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Polynomial, Rational, and Radical Relationships

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Algebra II • Module 1

Polynomial, Rational, and Radical Relationships

OVERVIEW

In this module, students draw on their foundation of the analogies between polynomial arithmetic and base-ten computation, focusing on properties of operations, particularly the distributive property (**A-SSE.B.2**, **A-APR.A.1**). Students connect multiplication of polynomials with multiplication of multi-digit integers, and division of polynomials with long division of integers (**A-APR.A.1**, **A-APR.D.6**). Students identify zeros of polynomials, including complex zeros of quadratic polynomials, and make connections between zeros of polynomials and solutions of polynomial equations (**A-APR.B.3**). Students explore the role of factoring, as both an aid to the algebra and to the graphing of polynomials (**A-SSE.2**, **A-APR.B.2**, **A-APR.B.3**, **F-IF.C.7c**). Students continue to build upon the reasoning process of solving equations as they solve polynomial, rational, and radical equations, as well as linear and non-linear systems of equations (**A-REI.A.1**, **A-REI.A.2**, **A-REI.C.6**, **A-REI.C.7**). The module culminates with the fundamental theorem of algebra as the ultimate result in factoring. Students pursue connections to applications in prime numbers in encryption theory, Pythagorean triples, and modeling problems.

An additional theme of this module is that the arithmetic of rational expressions is governed by the same rules as the arithmetic of rational numbers. Students use appropriate tools to analyze the key features of a graph or table of a polynomial function and relate those features back to the two quantities that the function is modeling in the problem (**F-IF.C.7c**).

Focus Standards

Reason quantitatively and use units to solve problems.

N-Q.A.2² Define appropriate quantities for the purpose of descriptive modeling.*

Perform arithmetic operations with complex numbers.

N-CN.A.1 Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.

² This standard will be assessed in Algebra II by ensuring that some modeling tasks (involving Algebra II content or securely held content from previous grades and courses) require the student to create a quantity of interest in the situation being described (i.e., this is not provided in the task). For example, in a situation involving periodic phenomena, the student might autonomously decide that amplitude is a key variable in a situation and then choose to work with peak amplitude.

- N-CN.A.2** Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

Use complex numbers in polynomial identities and equations.

- N-CN.C.7** Solve quadratic equations with real coefficients that have complex solutions.

Interpret the structure of expressions.

- A-SSE.A.2³** Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.*

Understand the relationship between zeros and factors of polynomials.

- A-APR.B.2⁴** Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a , the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.
- A-APR.B.3⁵** Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

Use polynomial identities to solve problems.

- A-APR.C.4** Prove⁶ polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ can be used to generate Pythagorean triples.*

Rewrite rational expressions.

- A-APR.D.6⁷** Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.

Understand solving equations as a process of reasoning and explain the reasoning.

- A-REI.A.1⁸** Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

³ In Algebra II, tasks are limited to polynomial, rational, or exponential expressions. Examples: see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$. In the equation $x^2 + 2x + 1 + y^2 = 9$, see an opportunity to rewrite the first three terms as $(x+1)^2$, thus recognizing the equation of a circle with radius 3 and center $(-1, 0)$. See $(x^2 + 4)/(x^2 + 3)$ as $((x^2+3) + 1)/(x^2+3)$, thus recognizing an opportunity to write it as $1 + 1/(x^2 + 3)$.

⁴ Include problems that involve interpreting the Remainder Theorem from graphs and in problems that require long division.

⁵ In Algebra II, tasks include quadratic, cubic, and quadratic polynomials and polynomials for which factors are not provided. For example, find the zeros of $(x^2 - 1)(x^2 + 1)$.

⁶ Prove and apply (in preparation for Regents Exams).

⁷ Include rewriting rational expressions that are in the form of a complex fraction.

⁸ In Algebra II, tasks are limited to simple rational or radical equations.

- A-REI.A.2** Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Solve equations and inequalities in one variable.

- A-REI.B.4⁹** Solve quadratic equations in one variable.
- b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b .

Solve systems of equations.

- A-REI.C.6¹⁰** Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.
- A-REI.C.7** Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. *For example, find the points of intersection between the line $y = -3x$ and the circle $x^2 + y^2 = 3$.*

Analyze functions using different representations.

- F-IF.C.7** Graph functions expressed symbolically and show key features of the graph (by hand in simple cases and using technology for more complicated cases).^{*}
- c. Graph polynomial functions, identifying zeros when suitable factorizations are available and showing end behavior.

Translate between the geometric description and the equation for a conic section.

- G-GPE.A.2** Derive the equation of a parabola given a focus and directrix.

Extension Standards

The (+) standards below are provided as an extension to Module 1 of the Algebra II course to provide coherence to the curriculum. They are used to introduce themes and concepts that will be fully covered in the Precalculus course. *Note: None of the (+) standards below will be assessed on the Regents Exam or PARCC Assessments until Precalculus.*

Use complex numbers in polynomial identities and equations.

- N-CN.C.8** (+) Extend polynomial identities to the complex numbers. *For example, rewrite $x^2 + 4$ as $(x + 2i)(x - 2i)$.*

⁹ In Algebra II, in the case of equations having roots with nonzero imaginary parts, students write the solutions as $a \pm bi$, where a and b are real numbers.

¹⁰ In Algebra II, tasks are limited to 3×3 systems.

- N-CN.C.9** (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

Rewrite rational expressions.

- A-APR.C.7** (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Foundational Standards

Use properties of rational and irrational numbers.

- N-RN.B.3** Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Reason quantitatively and use units to solve problems.

- N-Q.A.1** Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.*

Interpret the structure of expressions.

- A-SSE.A.1** Interpret expressions that represent a quantity in terms of its context.*
- Interpret parts of an expression, such as terms, factors, and coefficients.
 - Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1 + r)^n$ as the product of P and a factor not depending on P .*

Write expressions in equivalent forms to solve problems.

- A-SSE.B.3** Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.*
- Factor a quadratic expression to reveal the zeros of the function it defines.

Perform arithmetic operations on polynomials.

- A-APR.A.1** Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Create equations that describe numbers or relationships.

- A-CED.A.1** Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.**
- A-CED.A.2** Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.*
- A-CED.A.3** Represent constraints by equations or inequalities and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.**
- A-CED.A.4** Rearrange formulas to highlight a quantity of interest, using the same reasoning used in solving equations. *For example, rearrange Ohm's law $V = IR$ to highlight resistance R .**

Solve equations and inequalities in one variable.

- A-REI.B.3** Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.
- A-REI.B.4** Solve quadratic equations in one variable.
- a. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.

Solve systems of equations.

- A-REI.C.5** Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.

Represent and solve equations and inequalities graphically.

- A-REI.D.10** Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).
- A-REI.D.11** Explain why the x -coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.*

Translate between the geometric description and the equation for a conic section.

- G-GPE.A.1** Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

Focus Standards for Mathematical Practice

- MP.1** **Make sense of problems and persevere in solving them.** Students discover the value of equating factored terms of a polynomial to zero as a means of solving equations involving polynomials. Students solve rational equations and simple radical equations, while considering the possibility of extraneous solutions and verifying each solution before drawing conclusions about the problem. Students solve systems of linear equations and linear and quadratic pairs in two variables. Further, students come to understand that the complex number system provides solutions to the equation $x^2 + 1 = 0$ and higher-degree equations.
- MP.2** **Reason abstractly and quantitatively.** Students apply polynomial identities to detect prime numbers and discover Pythagorean triples. Students also learn to make sense of remainders in polynomial long division problems.
- MP.4** **Model with mathematics.** Students use primes to model encryption. Students transition between verbal, numerical, algebraic, and graphical thinking in analyzing applied polynomial problems. Students model a cross-section of a riverbed with a polynomial, estimate fluid flow with their algebraic model, and fit polynomials to data. Students model the locus of points at equal distance between a point (focus) and a line (directrix) discovering the parabola.
- MP.7** **Look for and make use of structure.** Students connect long division of polynomials with the long-division algorithm of arithmetic and perform polynomial division in an abstract setting to derive the standard polynomial identities. Students recognize structure in the graphs of polynomials in factored form and develop refined techniques for graphing. Students discern the structure of rational expressions by comparing to analogous arithmetic problems. Students perform geometric operations on parabolas to discover congruence and similarity.
- MP.8** **Look for and express regularity in repeated reasoning.** Students understand that polynomials form a system analogous to the integers. Students apply polynomial identities to detect prime numbers and discover Pythagorean triples. Students recognize factors of expressions and develop factoring techniques. Further, students understand that all quadratics can be written as a product of linear factors in the complex realm.

Terminology

New or Recently Introduced Terms

- **A Square Root of a Number** (A *square root of a number* x is a number whose square is x . In symbols, a square root of x is a number a such that $a^2 = x$. Negative numbers do not have any square roots, zero has exactly one square root, and positive numbers have two square roots.)
- **The Square Root of a Number** (Every positive real number x has a unique positive square root called the *square root* or *principle square root* of x ; it is denoted \sqrt{x} . The square root of zero is zero.)
- **Pythagorean Triple** (A *Pythagorean triple* is a triplet of positive integers (a, b, c) such that $a^2 + b^2 = c^2$. The triplet $(3, 4, 5)$ is a Pythagorean triple but $(1, 1, \sqrt{2})$ is not, even though the numbers are side lengths of an isosceles right triangle.)

- **End Behavior** (Let f be a function whose domain and range are subsets of the real numbers. The end behavior of a function f is a description of what happens to the values of the function
 - as x approaches positive infinity, and
 - as x approaches negative infinity.)
- **Even Function** (Let f be a function whose domain and range is a subset of the real numbers. The function f is called *even* if the equation, $f(x) = f(-x)$, is true for every number x in the domain. Even-degree polynomial functions are sometimes even functions, such as $f(x) = x^{10}$, and sometimes not, such as $g(x) = x^2 - x$.)
- **Odd Function** (Let f be a function whose domain and range is a subset of the real numbers. The function f is called *odd* if the equation, $f(-x) = -f(x)$, is true for every number x in the domain. Odd-degree polynomial functions are sometimes odd functions, such as $f(x) = x^{11}$, and sometimes not, such as $h(x) = x^3 - x^2$.)
- **Rational Expression** (A *rational expression* is either a numerical expression or a variable symbol, or the result of placing two previously generated rational expressions into the blanks of the addition operator ($_ + _$), the subtraction operator ($_ - _$), the multiplication operator ($_ \times _$), or the division operator ($_ \div _$).)
- **Parabola** (A *parabola* with *directrix line* L and *focus point* F is the set of all points in the plane that are equidistant from the point F and line L .)
- **Axis of Symmetry** (The *axis of symmetry of a parabola* given by a focus point and a directrix is the perpendicular line to the directrix that passes through the focus.)
- **Vertex of a Parabola** (The *vertex of a parabola* is the point where the axis of symmetry intersects the parabola.)
- **Dilation at the Origin** (A dilation at the origin D_k is a horizontal scaling by $k > 0$ followed by a vertical scaling by the same factor k . In other words, this dilation of the graph of $y = f(x)$ is the graph of the equation $y = kf\left(\frac{1}{k}x\right)$. A dilation at the origin is a special type of a dilation.)

Familiar Terms and Symbols¹¹

- Sequence
- Arithmetic Sequence
- Numerical Symbol
- Variable Symbol
- Algebraic Expression
- Numerical Expression
- Polynomial Expression
- Monomial
- Degree of a Monomial
- Binomial

¹¹ These are terms and symbols students have seen previously.

- Trinomial
- Coefficient of a Monomial
- Terms of a Polynomial
- Like Terms of a Polynomial
- Standard Form of a Polynomial in One Variable
- Degree of a Polynomial in One Variable
- Equivalent Polynomial Expressions
- Polynomial Identity
- Function
- Polynomial Function
- Degree of a Polynomial Function
- Constant Function
- Linear Function
- Quadratic Function
- Discriminant of a Quadratic Function
- Cubic Function
- Zeros or Roots of a Function
- Increasing/Decreasing
- Relative Maximum
- Relative Minimum
- Graph of f
- Graph of $y = f(x)$

Suggested Tools and Representations

- Graphing Calculator
- Wolfram Alpha Software
- Geometer's Sketchpad Software

Assessment Summary

| Assessment Type | Administered | Format | Standards Addressed |
|-------------------------------|---------------|----------------------------------|--|
| Mid-Module Assessment Task | After Topic B | Constructed response with rubric | N-Q.A.2, A-SSE.A.2, A-APR.B.2, A-APR.B.3, A-APR.C.4, A-REI.A.1, A-REI.B.4b, F-IF.C.7c |
| End-of-Module Assessment Task | After Topic D | Constructed response with rubric | N-Q.A.2, A-SSE.A.2, A-APR.B.2, A-APR.B.3, A-APR.C.4, A-APR.D.6, A-REI.A.1, A-REI.A.2, A-REI.B.4b, A-REI.C.6, A-REI.C.7, F-IF.C.7c, G-GPE.A.2 |