutes less time to travel 120 miles by car at night than by day because the lack of traffic allows the average speed at night to be 10 miles per hour faster than in the daytime. Find the average speed in the daytime.

4. The difference in the average speed of two trains is 16 miles per hour. The slower train takes 2 hours longer to travel 170 miles than the faster train takes to travel 150 miles. Find the speed of the slower train.

5. A school library spends $\$80$ a month on magazines. The average price for magazines bought in January was 70 cents more than the average price in December. Because of the price increase, the school library was forced to subscribe to 7 fewer magazines. How many magazines did the school library subscribe to in December?

6. An investor bought a number of shares of stock for $\$1{,}600$. After the price dropped by $\$10$ per share, the investor sold all but 4 of her shares for $\$1{,}120$. How many shares did she originally buy?

7. Newton's law of universal gravitation, $F = \frac{Gm_1m_2}{r^2}$, measures the force of gravity between two masses m_1 and m_2, where r is the distance between the centers of the masses, and G is universal gravitational constant. Solve this equation for G.

8. Suppose that $t = \frac{x+y}{1-xy}$.
   a. Show that when $x = \frac{1}{a}$ and $y = \frac{2a-1}{a+2}$, the value of t does not depend on the value of a.
   b. For which values of a do these relationships have no meaning?

9. Consider the rational equation $\frac{1}{R} = \frac{1}{x} + \frac{1}{y}$.
   a. Find the value of R when $x = \frac{2}{5}$ and $y = \frac{3}{4}$.
   b. Solve this equation for R and simplify.

10. Consider an ecosystem of rabbits in a park that starts with 10 rabbits and can sustain up to 60 rabbits. An equation that roughly models this scenario is

$$P = \frac{60}{1 + \frac{5}{t+1}},$$

where P represents the rabbit population in year t of the study.
   a. What is the rabbit population in year 10? Round your answer to the nearest whole rabbit.
   b. Solve this equation for t. Describe what this equation represents in the context of this problem.
   c. At what time does the population reach 50 rabbits?

11. Suppose that Huck Finn can paint a fence in 5 hours. If Tom Sawyer helps him pain the fence, they can do it in 3 hours. How long would it take for Tom to paint the fence by himself?

12. Huck Finn can paint a fence in 5 hours. After some practice, Tom Sawyer can now paint the fence in 6 hours.
   a. How long would it take Huck and Tom to paint the fence together?
   b. Tom demands a half hour break while Huck continues to pain, and they finish the job together. How long does it take them to paint the fence?
   c. Suppose that they have to finish the fence in $3\frac{1}{2}$ hours. What's the longest break that Tom can take?

Lesson 28: A Focus on Square Roots

Classwork

Exercises 1–4

For Exercises 1–4, describe each step taken to solve the equation. Then, check the solution to see if it is valid. If it is not a valid solution, explain why.

1. $\sqrt{x} - 6 = 4$
   $\sqrt{x} = 10$
   $x = 100$

2. $\sqrt[3]{x} - 6 = 4$
   $\sqrt[3]{x} = 10$
   $x = 1000$

3. $\sqrt{x} + 6 = 4$

4. $\sqrt[3]{x} + 6 = 4$

Example 1

Solve the radical equation. Be sure to check your solutions.

$$\sqrt{3x+5} - 2 = -1$$

Exercises 5–15

Solve each radical equation. Be sure to check your solutions.

5. $\sqrt{2x-3}=11$

6. $\sqrt[3]{6-x}=-3$

7. $\sqrt{x+5}-9=-12$

8. $\sqrt{4x-7}=\sqrt{3x+9}$

9. $-12\sqrt{x-6}=18$

10. $3\sqrt[3]{x+2}=12$

11. $\sqrt{x^2-5}=2$

12. $\sqrt{x^2+8x}=3$

Multiply each expression.

13. $(\sqrt{x}+2)(\sqrt{x}-2)$

14. $(\sqrt{x}+4)(\sqrt{x}+4)$

15. $(\sqrt{x-5})(\sqrt{x-5})$

Example 2

Rationalize the denominator in each expression. That is, rewrite each expression so that the fraction has a rational expression in the denominator.

a. $\frac{x-9}{\sqrt{x-9}}$

b. $\frac{x-9}{\sqrt{x}+3}$

Exercises 16–18

16. Rewrite $\frac{1}{\sqrt{x}-5}$ in an equivalent form with a rational expression in the denominator.

17. Solve the radical equation $\frac{3}{\sqrt{x+3}} = 1$. Be sure to check for extraneous solutions.

18. Without solving the radical equation $\sqrt{x+5} + 9 = 0$, how could you tell that it has no real solution?

Problem Set

1. a. If $\sqrt{x} = 9$, then what is the value of x?
   b. If $x^2 = 9$, then what is the value of x?
   c. Is there a value of x such that $\sqrt{x+5} = 0$? If yes, what is the value? If no, explain why not.
   d. Is there a value of x such that $\sqrt{x} + 5 = 0$? If yes, what is the value? If no, explain why not.

2. a. Is the statement $\sqrt{x^2} = x$ true for all x-values? Explain.
   b. Is the statement $\sqrt[3]{x^3} = x$ true for all x-values? Explain.

Rationalize the denominator in each expression.

3. $\frac{4-x}{2+\sqrt{x}}$

4. $\frac{2}{\sqrt{x-12}}$

5. $\frac{1}{\sqrt{x+3}-\sqrt{x}}$

Solve each equation and check the solutions.

6. $\sqrt{x+6} = 3$

7. $2\sqrt{x+3} = 6$

8. $\sqrt{x+3} + 6 = 3$

9. $\sqrt{x+3} - 6 = 3$

10. $16 = 8 + \sqrt{x}$

11. $\sqrt{3x-5} = 7$

12. $\sqrt{2x-3} = \sqrt{10-x}$

13. $3\sqrt{x+2} + \sqrt{x-4} = 0$

14. $\frac{\sqrt{x+9}}{4} = 3$

15. $\frac{12}{\sqrt{x+9}} = 3$

16. $\sqrt{x^2+9} = 5$

17. $\sqrt{x^2-6x} = 4$

18. $\frac{5}{\sqrt{x-2}} = 5$

19. $\frac{5}{\sqrt{x}-2} = 5$

20. $\sqrt[3]{5x-3} + 8 = 6$

21. $\sqrt[3]{9-x} = 6$

22. Consider the inequality $\sqrt{x^2+4x} > 0$. Determine whether each x-value is a solution to the inequality.
    a. $x = -10$ b. $x = -4$ c. $x = 10$ d. $x = 4$

23. Show that $\frac{a-b}{\sqrt{a}-\sqrt{b}} = \sqrt{a} + \sqrt{b}$ for all values of a and b such that $a > 0$ and $b > 0$ and $a \neq b$.

24. Without actually solving the equation, explain why the equation $\sqrt{x+1} + 2 = 0$ has no solution.

Lesson 29: Solving Radical Equations

Classwork

Example 1

Solve the equation $6 = x + \sqrt{x}$.

Exercises 1–4

Solve.

1. $3x = 1 + 2\sqrt{x}$

2. $3 = 4\sqrt{x} - x$

3. $\sqrt{x+5} = x - 1$

4. $\sqrt{3x+7} + 2\sqrt{x-8} = 0$

Example 2

Solve the equation $\sqrt{x} + \sqrt{x+3} = 3$

Exercises 5–6

Solve the following equations.

5. $\sqrt{x-3}+\sqrt{x+5}=4$

6. $3+\sqrt{x}=\sqrt{x+81}$

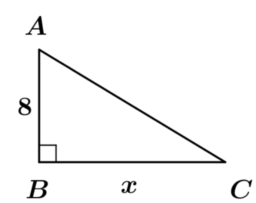
Lesson Summary

If $a = b$ and n is an integer, then $a^n = b^n$. However, the converse is not necessarily true. The statement $a^n = b^n$ does not imply that $a = b$. Therefore, it is necessary to check for extraneous solutions when both sides of an equation are raised to an exponent.

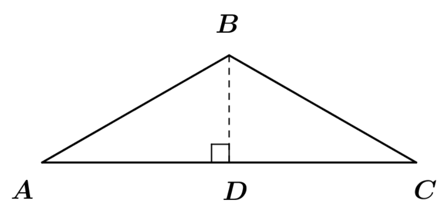
Problem Set

Solve.

1. $\sqrt{2x-5} - \sqrt{x+6} = 0$
2. $\sqrt{2x-5} + \sqrt{x+6} = 0$
3. $\sqrt{x-5} - \sqrt{x+6} = 2$
4. $\sqrt{2x-5} - \sqrt{x+6} = 2$
5. $\sqrt{x+4} = 3 - \sqrt{x}$
6. $\sqrt{x+4} = 3 + \sqrt{x}$
7. $\sqrt{x+3} = \sqrt{5x+6} - 3$
8. $\sqrt{2x+1} = x - 1$
9. $\sqrt{x+12} + \sqrt{x} = 6$
10. $2\sqrt{x} = 1 - \sqrt{4x-1}$
11. $2x = \sqrt{4x-1}$
12. $\sqrt{4x-1} = 2 - 2x$
13. $x + 2 = 4\sqrt{x-2}$
14. $\sqrt{2x-8} + \sqrt{3x-12} = 0$
15. $x = 2\sqrt{x-4} + 4$
16. $x - 2 = \sqrt{9x-36}$
17. Consider the right triangle ABC shown to the right, with $AB = 8$ and $BC = x$.
    a. Write an expression for the length of the hypotenuse in terms of x.
    b. Find the value of x for which $AC - AB = 9$.



18. Consider the right triangle ABC shown to the right, where $AD = DC$ and $\overline{BD}$ is the altitude of the triangle.
    a. If the length of $\overline{BD}$ is x cm and the length of $\overline{AC}$ is 18 cm, write an expression for the lengths of $\overline{AB}$ and $\overline{BC}$ in terms of x.
    b. Write an expression for the perimeter of ΔABC in terms of x.
    c. Find the value of x for which the perimeter of ΔABC is equal to 38 cm.



Lesson 30: Linear Systems in Three Variables

Classwork

Exercises 1–3

Determine the value of x and y in the following systems of equations.

1. $2x + 3y = 7$
   $2x + y = 3$

2. $5x - 2y = 4$
   $-2x + y = 2$

3. A scientist wants to create 120 ml of a solution that is 30% acidic. To create this solution, she has access to a 20% solution and a 45% solution. How many milliliters of each solution should she combine to create the 30% solution?

Example 1

Determine the values for x, y, and z in the following system.

$$2x + 3y - z = 5$$
$$4x - y - z = -1$$
$$x + 4y + z = 12$$

Exercises 4–5

4. Given the system below, determine the values of r, s, and u that satisfy all three equations.

$$\begin{aligned} r + 2x - u &= 8 \\ s + u &= 4 \\ r - s - u &= 2 \end{aligned}$$

5. Find the equation of the form $y = ax^2 + bx + c$ that satisfies the points $(1, 6)$, $(3, 20)$, and $(-2, 15)$.

Problem Set

Solve the following systems.

1. $$\begin{aligned} x + y &= 3 \\ y + z &= 6 \\ x + z &= 5 \end{aligned}$$

2. $$\begin{aligned} r &= 2(s - t) \\ 2t &= 3(s - r) \\ r + t &= 2s - 3 \end{aligned}$$

3. $$\begin{aligned} 2a + 4b + c &= 5 \\ a - 4b &= -6 \\ 2b + c &= 7 \end{aligned}$$

4. $$\begin{aligned} 2x + y - z &= -5 \\ 4x - 2y + z &= 10 \\ 2x + 3y + 2z &= 3 \end{aligned}$$

5. $$\begin{aligned} r + 3s + t &= 3 \\ 2r - 3s + 2t &= 3 \\ -r + 3s - 3t &= 1 \end{aligned}$$

6. $$\begin{aligned} x - y &= 1 \\ 2y + z &= -4 \\ x - 2z &= -6 \end{aligned}$$

7. $$\begin{aligned} x &= 3(y - z) \\ y &= 5(z - x) \\ x + y &= z + 4 \end{aligned}$$

8. $$\begin{aligned} p + q + 3r &= 4 \\ 2q + 3r &= 7 \\ p - q - r &= -2 \end{aligned}$$

9. $$\begin{aligned} \frac{1}{x} + \frac{1}{y} + \frac{1}{z} &= 5 \\ \frac{1}{x} + \frac{1}{y} &= 2 \\ \frac{1}{x} - \frac{1}{z} &= -2 \end{aligned}$$

10. $$\begin{aligned} \frac{1}{a} + \frac{1}{b} + \frac{1}{c} &= 6 \\ \frac{1}{b} + \frac{1}{c} &= 5 \\ \frac{1}{a} - \frac{1}{b} &= -1 \end{aligned}$$

11. Find the equation of the form $y = ax^2 + bx + c$ whose graph passes through the points $(1, -1)$, $(3, 23)$, and $(-1, 7)$.

12. Show that for any number t, the values $x = t + 2$, $y = 1 - t$, and $z = t + 1$ are solutions to the system of equations below.

$$\begin{aligned} x + y &= 3 \\ y + z &= 2 \end{aligned}$$

(In this situation, we say that t *parameterizes the solution set of the system.*)

13. Some rational expressions can be written as the sum of two or more rational expressions whose denominators are the factors of its denominator (called a *partial fraction decomposition*). Find the partial fraction decomposition for the following example by filling in the blank to make the equation true for all n except 0 and -1.

$$\frac{1}{n(n+1)} = \frac{\_\_\_\_\_\_}{n} - \frac{1}{n+1}$$

14. A chemist needs to make 30 ml of a 15% acid solution. He has a 5% acid solution and a 30% acid solution on hand. If he uses the 5% and 30% solutions to create the 15% solution, how many ml of each will he need?

15. An airplane makes a 400 mile trip against a head wind in 4 hours. The return trip takes 2.5 hours, the wind now being a tall wind. If the plane maintains a constant speed with respect to still air, and the speed of the wind is also constant and does not vary, find the still-air speed of the plane and the speed of the wind.

16. A restaurant owner estimates that she needs in small change the same number of dimes as pennies and nickels together and the same number of pennies as nickels. If she gets $\$26$ worth of pennies, nickels, and dimes, how should they be distributed?

Lesson 31: Systems of Equations

Classwork

Exploratory Challenge 1

a. Sketch the lines given by $x + y = 6$ and $-3x + y = 2$ on the same set of axes, and then solve the pair of equations algebraically to verify your graphical solution.

b. Suppose the second line is replaced by the line with equation $x + y = 2$. Plot the two lines on the same set of axes, and solve the pair of equations algebraically to verify your graphical solution.

c. Suppose the second line is replaced by the line with equation $2x = 12 - 2y$. Plot the lines on the same set of axes, and solve the pair of equations algebraically to verify your graphical solution.

d. We have seen that a pair of lines can intersect in 1, 0, or an infinite number of points. Are there any other possibilities?

Exploratory Challenge 2

a. Suppose that instead of equations for a pair of lines, you were given an equation for a circle and an equation for a line. What possibilities are there for the two figures to intersect? Sketch a graph for each possibility.

b. Graph the parabola with equation $y = x^2$. What possibilities are there for a line to intersect the parabola? Sketch each possibility.

c. Sketch the circle given by $x^2 + y^2 = 1$ and the line given by $y = 2x + 2$ on the same set of axes. One solution to the pair of equations has a value of y that is easily identifiable from the sketch. What is it?

d. Solve $x^2 + (2x + 2)^2 = 1$.

Exercises 1–6

1. Draw a graph of the circle with equation $x^2 + y^2 = 9$.

   a. What are the solutions to the system of circle and line when the circle is given by $x^2 + y^2 = 9$ and the line is given by $y = 2$?

   b. What happens when the line is given by $y = 3$?

   c. What happens when the line is given by $y = 4$?

2. By solving the equations as a system, find the points common to the line with equation $x - y = 6$ and the circle with equation $x^2 + y^2 = 26$. Graph the line and the circle to show those points.

3. Graph the line given by $5x + 6y = 12$ and the circle given by $x^2 + y^2 = 1$. Find all solutions to the system of equations.

4. Graph the line given by $3x + 4y = 25$ and the circle given by $x^2 + y^2 = 25$. Find all solutions to the system of equations. Verify your result both algebraically and graphically.

5. Graph the line given by $2x + y = 1$ and the circle given by $x^2 + y^2 = 10$. Find all solutions to the system of equations. Verify your result both algebraically and graphically.

6. Graph the line given by $x + y = -2$ and the quadratic curve given by $y = x^2 - 4$. Find all solutions to the system of equations. Verify your result both algebraically and graphically.

Lesson Summary

Here are some steps to consider when solving systems of equations that represent a line and a quadratic curve.

1. Solve the linear equation for y in terms of x. This is equivalent to rewriting the equation in slope-intercept form. Note that working with the quadratic equation first would likely be more difficult and might cause the loss of a solution.
2. Replace y in the quadratic equation with the expression involving x from the slope-intercept form of the linear equation. That will yield an equation in one variable.
3. Solve the quadratic equation for x.
4. Substitute x into the linear equation to find the corresponding value of y.
5. Sketch a graph of the system to check your solution.

Problem Set

1. Where do the lines given by $y = x + b$ and $y = 2x + 1$ intersect?

2. Find all solutions to the following system of equations.
$$(x-2)^2 + (y+3)^2 = 4$$
$$x - y = 3$$
Illustrate with a graph.

3. Find all solutions to the following system of equations.
$$x + 2y = 0$$
$$x^2 - 2x + y^2 - 2y - 3 = 0$$
Illustrate with a graph.

4. Find all solutions to the following system of equations.
$$x + y = 4$$
$$(x+3)^2 + (y-2)^2 = 10$$
Illustrate with a graph.

5. Find all solutions to the following system of equations.
$$y = -2x + 3$$
$$y = x^2 - 6x + 3$$
Illustrate with a graph.

6. Find all solutions to the following system of equations.
$$-y^2 + 6y + x - 9 = 0$$
$$6y = x + 27$$
Illustrate with a graph.

7. If the following system of equations has two solutions, what is the value of k?

$$x^2 + y^2 = 25$$
$$y = k$$

Illustrate with a graph.

8. If the following system of equations has exactly one solution, what is the value of k?

$$y = 5 - (x - 3)^2$$
$$y = k$$

Illustrate with a graph.

9. If the following system of equations no solutions, what is the value of k?

$$x^2 + (y - k)^2 = 36$$
$$y = 5x + k$$

Illustrate with a graph.

Lesson 32: Graphing Systems of Equations

Classwork

Opening Exercise

Given the line $y = 2x$, is there a point on the line at a distance 3 from $(1, 3)$? Explain how you know.

Draw a graph showing where the point is.

Exercise 1

Solve the system $(x-1)^2 + (y-2)^2 = 2^2$ and $y = 2x + 2$.

What are the coordinates of the center of the circle?

What can you say about the distance from the intersection points to the center of the circle?

Using your graphing tool, graph the line and the circle.

Example 1

Rewrite $x^2 + y^2 - 4x + 2y = -1$ by completing the square in both x and y. Describe the circle represented by this equation.

Using your graphing tool, graph the circle.

In contrast, consider the following equation: $x^2 + y^2 - 2x - 8y = -19$

What happens when you use your graphing tool with this equation?

Exercise 2

Consider a circle with radius 5 and another circle with radius 3. Let d represent the distance between the two centers. We want to know how many intersections there are of these two circles for different values of d. Draw figures for each case.

a. What happens if $d = 8$?

b. What happens if $d = 10$?

c. What happens if $d = 1$?

d. What happens if $d = 2$?

e. For which values of d do the circles intersect in exactly one point. Generalize this result to circles of any radius.

f. For which values of d do the circles intersect in two points? Generalize this result to circles of any radius.

g. For which values of d do the circles not intersect? Generalize this result to circles of any radius

Example 2

Find the distance between the centers of the two circles with equations below, and use that distance to determine in how many points these circles intersect.

$$x^2 + y^2 = 5$$
$$(x-2)^2 + (y-1)^2 = 3$$

Exercise 3

Use the distance formula to show algebraically and graphically that the following two circles do not intersect.

$$(x-1)^2 + (y+2)^2 = 1$$
$$(x+5)^2 + (y-4)^2 = 4$$

Example 3

Point $A\ (3, 2)$ is on a circle whose center is $C\ (-2, 3)$. What is the radius of the circle?

What is the equation of the circle? Graph it.

Use the fact that the tangent at $A\ (3, 2)$ is perpendicular to the radius at that point to find the equation of the tangent line. Then graph it.

Find the coordinates of point B, the second intersection of the line $\overleftrightarrow{AC}$ and the circle.

What is the equation of the tangent to the circle at $(-7, 4)$? Graph it as a check.

The lines $y = 5x + b$ are parallel to the tangent lines to the circle at points A and B. How is the y-intercept b for these lines related to the number of times each line intersects the circle?

Problem Set

1. Use the distance formula to find the distance between the points $(-1, -13)$ and $(3, -9)$.

2. Use the distance formula to find the length of the longer side of the rectangle whose vertices are $(1, 1)$, $(3, 1)$, $(3, 7)$, and $(1, 7)$.

3. Use the distance formula to find the length of the diagonal of the square whose vertices are $(0, 0)$, $(0, 5)$, $(5, 5)$, and $(5, 0)$.

Write an equation for the circles in Exercises 4–6 in the form $(x-h)^2 + (y-k)^2 = r^2$, where the center is (h, k) and the radius is r units. Then write the equation in the standard form $x^2 + ax + y^2 + by + c = 0$, and construct the graph of the equation.

4. A circle with center $(4, -1)$ and radius 6 units.

5. A circle with center $(-3, 5)$ tangent to the x-axis.

6. A circle in the third quadrant, radius 1 unit, tangent to both axes.

7. By finding the radius of each circle and the distance between their centers, show that the circles $x^2 + y^2 = 4$ and $x^2 - 4x + y^2 - 4y + 4 = 0$ intersect. Illustrate graphically.

8. Find the point of intersection of the circles $x^2 + y^2 - 15 = 0$ and $x^2 - 4x + y^2 + 2y - 5 = 0$. Check by graphing the equations.

9. Solve the system $y = x^2 - 2$ and $x^2 + y^2 = 4$. Illustrate graphically.

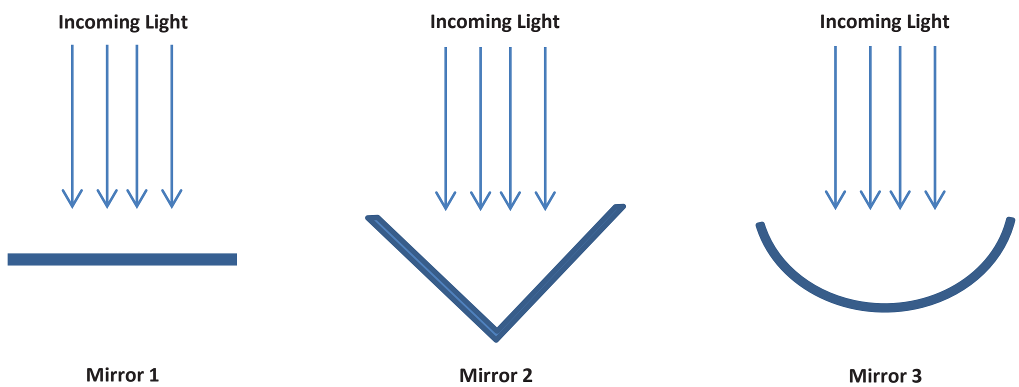
10. Solve the system $y = 2x - 13$ and $y = x^2 - 6x + 3$. Illustrate graphically.

Lesson 33: The Definition of a Parabola

Classwork

Opening Exercise

Suppose you are viewing the cross-section of a mirror. Where would the incoming light be reflected in each type of design? Sketch your ideas below.



Discussion

When Newton designed his reflector telescope he understood two important ideas. Figure 1 shows a diagram of this type of telescope.

- The curved mirror needs to focus all the light to a single point that we will call the focus. An angled flat mirror is placed near this point and reflects the light to the eyepiece of the telescope.
- The reflected light needs to arrive at the focus at the same time, otherwise the image is distorted.

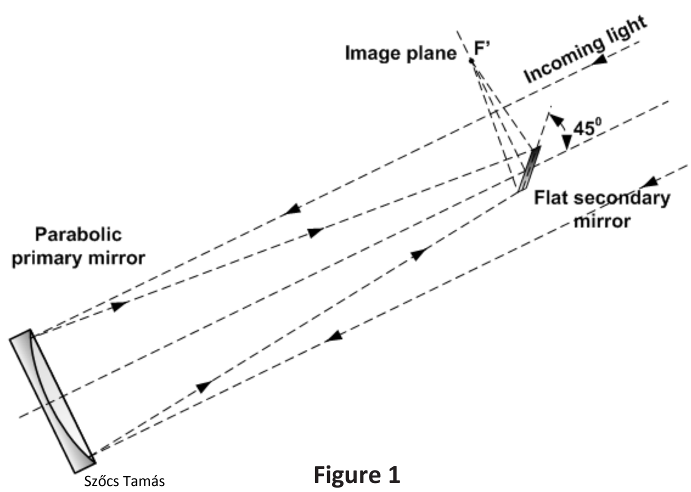


Figure 1

Definition: A *parabola with directrix* L *and focus point* F is the set of all points in the plane that are equidistant from the point F and line L.

Figure 2 to the right illustrates this definition of a parabola. In this diagram, $FP_1 = P_1Q_1$, $FP_2 = P_2Q_2$, $FP_3 = P_3Q_3$ showing that for any point P on the parabola, the distance between P and F is equal to the distance between P and the line L.

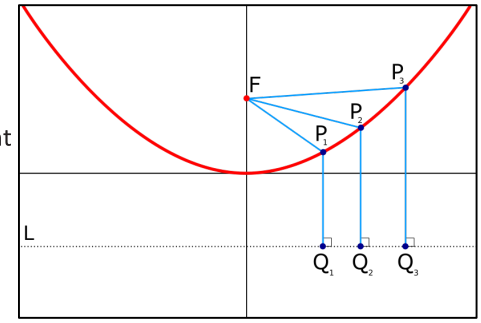


Figure 2

All parabolas have the reflective property illustrated in Figure 3 Rays parallel to the axis will reflect off the parabola and through the focus point, F.

Thus, a mirror shaped like a rotated parabola would satisfy Newton's requirements for his telescope design.

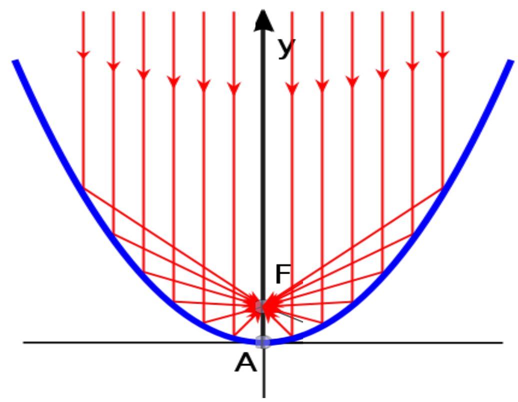


Figure 3

Figure 4 below shows several different line segments representing the reflected light with one endpoint on the curved mirror that is a parabola and the other endpoint at the focus. Anywhere the light hits this type of curved surface, it always reflects to the focus, F, at exactly the same time.

Figure 5 shows the same image with a directrix. Imagine for a minute that the mirror was not there. Then, the light would arrive at the directrix all at the same time. Since the distance from each point on the parabolic mirror to the directrix is the same as the distance from the point on the mirror to the focus, and the speed of light is constant, it takes the light the same amount of time to travel to the focus as it would have taken it to travel to the directrix. In the diagram, this means that $AF = AF_A$, $BF = BF_B$, and so on. Thus, the light rays arrive at the focus at the same time, and the image is not distorted.

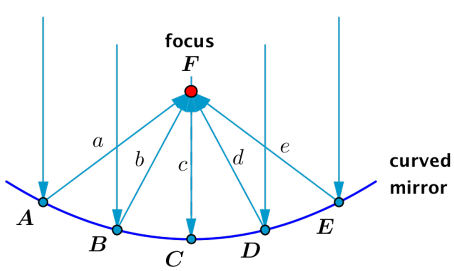


Figure 4

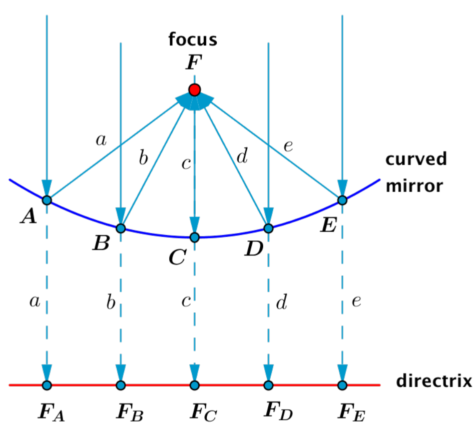
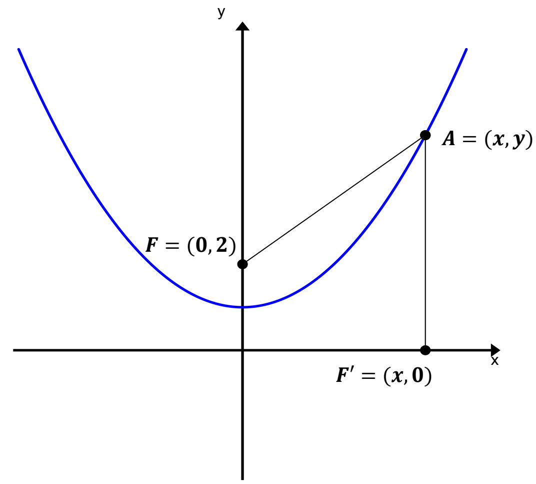


Figure 5

Example 1

Given a focus and a directrix, create an equation for a parabola.



Focus: $F = (0,2)$

Directrix: x-axis

Parabola: $P = \{(x, y)|$
(x, y) is equidistant to F and to the x-axis.$\}$

Let A be any point (x, y) on the parabola P. Let F' be a point on the directrix with the same x-coordinate as point A.

What is the length of AF'?

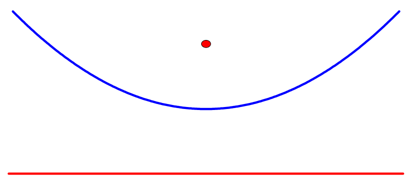
Use the distance formula to create an expression that represents the length of AF.

Create an equation that relates the two lengths and solve it for y.

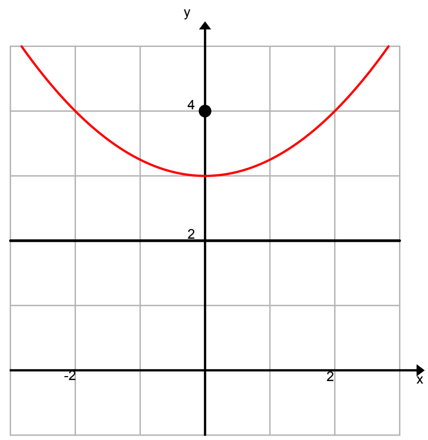
Verify that this equation appears to match the graph shown.

Exercises 1–2

1. Demonstrate your understanding of the definition of a parabola by drawing several pairs of congruent segments given the parabola, its focus, and directrix. Measure the segments that you drew to confirm the accuracy of your sketches in either centimeters or inches.



2. Derive the analytic equation of a parabola given the focus of $(0,4)$ and the directrix $y = 2$. Use the diagram to help you work this problem.



a. Label a point (x, y) anywhere on the parabola.

b. Write an expression for the distance from the point (x, y) to the directrix.

c. Write an expression for the distance from the point (x, y) to the focus.

d. Apply the definition of a parabola to create an equation in terms of x and y. Solve this equation for y.

e. What is the translation that takes the graph of this parabola to the graph of the equation derived in Example 1?

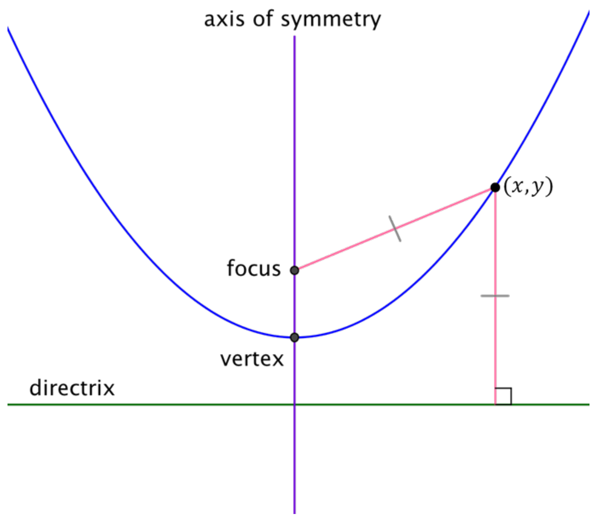
Lesson Summary

Parabola: A parabola with directrix line L and focus point F is the set of all points in the plane that are equidistant from the point F and line L

Axis of symmetry: The axis of symmetry of a parabola given by a focus point and a directrix is the perpendicular line to the directrix that passes through the focus

Vertex of a parabola: The vertex of a parabola is the point where the axis of symmetry intersects the parabola.

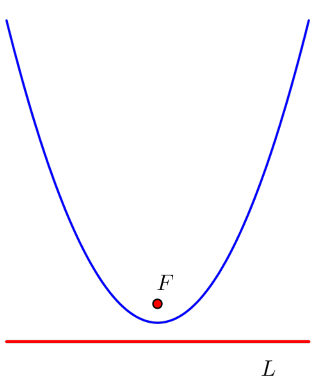
In the Cartesian plane, the distance formula can help us to derive an analytic equation for the parabola.



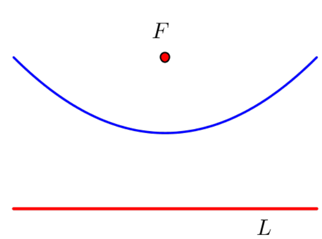
Problem Set

1. Demonstrate your understanding of the definition of a parabola by drawing several pairs of congruent segments given each parabola, its focus, and directrix. Measure the segments that you drew in either inches or centimeters to confirm the accuracy of your sketches.

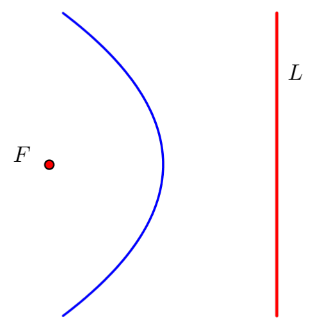
a.



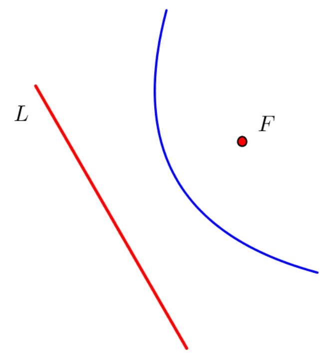
b.



c.



d.



2. Find the distance from the point $(4,2)$ to the point $(0,1)$.

3. Find the distance from the point $(4,2)$ to the line $y = -2$.

4. Find the distance from the point $(-1,3)$ to the point $(3,-4)$.

5. Find the distance from the point $(-1,3)$ to the line $y = 5$.

6. Find the distance from the point $(x, 4)$ to the line $y = -1$.

7. Find the distance from the point $(x, -3)$ to the line $y = 2$.

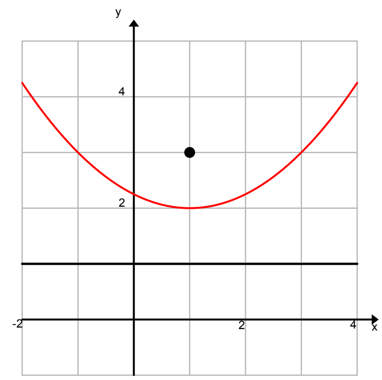
8. Find the values of x for which the point $(x, 4)$ is equidistant from $(0,1)$ and the line $y = -1$.

9. Find the values of x for which the point $(x, -3)$ is equidistant from $(1, -2)$ and the line $y = 2$.

10. Consider the equation $y = x^2$.
    a. Find the coordinates of the three points on the graph of $y = x^2$ whose x-values are 1, 2, and 3.
    b. Show that each of the three points in part (a) is equidistant from the point $\left(0, \frac{1}{4}\right)$ and the line $y = -\frac{1}{4}$.
    c. Show that if the point with coordinates (x, y) is equidistant from the point $\left(0, \frac{1}{4}\right)$, and the line $y = -\frac{1}{4}$, then $y = x^2$.

11. Given the equation $y = \frac{1}{2}x^2 - 2x$,
    a. Find the coordinates of the three points on the graph of $y = \frac{1}{2}x^2 - 2x$ whose x-values are -2, 0, and 4.
    b. Show that each of the three points in part (a) is equidistant from the point $\left(2, -\frac{3}{2}\right)$, and the line $y = -\frac{5}{2}$.
    c. Show that if the point with coordinates (x, y) is equidistant from the point $\left(2, -\frac{3}{2}\right)$ and the line $y = -\frac{5}{2}$ then $y = \frac{1}{2}x^2 - 2x$.

12. Derive the analytic equation of a parabola with focus $(1,3)$ and directrix $y = 1$. Use the diagram to help you work this problem.
    a. Label a point (x, y) anywhere on the parabola.
    b. Write an expression for the distance from the point (x, y) to the directrix.
    c. Write an expression for the distance from the point (x, y) to the focus $(1,3)$.
    d. Apply the definition of a parabola to create an equation in terms of x and y. Solve this equation for y.
    e. Describe a sequence of transformations that would take this parabola to the parabola with equation $y = \frac{1}{4}x^2 + 1$ derived in Example 1.



13. Consider a parabola with focus $(0, -2)$ and directrix on the x-axis.
    a. Derive the analytic equation for this parabola.
    b. Describe a sequence of transformations that would take the parabola with equation $y = \frac{1}{4}x^2 + 1$ derived in Example 1 to the graph of the parabola in part (a).

14. Derive the analytic equation of a parabola with focus $(0,10)$ and directrix on the x-axis.

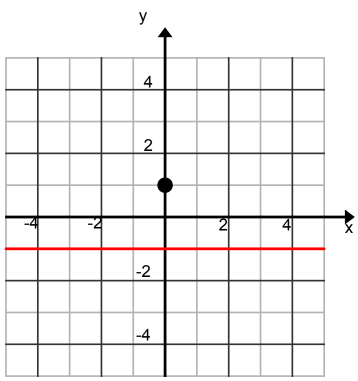
Lesson 34: Are All Parabolas Congruent?

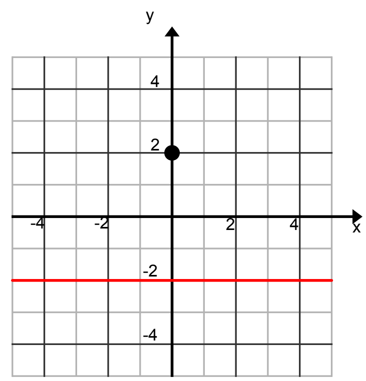
Classwork

Opening Exercise

Are all parabolas congruent? Use the following questions to support your answer.

a. Draw the parabola for each focus and directrix given below.





b. What do we mean by congruent parabolas?

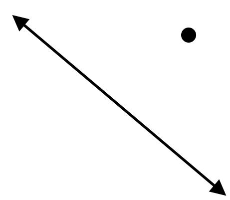
c. Are the two parabolas from part (a) congruent? Explain how you know.

d. Are all parabolas congruent?

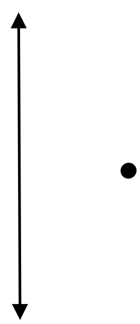
e. Under what conditions might two parabolas be congruent? Explain your reasoning.

Exercises 1–5

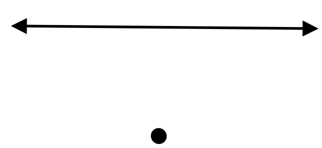
1. Draw the parabola with the given focus and directrix.



2. Draw the parabola with the given focus and directrix.

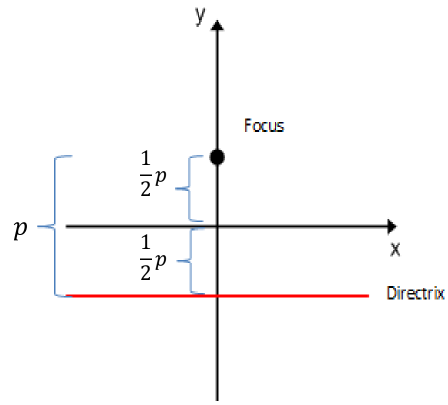


3. Draw the parabola with the given focus and directrix.



4. What can you conclude about the relationship between the parabolas in Exercises 1–3?

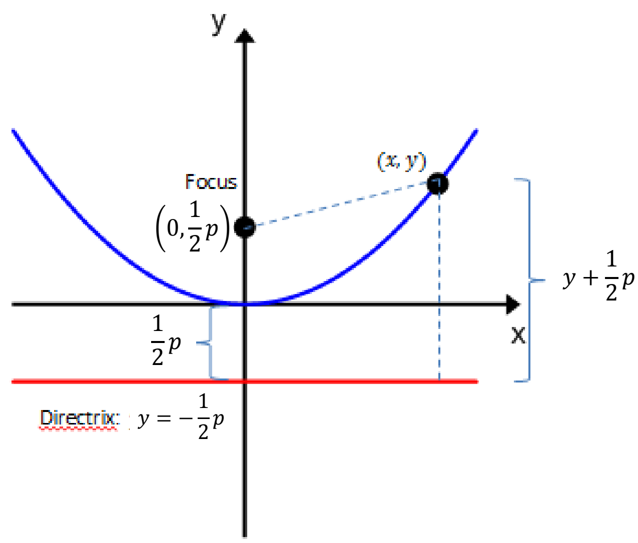
5. Let p be the number of units between the focus and the directrix, as shown. As the value of p increases, what happens to the shape of the resulting parabola?



Example 1

Consider a parabola P with distance $p > 0$ between the focus with coordinates $\left(0, \frac{1}{2}p\right)$, and directrix $y = -\frac{1}{2}p$.

What is the equation that represents this parabola?



Discussion

We have shown that any parabola with a distance $p > 0$ between the focus $\left(0, \frac{1}{2}p\right)$ and directrix $y = -\frac{1}{2}p$ has a vertex at the origin and is represented by a quadratic equation of the form $y = \frac{1}{2p}x^2$.

Suppose that the vertex of a parabola with a horizontal directrix that opens upward is (h, k), and the distance from the focus to directrix is $p > 0$. Then, the focus has coordinates $\left(h, k + \frac{1}{2}p\right)$, and the directrix has equation $y = k - \frac{1}{2}p$. If we go through the above derivation with focus $\left(h, k + \frac{1}{2}p\right)$ and directrix $y = k - \frac{1}{2}p$, we should not be surprised to get a quadratic equation. In fact, if we complete the square on that equation, we can write it in the form $y = \frac{1}{2p}(x - h)^2 + k$.

In Algebra I, Module 4, Topic B, we saw that any quadratic function can be put into vertex form: $f(x) = a(x - h)^2 + k$. Now we see that any parabola that opens upward can be described by a quadratic function in vertex form, where $a = \frac{1}{2p}$.

If the parabola opens downward, then the equation is $y = -\frac{1}{2p}(x - h)^2 + k$, and the graph of any quadratic equation of this form is a parabola with vertex at (h, k), distance p between focus and directrix, and opening downward. Likewise, we can derive analogous equations for parabolas that open to the left and right. This discussion is summarized in the box below.

<u>Vertex Form of a Parabola</u>

Given a parabola P with vertex (h, k), horizontal directrix, and distance $p > 0$ between focus and directrix, the analytic equation that describes the parabola P is:

- $y = \frac{1}{2p}(x - h)^2 + k$ if the parabola opens upward, and
- $y = -\frac{1}{2p}(x - h)^2 + k$ if the parabola opens downward.

Conversely, if $p > 0$, then

- The graph of the quadratic equation $y = \frac{1}{2p}(x - h)^2 + k$ is a parabola that opens upward with vertex at (h, k) and distance p from focus to directrix, and
- The graph of the quadratic equation $y = -\frac{1}{2p}(x - h)^2 + k$ is a parabola that opens downward with vertex at (h, k) and distance p from focus to directrix.

Given a parabola P with vertex (h, k), vertical directrix, and distance $p > 0$ between focus and directrix, the analytic equation that describes the parabola P is:

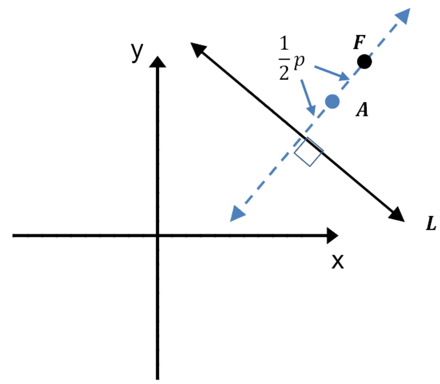
- $x = \frac{1}{2p}(y - k)^2 + h$ if the parabola opens to the right, and
- $x = -\frac{1}{2p}(y - k)^2 + h$ if the parabola opens to the left.

Conversely, if $p > 0$, then

- The graph of the quadratic equation $x = \frac{1}{2p}(y - k)^2 + h$ is a parabola that opens to the right with vertex at (h, k) and distance p from focus to directrix, and
- The graph of the quadratic equation $x = -\frac{1}{2p}(y - k)^2 + h$ is a parabola that opens to the left with vertex at (h, k) and distance p from focus to directrix.

Example 2

Theorem: Given a parabola P given by a directrix L and a focus F in the Cartesian plane, then P is congruent to the graph of $y = \frac{1}{2p}x^2$, where p is the distance from F to L.



Proof

Exercises 6–9

6. Restate the results of the theorem from Example 2 in your own words.

7. Create the equation for a parabola that is congruent to $y = 2x^2$. Explain how you determined your answer.

8. Create an equation for a parabola that IS NOT congruent to $y = 2x^2$. Explain how you determined your answer.

9. Write the equation for two different parabolas that are congruent to the parabola with focus point $(0,3)$ and directrix line $y = -3$.

Problem Set

1. Show that if the point with coordinates (x, y) is equidistant from $(4,3)$ and the line $y = 5$, then $y = -\frac{1}{4}x^2 + 2x$.

2. Show that if the point with coordinates (x, y) is equidistant from the point $(2,0)$ and the line $y = -4$, then $y = \frac{1}{8}(x-2)^2 - 2$.

3. Find the equation of the set of points which are equidistant from $(0,2)$ and the x-axis. Sketch this set of points.

4. Find the equation of the set of points which are equidistant from the origin and the line $y = 6$. Sketch this set of points.

5. Find the equation of the set of points which are equidistant from $(4, -2)$ and the line $y = 4$. Sketch this set of points.

6. Find the equation of the set of points which are equidistant from $(4,0)$ and the y-axis. Sketch this set of points.

7. Find the equation of the set of points which are equidistant from the origin and the line $x = -2$. Sketch this set of points.

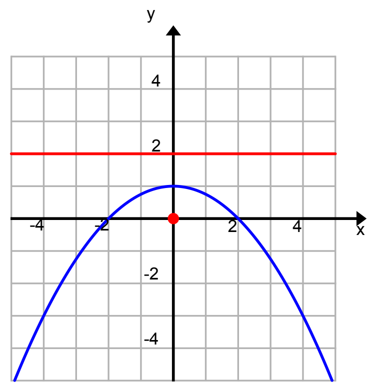
8. Use the definition of a parabola to sketch the parabola defined by the given focus and directrix.
   a. Focus: $(0,5)$ Directrix: $y = -1$
   b. Focus: $(-2,0)$ Directrix: y-axis
   c. Focus: $(4, -4)$ Directrix: x-axis
   d. Focus: $(2,4)$ Directrix: $y = -2$

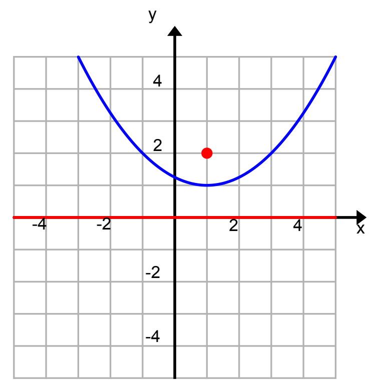
9. Find an analytic equation for each parabola described in Problem 8.

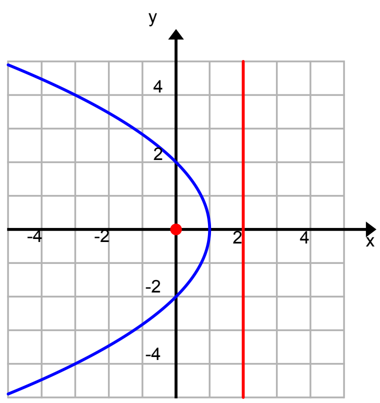
10. Are any of the parabolas described in Problem 9 congruent? Explain your reasoning.

11. Sketch each parabola, labeling its focus and directrix.
    a. $y = \frac{1}{2}x^2 + 2$
    b. $y = -\frac{1}{4}x^2 + 1$
    c. $x = \frac{1}{8}y^2$
    d. $x = \frac{1}{2}y^2 + 2$
    e. $y = \frac{1}{10}(x-1)^2 - 2$

12. Determine which parabolas are congruent to the parabola that is the graph of the equation $y = -\frac{1}{4}x^2$.

a.


c.


b.


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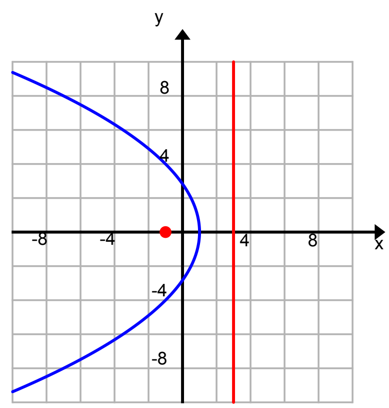
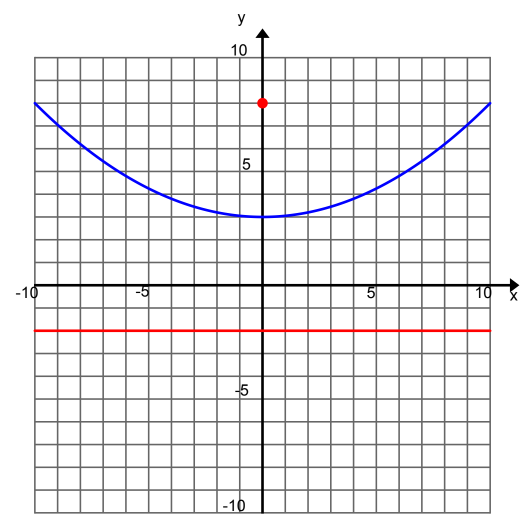


13. Determine which equations represent the graph of a parabola that is congruent to the parabola shown to right.



a. $y = \frac{1}{20}x^2$

b. $y = \frac{1}{10}x^2 + 3$

c. $y = -\frac{1}{20}x^2 + 8$

d. $y = \frac{1}{5}x^2 + 5$

e. $x = \frac{1}{10}y^2$

f. $x = \frac{1}{5}(y-3)^2$

g. $x = \frac{1}{20}y^2 + 1$

14. Jemma thinks that the parabola whose graph is the equation $y = \frac{1}{3}x^2$ is NOT congruent to the parabola whose graph is the equation $y = -\frac{1}{3}x^2 + 1$. Do you agree or disagree? Create a convincing argument to support your reasoning.

15. Let P be the parabola with focus $(2,6)$ and directrix $y = -2$.
    a. Write an equation whose graph is a parabola congruent to P with a focus $(0,4)$.
    b. Write an equation whose graph is a parabola congruent to P with a focus $(0,0)$.
    c. Write an equation whose graph is a parabola congruent to P with the same directrix, but different focus.
    d. Write an equation whose graph is a parabola congruent to P with the same focus, but with a vertical directrix.

16. Let P be the parabola with focus $(0,4)$ and directrix $y = x$.
    a. Sketch this parabola.
    b. By how many degrees would you have to rotate P about the focus to make the directrix line horizontal?
    c. Write an equation in the form $y = \frac{1}{2a}x^2$ whose graph is a parabola that is congruent to P.
    d. Write an equation whose graph is a parabola with a vertical directrix that is congruent to P.
    e. Write an equation whose graph is P', the parabola congruent to P that results after P is rotated clockwise $45°$ about the focus.
    f. Write an equation whose graph is P'', the parabola congruent to P that results after P's directrix is rotated $45°$ about the origin.

Extension:

17. Consider the function $f(x) = \frac{2x^2-8x+9}{-x^2+4x-5}$, where x is a real number.
    a. Use polynomial division to rewrite f in the form $f(x) = q + \frac{r}{-x^2+4x-5}$ for some real numbers q and r.
    b. Find the x-value where the maximum occurs for the function f, without using graphing technology. Explain how you know.

Lesson 35: Are All Parabolas Similar?

Classwork

Exercises 1–6

1. Write the equation of two parabolas that are congruent to $y = x^2$ and explain how you determined your equations.

2. Sketch the graph of $y = x^2$ and the two parabolas you created on the same coordinate axes.

3. Write the equation of two parabolas that are NOT congruent to $y = x^2$. Explain how you determined your equations.

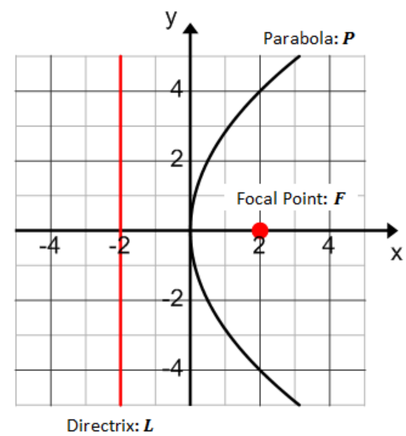
4. Sketch the graph of $y = x^2$ and the two non-congruent parabolas you created on the same coordinate axes.

5. Use your work on Exercises 1–4 to answer the question posed in the lesson title: Are all parabolas similar? Explain your reasoning.

6. The parabola at right is the graph of what equation?

   a. Label a point (x, y) on the graph of P.

   b. What does the definition of a parabola tell us about the distance between the point (x, y) and the directrix L, and the distance between the point (x, y) and the focus F?



   c. Create an equation that relates these two distances.

   d. Solve this equation for x.

   e. Find two points on the parabola P, and show that they satisfy the equation found in part (d).

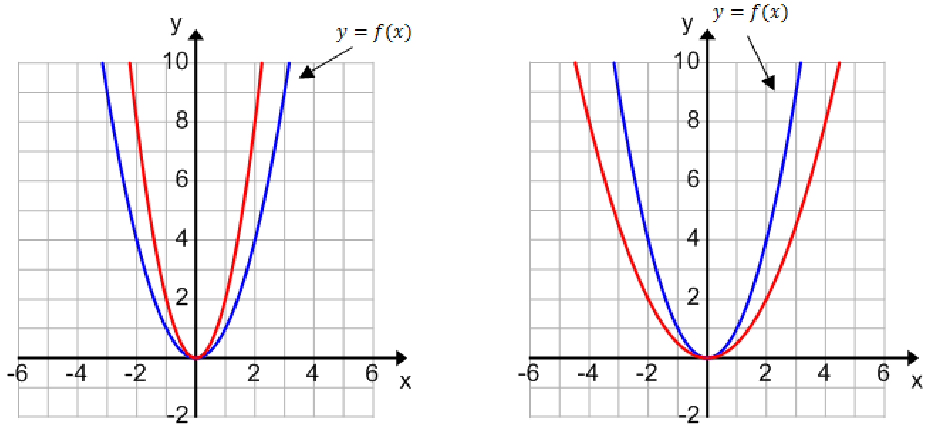
Discussion

How many of you think that all parabolas are similar? Explain why you think so.

What could we do to show that two parabolas are similar? How might you show this?

Exercises 7–10

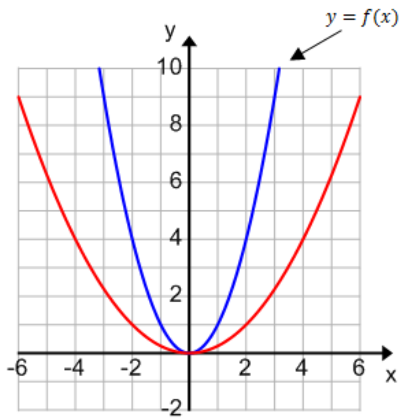
Use the graphs below to answer Exercises 7 and 8.

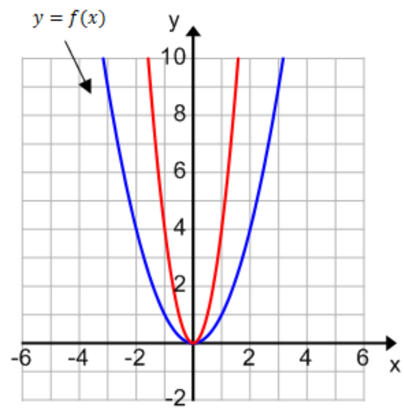


7. Suppose the unnamed red graph on the left coordinate plane is the graph of the function g. Describe g as a vertical scaling of the graph of $y = f(x)$; that is, find a value of k so that $g(x) = kf(x)$. What is the value of k? Explain how you determined your answer.

8. Suppose the unnamed red graph on the right coordinate plane is the graph of the function h. Describe h as a vertical scaling of the graph of $y = f(x)$; that is, find a value of k so that $h(x) = kf(x)$. Explain how you determined your answer.

Use the graphs below to answer Exercises 9–10.





9. Suppose the unnamed function graphed in red on the left coordinate plane is g. Describe g as a horizontal scaling of the graph of $y = f(x)$. What is the value of k? Explain how you determined your answer.

10. Suppose the unnamed function graphed in red on the right coordinate plane is h. Describe h as a horizontal scaling of the graph of $y = f(x)$. What is the value of k? Explain how you determined your answer.

Definition: A dilation at the origin D_k is a horizontal scaling by $k > 0$ followed by a vertical scaling by the same factor k. In other words, this dilation of the graph of $y = f(x)$ is the graph of the equation $y = kf\left(\frac{1}{k}x\right)$.

Example 1

Let $f(x) = x^2$ and let $k = 2$. Write a formula for the function g that results from dilating f at the origin by a factor of 2.

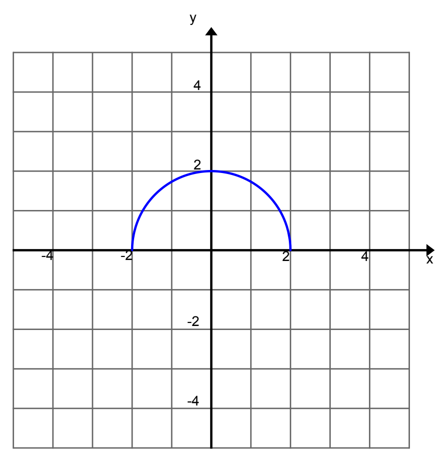
What would the results be for $k = 3$, 4, or 5? What about $k = \frac{1}{2}$?

Lesson Summary

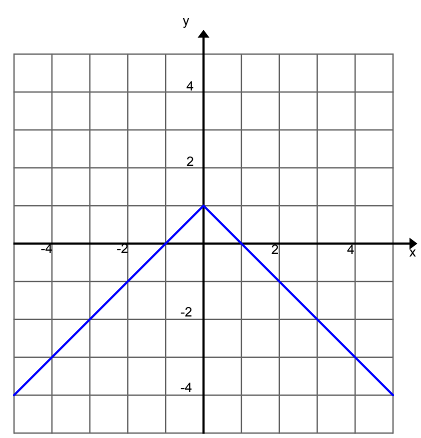
- We started with a geometric figure of a parabola defined by geometric requirements and recognized that it involved the graph of an equation we studied in algebra.
- We used algebra to prove that all parabolas with the same distance between the focus and directrix are congruent to each other, and in particular, they are congruent to a parabola with vertex at the origin, axis of symmetry along the y-axis, and equation of the form $y = \frac{1}{2p}x^2$.
- Noting that the equation for a parabola with axis of symmetry along the y-axis is of the form $y = f(x)$ for a quadratic function f, we proved that all parabolas are similar using transformations of functions.

Problem Set

1. Let $(x) = \sqrt{4 - x^2}$. The graph of f is shown below. On the same axes, graph the function g, where $g(x) = f\left(\frac{1}{2}x\right)$. Then, graph the function h, where $h(x) = 2g(x)$.



2. Let $f(x) = -|x| + 1$. The graph of f is shown below. On the same axes, graph the function g, where $g(x) = f\left(\frac{1}{3}x\right)$. Then, graph the function h, where $h(x) = 3g(x)$.



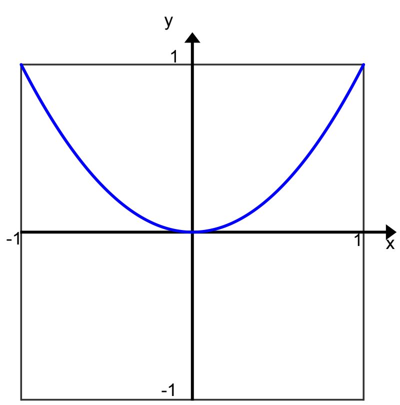
3. Based on your work in Problems 1 and 2, describe the resulting function when the original function is transformed with a horizontal and then a vertical scaling by the same factor, k.

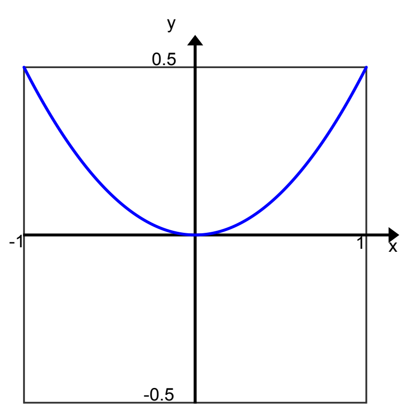
4. Let $f(x) = x^2$.
   a. What are the focus and directrix of the parabola that is the graph of the function $f(x) = x^2$?
   b. Describe the sequence of transformations that would take the graph of f to each parabola described below.
      i. Focus: $\left(0, -\frac{1}{4}\right)$, directrix: $y = \frac{1}{4}$
      ii. Focus: $\left(\frac{1}{4}, 0\right)$, directrix: $x = -\frac{1}{4}$
      iii. Focus: $(0,0)$, directrix: $y = -\frac{1}{2}$
      iv. Focus: $\left(0, \frac{1}{4}\right)$, directrix: $y = -\frac{3}{4}$
      v. Focus: $(0,3)$, directrix: $y = -1$
   c. Which parabolas are similar to the parabola that is the graph of f? Which are congruent to the parabola that is the graph of f?

5. Derive the analytic equation for each parabola described in Problem 4(b) by applying your knowledge of transformations.

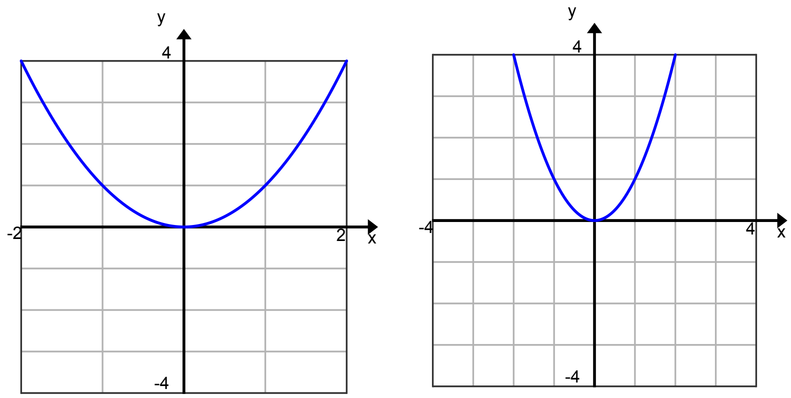
6. Are all parabolas the graph of a function of x in the xy-plane? If so, explain why, and if not, provide an example (by giving a directrix and focus) of a parabola that is not.

7. Are the following parabolas congruent? Explain your reasoning.





8. Are the following parabolas congruent? Explain your reasoning.



9. Write the equation of a parabola congruent to $y = 2x^2$ that contains the point $(1, -2)$. Describe the transformations that would take this parabola to your new parabola.

10. Write the equation of a parabola similar to $y = 2x^2$ that does NOT contain the point $(0,0)$, but does contain the point $(1,1)$.

Lesson 36: Overcoming a Third Obstacle to Factoring—What If There Are No Real Number Solutions?

Classwork

Opening Exercise

Find all solutions to each of the systems of equations below using any method.

$2x - 4y = -1$ $3x - 6y = 4$	$y = x^2 - 2$ $y = 2x - 5$	$x^2 + y^2 = 1$ $x^2 + y^2 = 4$

Exercises 1–4

1. Are there any real number solutions to the system $\begin{cases} y = 4 \\ x^2 + y^2 = 2 \end{cases}$? Support your findings both analytically and graphically.

2. Does the line $y = x$ intersect the parabola $y = -x^2$? If so, how many times, and where? Draw graphs on the same set of axes.

3. Does the line $y = -x$ intersect the circle $x^2 + y^2 = 1$? If so, how many times, and where? Draw graphs on the same set of axes.

4. Does the line $y = 5$ intersect the parabola $y = 4 - x^2$? Why or why not? Draw the graphs on the same set of axes.

Lesson Summary

An equation or a system of equations may have one or more solutions in the real numbers, or it may have no real number solution.

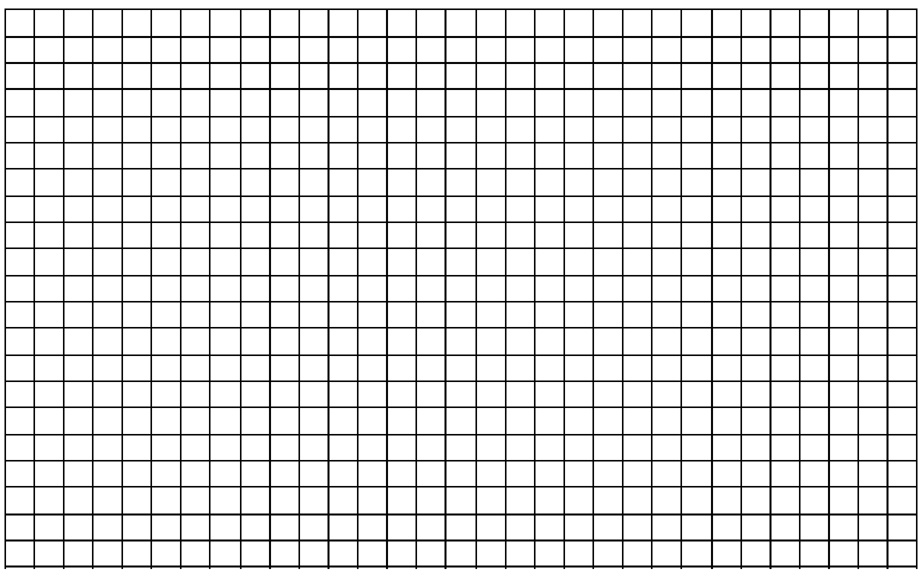
Two graphs that do not intersect in the real plane describe a system of two equations without a real solution. If a system of two equations does not have a real solution, the graphs of the two equations do not intersect in the real plane.

A quadratic equation in the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$, that has no real solution indicates that the graph of $y = ax^2 + bx + c$ does not intersect the x-axis.

Problem Set

1. For each part, solve the system of linear equations, or show that no real solution exists. Graphically support your answer.

   a. $$\begin{aligned} 4x + 2y &= 9 \\ x + y &= 3 \end{aligned}$$



b. $$2x - 8y = 9$$
$$3x - 12y = 0$$

2. Solve the following system of equations, or show that no real solution exists. Graphically confirm your answer.

$$3x^2 + 3y^2 = 6$$
$$x - y = 3$$

3. Find the value of k so that the graph of the following system of equations has no solution.

$$3x - 2y - 12 = 0$$
$$kx + 6y - 10 = 0$$

4. Offer a geometric explanation to why the equation $x^2 - 6x + 10 = 0$ has no real solutions.

5. Without his pencil or calculator, Joey knows that $2x^3 + 3x^2 - 1 = 0$ has at least one real solution. How does he know?

6. The graph of the quadratic equation $y = x^2 + 1$ has no x-intercepts. However, Gia claims that when the graph of $y = x^2 + 1$ is translated by a distance of 1 in a certain direction, the new (translated) graph would have exactly one x-intercept. Further, if $y = x^2 + 1$ is translated by a distance greater than 1 in the same direction, the new (translated) graph would have exactly two x-intercepts. Support or refute Gia's claim. If you agree with her, in which direction did she translate the original graph? Draw graphs to illustrate.

7. In the previous problem, we mentioned that the graph of $y = x^2 + 1$ has no x-intercepts. Suppose that $y = x^2 + 1$ is one of two equations in a system of equations and that the other equation is a line. Give an example of a linear equation such that this system has exactly one solution.

8. In prior problems, we mentioned that the graph of $y = x^2 + 1$ has no x-intercepts. Does the graph of $y = x^2 + 1$ intersect the graph of $y = x^3 + 1$?

Lesson 37: A Surprising Boost from Geometry

Classwork

Opening Exercise

Solve each equation for x.

a. $x - 1 = 0$

b. $x + 1 = 0$

c. $x^2 - 1 = 0$

d. $x^2 + 1 = 0$

Example 1: Addition with Complex Numbers

Compute $(3 + 4i) + (7 - 20i)$.

Example 2: Subtraction with Complex Numbers

Compute $(3 + 4i) - (7 - 20i)$.

Example 3: Multiplication with Complex Numbers

Compute $(1 + 2i)(1 - 2i)$.

Example 4: Multiplication with Complex Numbers

Verify that $-1 + 2i$ and $-1 - 2i$ are solutions to $x^2 + 2x + 5 = 0$.

Lesson Summary

Multiplying by i rotates every complex number in the complex plane by 90° about the origin.

Every complex number is in the form $a + bi$, where a is the real part and b is the imaginary part of the number. Real numbers are also complex numbers; the real number a can be written as the complex number $a + 0i$.

Adding two complex numbers is analogous to combining like terms in a polynomial expression.

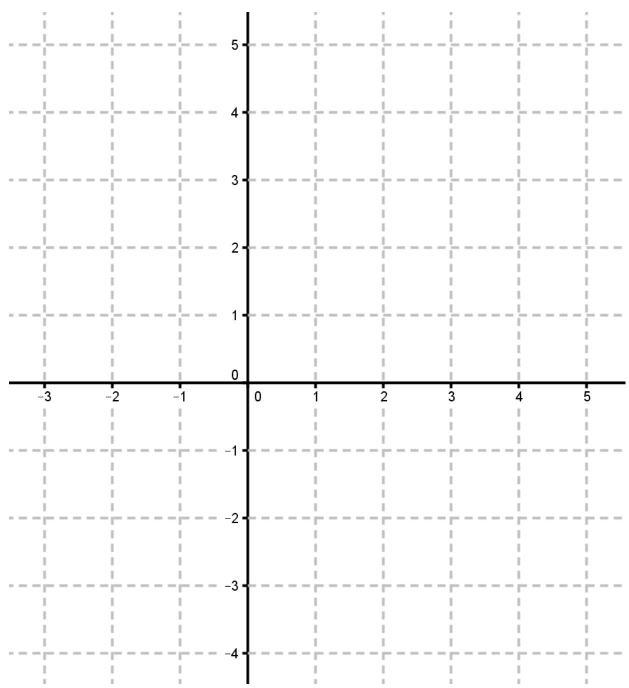
Multiplying two complex numbers is like multiplying two binomials, except one can use $i^2 = -1$ to further write the expression in simpler form.

Complex numbers satisfy the associative, commutative, and distributive properties.

Complex numbers can now allow us to find solutions to equations that previously had no real number solutions.

Problem Set

1. Locate the point on the complex plane corresponding to the complex number given in parts (a)–(h). On one set of axes, label each point by its identifying letter. For example, the point corresponding to $5 + 2i$ should be labeled "a."
   a. $5 + 2i$
   b. $3 - 2i$
   c. $-2 - 4i$
   d. $-i$
   e. $\frac{1}{2} + i$
   f. $\sqrt{2} - 3i$
   g. 0
   h. $-\frac{3}{2} + \frac{\sqrt{3}}{2}i$



2. Express each of the following in $a + bi$ form.
   a. $(13 + 4i) + (7 + 5i)$
   b. $(5 - i) - 2(1 - 3i)$
   c. $\left((5 - i) - 2(1 - 3i)\right)^2$
   d. $(3 - i)(4 + 7i)$
   e. $(3 - i)(4 + 7i) - \left((5 - i) - 2(1 - 3i)\right)$

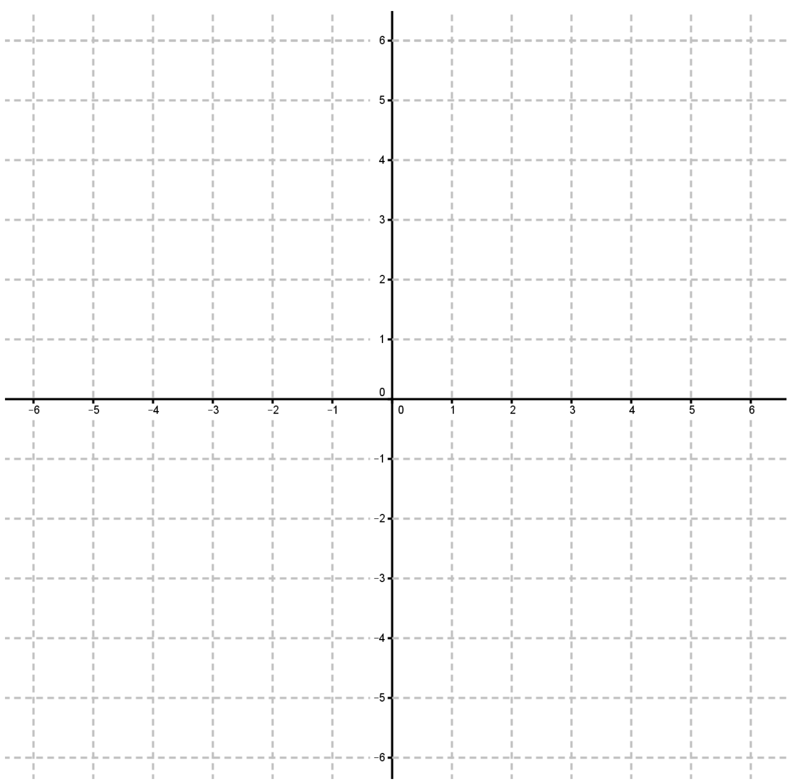
3. Express each of the following in a + bi form.
   a. $(2 + 5i) + (4 + 3i)$
   b. $(-1 + 2i) - (4 - 3i)$
   c. $(4 + i) + (2 - i) - (1 - i)$
   d. $(5 + 3i)(3 + 5i)$
   e. $-i(2 - i)(5 + 6i)$
   f. $(1 + i)(2 - 3i) + 3i(1 - i) - i$

4. Find the real values of x and y in each of the following equations using the fact that if $a + bi = c + di$, then $a = c$ and $b = d$.
   a. $5x + 3yi = 20 + 9i$
   b. $2(5x + 9) = (10 - 3y)i$
   c. $3(7 - 2x) - 5(4y - 3)i = x - 2(1 + y)i$

5. Since $i^2 = -1$, we see that

$$i^3 = i^2 \cdot i = -1 \cdot i = -i$$
$$i^4 = i^2 \cdot i^2 = -1 \cdot -1 = 1.$$

Plot i, i^2, i^3, and i^4 on the complex plane and describe how multiplication by each rotates points in the complex plane.



6. Express each of the following in $a + bi$ form.
   a. i^5
   b. i^6
   c. i^7
   d. i^8
   e. i^{102}

7. Express each of the following in $a + bi$ form.
   a. $(1 + \mathrm{i})^2$
   b. $(1 + \mathrm{i})^4$
   c. $(1 + \mathrm{i})^6$

8. Evaluate $x^2 - 6x$ when $x = 3 - i$.

9. Evaluate $4x^2 - 12x$ when $x = \frac{3}{2} - \frac{i}{2}$.

10. Show by substitution that $\frac{5 - i\sqrt{5}}{5}$ is a solution to $5x^2 - 10x + 6 = 0$.

11. a. Evaluate the four products below.

    Evaluate $\sqrt{9} \cdot \sqrt{4}$.

    Evaluate $\sqrt{9} \cdot \sqrt{-4}$.

    Evaluate $\sqrt{-9} \cdot \sqrt{4}$.

    Evaluate $\sqrt{-9} \cdot \sqrt{-4}$.

    b. Suppose a and b are positive real numbers. Determine whether the following quantities are equal or not equal.

    $\sqrt{a} \cdot \sqrt{b}$ and $\sqrt{-a} \cdot \sqrt{-b}$

    $\sqrt{-a} \cdot \sqrt{b}$ and $\sqrt{a} \cdot \sqrt{-b}$

Lesson 38: Complex Numbers as Solutions to Equations

Classwork

Opening Exercises

1. Use the quadratic formula to solve the following quadratic equations. Calculate the discriminant for each equation.
   a. $x^2 - 9 = 0$

   b. $x^2 - 6x + 9 = 0$

   c. $x^2 + 9 = 0$

2. How does the value of the discriminant for each equation relate the number of solutions you found?

Example 1

Consider the equation $3x + x^2 = -7$.

What does the value of the discriminant tell us about number of solutions to this equation?

Solve the equation. Does the number of solutions match the information provided by the discriminant? Explain.

Exercise

Compute the value of the discriminant of the quadratic equation in each part. Use the value of the discriminant to predict the number and type of solutions. Find all real and complex solutions.

a. $x^2 + 2x + 1 = 0$

b. $x^2 + 4 = 0$

c. $9x^2 - 4x - 14 = 0$

d. $3x^2 + 4x + 2 = 0$

e. $x = 2x^2 + 5$

f. $8x^2 + 4x + 32 = 0$

Lesson Summary

- A quadratic equation with real coefficients and a real constant may have real or complex solutions.
- Given a quadratic equation $ax^2 + bx + c = 0$, the discriminant $b^2 - 4ac$ indicates whether the equation has two distinct real solutions, one real solution, or two complex solutions.
  - If $b^2 - 4ac > 0$, there are two real solutions to $ax^2 + bx + c = 0$.
  - If $b^2 - 4ac = 0$, there is one real solution to $ax^2 + bx + c = 0$.
  - If $b^2 - 4ac < 0$, there are two complex solutions to $ax^2 + bx + c = 0$.

Problem Set

1. Give an example of a quadratic equation in standard form that has…
   a. Exactly two distinct real solutions.
   b. Exactly one distinct real solution.
   c. Exactly two complex (non-real) solutions.

2. Suppose we have a quadratic equation $ax^2 + bx + c = 0$ so that $a + c = 0$. Does the quadratic equation have one solution or two distinct solutions? Are they real or complex? Explain how you know.

3. Solve the equation $5x^2 - 4x + 3 = 0$.

4. Solve the equation $2x^2 + 8x = -9$.

5. Solve the equation $9x - 9x^2 = 3 + x + x^2$.

6. Solve the equation $3x^2 - x + 1 = 0$.

7. Solve the equation $6x^4 + 4x^2 - 3x + 2 = 2x^2(3x^2 - 1)$.

8. Solve the equation $25x^2 + 100x + 200 = 0$.

9. Write a quadratic equation in standard form such that -5 is its only solution.

10. Is it possible that the quadratic equation $ax^2 + bx + c = 0$ has a positive real solution if a, b, and c are all positive real numbers?

11. Is it possible that the quadratic equation $ax^2 + bx + c = 0$ has a positive real solution if a, b, and c are all negative real numbers?

Extension:

12. Show that if $k > 3.2$, the solutions of $5x^2 - 8x + k = 0$ are not real numbers.

13. Let k be a real number, and consider the quadratic equation $(k+1)x^2 + 4kx + 2 = 0$.
    a. Show that the discriminant of $(k+1)x^2 + 4kx + 2 = 0$ defines a quadratic function of k.
    b. Find the zeros of the function in part (a) and make a sketch of its graph.
    c. For what value of k are there two distinct real solutions to the given quadratic equation?
    d. For what value of k are there two complex solutions to the given quadratic equation?
    e. For what value of k is there one solution to the given quadratic equation?

14. We can develop two formulas that can help us find errors in calculated solutions of quadratic equations.
    a. Find a formula for the sum S of the solutions of the quadratic equation $ax^2 + bx + c = 0$.
    b. Find a formula for the product R of the solutions of the quadratic equation $ax^2 + bx + c = 0$.
    c. June calculated the solutions 7 and -1 to the quadratic equation $x^2 - 6x + 7 = 0$. Do the formulas from parts (a) and (b) detect an error in her solutions? If not, determine if her solution is correct.
    d. Paul calculated the solutions $3 - i\sqrt{2}$ and $3 + i\sqrt{2}$ to the quadratic equation $x^2 - 6x + 7 = 0$. Do the formulas from parts (a) and (b) detect an error in his solutions? If not, determine if his solutions are correct.
    e. Joy calculated the solutions $3 - \sqrt{2}$ and $3 + \sqrt{2}$ to the quadratic equation $x^2 - 6x + 7 = 0$. Do the formulas from parts (a) and (b) detect an error in her solutions? If not, determine if her solutions are correct.
    f. If you find solutions to a quadratic equations that match the results from parts (a) and (b), does that mean your solutions are correct?
    g. Summarize the results of this exercise.

Lesson 39: Factoring Extended to the Complex Realm

Classwork

Opening Exercise

Rewrite each expression as a polynomial in standard form.

a. $(x+i)(x-i)$

b. $(x+5i)(x-5i)$

c. $\big(x-(2+i)\big)\big(x-(2-i)\big)$

Exercises 1–4

Completely factor the following polynomial expressions.

1. x^2+9

2. x^2+5

3. Consider the polynomial $P(x) = x^4 - 3x^2 - 4$.
   a. What are the solutions to $x^4 - 3x^2 - 4 = 0$?

   b. How many x-intercepts does the graph of the equation $y = x^4 - 3x^2 - 4$ have? What are the coordinates of the x-intercepts?

   c. Are solutions to the polynomial equation $P(x) = 0$ the same as the x-intercepts of the graph of $y = P(x)$? Justify your reasoning.

4. Write a polynomial P with the lowest possible degree that has the given solutions. Explain how you generated each answer.

   a. $-2, 3, -4i, 4i$

   b. $-1, 3i$

   c. $0, 2, 1+i, 1-i$

   d. $\sqrt{2}, -\sqrt{2}, 3, 1+2i$

   e. $\sqrt{2}, -\sqrt{2}, 3, 1+2i$

Lesson Summary

- Polynomial equations with real coefficients can have real or complex solutions or they can have both.
- Complex solutions to polynomial equations with real coefficients are always members of a conjugate pair.
- Real solutions to polynomial equations are the same as x-intercepts of the associated graph, but complex solutions are not.

Problem Set

1. Rewrite each expression in standard form.
   a. $(x+3i)(x-3i)$
   b. $(x-a+bi)(x-(a+bi))$
   c. $(x+2i)(x-i)(x+i)(x-2i)$
   d. $(x+i)^2 \cdot (x-i)^2$

2. Suppose in Problem 1 that you had no access to paper, writing utensils, or technology. How do you know that the expressions in parts (a)–(d) are polynomials with real coefficients?

3. Write a polynomial equation of degree 4 in standard form that has the solutions $i, -i, 1, -1$.

4. Explain the difference between x-intercepts and solutions to an equation. Give an example of a polynomial with real coefficients that has twice as many solutions as x-intercepts. Write it in standard form.

5. Find the solutions to $x^4 - 5x^2 - 36 = 0$ and the x-intercepts of the graph of $y = x^4 - 5x^2 - 36$.

6. Find the solutions to $2x^4 - 24x^2 + 40 = 0$ and the x-intercepts of the graph of $y = 2x^4 - 24x^2 + 40$.

7. Find the solutions to $x^4 - 64 = 0$ and the x-intercepts of the graph of $y = x^4 - 64$.

8. Use the fact that $x^4 + 64 = (x^2 - 4x + 8)(x^2 + 4x + 8)$ to explain how you know that the graph of $y = x^4 + 64$ has no x-intercepts. You need not find the solutions.

Lesson 40: Obstacles Resolved—A Surprising Result

Classwork

Opening Exercise

Write each of the following quadratic expressions as a product of linear factors. Verify that the factored form is equivalent.

1. $x^2 + 12x + 27$

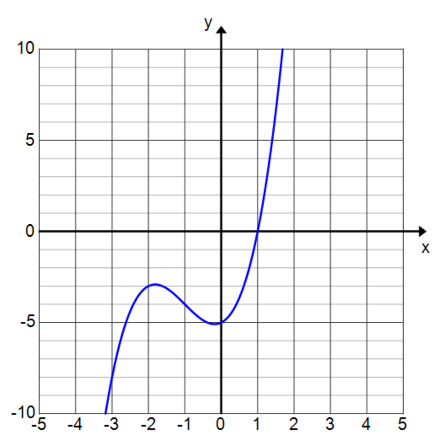
2. $x^2 - 16$

3. $x^2 + 16$

4. $x^2 + 4x + 5$

Example 1

Consider the polynomial $P(x) = x^3 + 3x^2 + x - 5$ whose graph is shown to the right.



a. Looking at the graph, how do we know that there is only one real solution?

b. Is it possible for a cubic polynomial function to have no zeros?

c. From the graph, what appears to be one solution to the equation $x^3 + 3x^2 + x - 5 = 0$?

d. How can we verify that this is a solution?

e. According to the Remainder Theorem, what is one factor of the cubic expression $x^3 + 3x^2 + x - 5$?

f. Factor out the expression you found in part (e) from $x^3 + 3x^2 + x - 5$.

g. What are all of the solutions to $x^3 + 3x^2 + x - 5 = 0$?

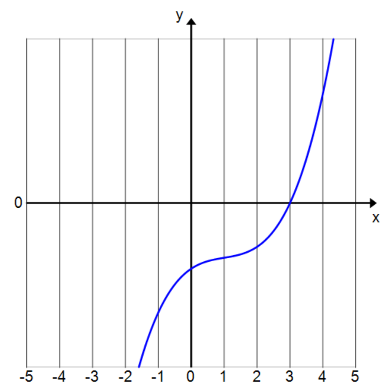
h. Write the expression $x^3 + 3x^2 + x - 5$ in terms of linear factors.

Exercises 1–2

Write each polynomial in terms of linear factors. The graph of $y = x^3 - 3x^2 + 4x - 12$ is provided for Exercise 2.

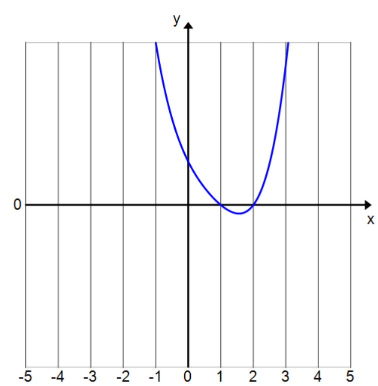
1. $f(x) = x^3 + 5x$

2. $g(x) = x^3 - 3x^2 + 4x - 12$



Example 2

Consider the polynomial function $P(x) = x^4 - 3x^3 + 6x^2 - 12x + 8$, whose corresponding graph $y = x^4 - 3x^3 + 6x^2 - 12x + 8$ is shown to the right. How many zeros does P have?



a. Part 1 of the Fundamental Theorem of Algebra says that this equation will have at least one solution in the complex numbers. How does this align with what we can see in the graph to the right?

b. Identify one zero from the graph.

c. Use polynomial division to factor out one linear term from the expression $x^4 - 3x^3 + 6x^2 - 12x + 8$.

d. Now we have a cubic polynomial to factor. We know by part 1 of the Fundamental Theorem of Algebra that a polynomial function will have at least one real zero. What is that zero in this case?

e. Use polynomial division to factor out another linear term of $x^4 - 3x^3 + 6x^2 - 12x + 8$.

f. Are we done? Can we factor this polynomial any further?

g. Now that the polynomial is in factored form, we can quickly see how many solutions there are to the original equation $x^4 - 3x^3 + 6x^2 - 12x + 8 = 0$.

h. What if we had started with a polynomial function of degree 8?

Lesson Summary

Every polynomial function of degree n, for $n \geq 1$, has n roots over the complex numbers, counted with multiplicity. Therefore, such polynomials can always be factored into n linear factors, and the obstacles to factoring we saw before have all disappeared in the larger context of allowing solutions to be complex numbers.

The Fundamental Theorem of Algebra:

1. If P is a polynomial function of degree $n \geq 1$, with real or complex coefficients, then there exists at least one number r (real or complex) such that $P(r) = 0$.
2. If P is a polynomial function of degree $n \geq 1$, given by $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with real or complex coefficients a_i, then P has exactly n zeros $r_1, r_2, \ldots, r_n$ (not all necessarily distinct), such that $P(x) = a(x - r_1)(x - r_2)\cdots(x - r_n)$.

Problem Set

1. Write each quadratic function below in terms of linear factors.
   a. $f(x) = x^2 - 25$
   b. $f(x) = x^2 + 25$
   c. $f(x) = 4x^2 + 25$
   d. $f(x) = x^2 - 2x + 1$
   e. $f(x) = x^2 - 2x + 4$

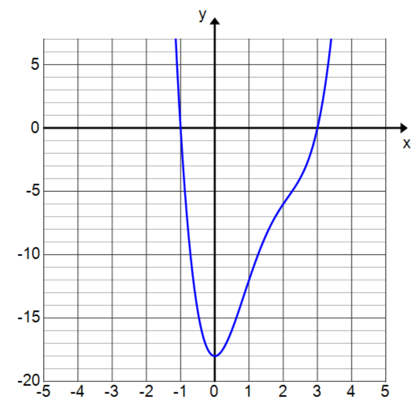
2. Consider the polynomial function $P(x) = (x^2 + 4)(x^2 + 1)(2x + 3)(3x - 4)$.
   a. Express P in terms of linear factors.
   b. Fill in the blanks of the following sentence.

      The polynomial P has degree ________ and can, therefore, be written in terms of ________ linear factors. The function P has ________ zeros. There are _______ real zeros and ________ complex zeros. The graph of $y = P(x)$ has ________ x-intercepts.

3. Express each cubic function below in terms of linear factors.
   a. $f(x) = x^3 - 6x^2 - 27x$
   b. $f(x) = x^3 - 16x^2$
   c. $f(x) = x^3 + 16x$

4. For each cubic function below, one of the zeros is given. Express each cubic function in terms of linear factors.
   a. $f(x) = 2x^3 - 9x^2 - 53x - 24; f(8) = 0$
   b. $f(x) = x^3 + x^2 + 6x + 6; f(-1) = 0$

5. Determine if each statement is always true or sometimes false. If it is sometimes false, explain why it is not always true.
   a. A degree 2 polynomial function will have two linear factors.
   b. The graph of a degree 2 polynomial function will have two x-intercepts.
   c. The graph of a degree 3 polynomial function might not cross the x-axis.
   d. A polynomial function of degree n can be written in terms of n linear factors.



6. Consider the polynomial function $f(x) = x^6 - 9x^3 + 8$.
   a. How many linear factors does $x^6 - 9x^3 + 8$ have? Explain.
   b. How is this information useful for finding the zeros of f?
   c. Find the zeros of f. (Hint: Let $Q = x^3$. Rewrite the equation in terms of Q to factor.)

7. Consider the polynomial function $P(x) = x^4 - 6x^3 + 11x^2 - 18$.
   a. Use the graph to find the real zeros of P
   b. Confirm that the zeros are correct by evaluating the function P at those values.
   c. Express P in terms of linear factors.
   d. Find all zeros of P.

8. Penny says that the equation $x^3 - 8 = 0$ has only one solution, $x = 2$. Use the Fundamental Theorem of Algebra to explain to her why she is incorrect.

9. Roger says that the equation $x^2 - 12x + 36 = 0$ has only one solution, 6. Regina says Roger is wrong and that the Fundamental Theorem of Algebra guarantees that a quadratic equation must have two solutions. Who is correct and why?