Table of Contents[1]

Integer Exponents and Scientific Notation

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[1] Each lesson is ONE day, and ONE day is considered a 45-minute period.

Grade 8 • Module 1

Integer Exponents and Scientific Notation

OVERVIEW

In Module 1, students' knowledge of operations on numbers will be expanded to include operations on numbers in integer exponents. Module 1 also builds on students' understanding from previous grades with regard to transforming expressions. Students were introduced to exponential notation in Grade 5 as they used whole number exponents to denote powers of ten (**5.NBT.A.2**). In Grade 6, students expanded the use of exponents to include bases other than ten as they wrote and evaluated exponential expressions limited to whole-number exponents (**6.EE.A1**). Students made use of exponents again in Grade 7 as they learned formulas for the area of a circle (**7.G.B.4**) and volume (**7.G.B.6**).

In this module, students build upon their foundation with exponents as they make conjectures about how zero and negative exponents of a number should be defined and prove the properties of integer exponents (**8.EE.A.1**). These properties are codified into three laws of exponents. They make sense out of very large and very small numbers, using the number line model to guide their understanding of the relationship of those numbers to each other (**8.EE.A.3**).

Having established the properties of integer exponents, students learn to express the magnitude of a positive number through the use of scientific notation and to compare the relative size of two numbers written in scientific notation (**8.EE.A.3**). Students explore use of scientific notation and choose appropriately sized units as they represent, compare, and make calculations with very large quantities, such as the U.S. national debt, the number of stars in the universe, and the mass of planets; and very small quantities, such as the mass of subatomic particles (**8.EE.A.4**).

The Mid-Module Assessment follows Topic A. The End-of-Module Assessment follows Topic B.

Focus Standards

Work with radicals and integer exponents.

8.EE.A.1   Know and apply the properties of integer exponents to generate equivalent numerical expressions. *For example, $3^2 \times 3^{-5} = 3^{-3} = 1/3^3 = 1/27$.*

8.EE.A.3   Use numbers expressed in the form of a single digit times an integer power of 10 to estimate very large or very small quantities, and to express how many times as much one is than the other. *For example, estimate the population of the United States as 3×10^8 and the population of the world as 7×10^9, and determine that the world population is more than 20 times larger.*

8.EE.A.4   Perform operations with numbers expressed in scientific notation, including problems where both decimal and scientific notation are used. Use scientific notation and choose units of appropriate size for measurements of very large or very small quantities (e.g., use millimeters per year for seafloor spreading). Interpret scientific notation that has been generated by technology.

Foundational Standards

Understand the place value system.

5.NBT.A.2   Explain patterns in the number of zeros of the product when multiplying a number by powers of 10, and explain patterns in the placement of the decimal point when a decimal is multiplied or divided by a power of 10. Use whole-number exponents to denote powers of 10.

Apply and extend previous understandings of arithmetic to algebraic expressions.

6.EE.A.1   Write and evaluate numerical expressions involving whole-number exponents.

Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.

7.G.B.4   Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle.

7.G.B.6   Solve real-world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.

Focus Standards for Mathematical Practice

MP.2   **Reason abstractly and quantitatively.** Students use concrete numbers to explore the properties of numbers in exponential form and then prove that the properties are true for all positive bases and all integer exponents using symbolic representations for bases and exponents. As lessons progress, students use symbols to represent integer exponents and make sense of those quantities in problem situations. Students refer to symbolic notation in order to contextualize the requirements and limitations of given statements (e.g., letting m, n represent positive integers, letting a, b represent all integers, both with respect to the properties of exponents).

MP.3   **Construct viable arguments and critique the reasoning of others.** Students reason through the acceptability of definitions and proofs (e.g., the definitions of x^0 and x^{-b} for all integers b and positive integers x). New definitions, as well as proofs, require students to analyze situations and break them into cases. Further, students examine the implications of these definitions and proofs on existing properties of integer exponents. Students keep the goal of a logical argument in mind while attending to details that develop during the reasoning process.

MP.6   **Attend to precision.** Beginning with the first lesson on exponential notation, students are required to attend to the definitions provided throughout the lessons and the limitations of symbolic statements, making sure to express what they mean clearly. Students are provided a hypothesis, such as $x < y$, for positive integers x, y, and then asked to evaluate whether a statement, like $-2 < 5$, contradicts this hypothesis.

MP.7   **Look for and make use of structure.** Students understand and make analogies to the distributive law as they develop properties of exponents. Students will know $x^m \cdot x^n = x^{m+n}$ as an analog of $mx + nx = (m+n)x$ and $(x^m)^n = x^{m \times n}$ as an analog of $n \times (m \times x) = (n \times m) \times x$.

MP.8   **Look for and express regularity in repeated reasoning.** While evaluating the cases developed for the proofs of laws of exponents, students identify when a statement must be proved or if it has already been proven. Students see the use of the laws of exponents in application problems and notice the patterns that are developed in problems.

Terminology

New or Recently Introduced Terms

- **Scientific Notation** (The scientific notation *for a finite decimal* is the representation of that decimal as the product of a decimal s and a power of 10, where s satisfies the property that it is at least 1, but smaller than 10, or in symbolic notation, $1 \le s < 10$. For example, the scientific notation for 192.7 is 1.927×10^2.)

- **Order of Magnitude** (The order of magnitude *of a finite decimal* is the exponent in the power of 10 when that decimal is expressed in scientific notation. For example, the order of magnitude of 192.7 is 2 because when 192.7 is expressed in scientific notation as 1.927×10^2, 2 is the exponent of 10^2. *Sometimes we also include the number* 10 *in the definition of order of magnitude* and say that the order of magnitude of 192.7 is 10^2.)

Module 1:   Integer Exponents and Scientific Notation
Date: 5/23/14

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Familiar Terms and Symbols[2]

- Exponential Notation
- Base, Exponent, Power
- Integer
- Whole Number
- Expanded Form (of decimal numbers)
- Square and Cube (of a number)
- Equivalent Fractions

Suggested Tools and Representations

- Scientific Calculator

Rapid White Board Exchanges

Implementing a RWBE requires that each student be provided with a personal white board, a white board marker, and a means of erasing his or her work. An economic choice for these materials is to place sheets of card stock inside sheet protectors to use as the personal white boards and to cut sheets of felt into small squares to use as erasers.

A RWBE consists of a sequence of 10 to 20 problems on a specific topic or skill that starts out with a relatively simple problem and progressively gets more difficult. The teacher should prepare the problems in a way that allows him or her to reveal them to the class one at a time. A flip chart or PowerPoint presentation can be used, or the teacher can write the problems on the board and either cover some with paper or simply write only one problem on the board at a time.

The teacher reveals, and possibly reads aloud, the first problem in the list and announces, "Go". Students work the problem on their personal white boards as quickly as possible and hold their work up for their teacher to see their answers as soon as they have the answer ready. The teacher gives immediate feedback to each student, pointing and/or making eye contact with the student and responding with an affirmation for correct work such as, "Good job!", "Yes!", or "Correct!", or responding with guidance for incorrect work such as "Look again," "Try again," "Check your work," etc. In the case of the RWBE, it is not recommended that the feedback include the name of the student receiving the feedback.

If many students have struggled to get the answer correct, go through the solution of that problem as a class before moving on to the next problem in the sequence. Fluency in the skill has been established when the class is able to go through each problem in quick succession without pausing to go through the solution of each problem individually. If only one or two students have not been able to successfully complete a problem, it is appropriate to move the class forward to the next problem without further delay; in this case find a time to provide remediation to that student before the next fluency exercise on this skill is given.

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[2] These are terms and symbols students have seen previously.

Sprints

Sprints are designed to develop fluency. They should be fun, adrenaline-rich activities that intentionally build energy and excitement. A fast pace is essential. During Sprint administration, teachers assume the role of athletic coaches. A rousing routine fuels students' motivation to do their personal best. Student recognition of increasing success is critical, and so every improvement is acknowledged. (See the Sprint Delivery Script for the suggested means of acknowledging and celebrating student success.)

One Sprint has two parts with closely-related problems on each. Students complete the two parts of the Sprint in quick succession with the goal of improving on the second part, even if only by one more.

Sprints are not to be used for a grade. Thus, there is no need for students to write their names on the Sprints. The low-stakes nature of the exercise means that even students with allowances for extended time can participate. When a particular student finds the experience undesirable, it is recommended that the student be allowed to opt-out and take the Sprint home. In this case, it is ideal if the student has a regular opportunity to express the desire to opt-in.

With practice, the Sprint routine takes about 8 minutes.

Sprint Delivery Script

Gather the following: stopwatch, a copy of Sprint A for each student, a copy of Sprint B for each student, answers for Sprint A and Sprint B. The following delineates a script for delivery of a pair of Sprints.

This sprint covers: *topic.*

Do not look at the Sprint, keep it turned face down on your desk.

There are xx problems on the Sprint. You will have 60 seconds. Do as many as you can. I do not expect any of you to finish.

On your mark, get set, GO.

60 seconds of silence.

STOP. Circle the last problem you completed.

I will read the answers. You say "YES" if your answer matches. Mark the ones you have wrong. Don't try to correct them.

Energetically, rapid-fire call the answers ONLY.

Stop reading answers after there are no more students answering, "Yes."

Fantastic! Count the number you have correct, and write it on the top of the page. This is your personal goal for Sprint B.

Raise your hand if you have 1 or more correct. 2 or more, 3 or more...

Let us all applaud our runner up, [insert name] with x correct. And let us applaud our winner, [insert name], with x correct.

You have a few minutes to finish up the page and get ready for the next Sprint.

Students are allowed to talk and ask for help; let this part last as long as most are working seriously.

Stop working. I will read the answers again so you can check your work. You say "YES" if your answer matches.

Energetically, rapid-fire call the answers ONLY.

Optionally, ask students to stand and lead them in an energy-expanding exercise that also keeps the brain going. Examples are jumping jacks or arm circles, etc. while counting by 15's starting at 15, going up to 150 and back down to 0. You can follow this first exercise with a cool down exercise of a similar nature, such as calf raises with counting by one-sixths $\left(\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1 \dots\right)$.

Hand out the second Sprint and continue reading the script.

Keep the Sprint face down on your desk.

There are xx problems on the Sprint. You will have 60 seconds. Do as many as you can. I do not expect any of you to finish.

On your mark, get set, GO.

60 seconds of silence.

STOP. Circle the last problem you completed.

I will read the answers. You say "YES" if your answer matches. Mark the ones you have wrong. Don't try to correct them.

Quickly read the answers ONLY.

Count the number you have correct, and write it on the top of the page.

Raise your hand if you have 1 or more correct. 2 or more, 3 or more, ...

Let us all applaud our runner up, [insert name] with x correct. And let us applaud our winner, {insert name], with x correct.

Write the amount by which your score improved at the top of the page.

Raise your hand if you improved your score by 1 or more. 2 or more, 3 or more...

Let us all applaud our runner up for most improved, [insert name]. And let us applaud our winner for most improved, [insert name].

You can take the Sprint home and finish it if you want.

Assessment Summary

Assessment Type	Administered	Format	Standards Addressed
Mid-Module Assessment Task	After Topic A	Constructed response with rubric	8.EE.A.1
End-of-Module Assessment Task	After Topic B	Constructed response with rubric	8.EE.A.3, 8.EE.A.4

Topic A:

Exponential Notation and Properties of Integer Exponents

8.EE.A.1

Focus Standard:	8.EE.A.1	Know and apply the properties of integer exponents to generate equivalent numerical expressions. *For example, $3^2 \times 3^{-5} = 3^{-3} = 1/3^3 = 1/27$.*
Instructional Days:	6	
	Lesson 1:	Exponential Notation (S)[1]
	Lesson 2:	Multiplication and Division of Numbers in Exponential Form (S)
	Lesson 3:	Numbers in Exponential Form Raised to a Power (S)
	Lesson 4:	Numbers Raised to the Zeroth Power (E)
	Lesson 5:	Negative Exponents and the Laws of Exponents (S)
	Lesson 6:	Proofs of Laws of Exponents (S)

In Topic A, students begin by learning the precise definition of exponential notation where the exponent is restricted to being a positive integer. In Lessons 2 and 3, students discern the structure of exponents by relating multiplication and division of expressions with the same base to combining like terms using the distributive property and by relating multiplying three factors using the associative property to raising a power to a power.

Lesson 4 expands the definition of exponential notation to include what it means to raise a nonzero number to a zero power; students verify that the properties of exponents developed in Lessons 2 and 3 remain true. Properties of exponents are extended again in Lesson 5 when a positive integer, raised to a negative exponent, is defined. In Lesson 5, students accept the properties of exponents as true for all integer exponents and are shown the value of learning them; in other words, if the three properties of exponents are known, then facts about dividing numbers in exponential notation with the same base and raising fractions to a power are also known.

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[1] Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson

Topic A culminates in Lesson 6 when students work to prove the laws of exponents for all integer exponents. Throughout Topic A, students generate equivalent numerical expressions by applying properties of integer exponents, first with positive integer exponents, then with whole number exponents, and concluding with integer exponents in general.

Lesson 1: Exponential Notation

Student Outcomes

- Students know what it means for a number to be raised to a power and how to represent the repeated multiplication symbolically.
- Students know the reason for some bases requiring parentheses.

Lesson Notes

This lesson is foundational for the topic of properties of integer exponents. However, if your students have already mastered the skills in this lesson, it is your option to move forward and begin with Lesson 2.

Classwork

Discussion (15 minutes)

When we add 5 copies of 3; we devise an abbreviation – a new notation, for this purpose:

$$3 + 3 + 3 + 3 + 3 = 5 \times 3$$

Now if we multiply the same number, 3, with itself 5 times, how should we abbreviate this?

$$3 \times 3 \times 3 \times 3 \times 3 = ?$$

MP.2 & MP.7

Allow students to make suggestions, see sidebar for scaffolds.

$$3 \times 3 \times 3 \times 3 \times 3 = 3^5$$

Similarly, we also write $3^3 = 3 \times 3 \times 3$; $3^4 = 3 \times 3 \times 3 \times 3$; etc.

We see that when we add 5 copies of 3, we write 5×3, but when we multiply 5 copies of 3, we write 3^5. Thus, the "multiplication by 5" in the context of addition corresponds exactly to the superscript 5 in the context of multiplication.

Make students aware of the correspondence between addition and multiplication because what they know about *repeated addition* will help them learn exponents as *repeated multiplication* as we go forward.

> *Scaffolding:*
>
> Remind students of their previous experiences:
>
> - The square of a number, e.g., 3×3 is denoted by 3^2.
> - From the expanded form of a whole number, we also learned, e.g., 10^3 stands for $10 \times 10 \times 10$.

5^6 means $5 \times 5 \times 5 \times 5 \times 5 \times 5$ and $\left(\frac{9}{7}\right)^4$ means $\frac{9}{7} \times \frac{9}{7} \times \frac{9}{7} \times \frac{9}{7}$.

You have seen this kind of notation before; it is called exponential notation. In general, for any number x and any positive integer n,

$$x^n = \underbrace{(x \cdot x \cdots x)}_{n\ times}.$$

The number x^n is called x raised to the nth power, where n is the exponent of x in x^n and x is the base of x^n.

Examples 1–5

Work through Examples 1–5 as a group, supplement with additional examples if needed.

Example 1

$$5 \times 5 \times 5 \times 5 \times 5 \times 5 = 5^6$$

Example 2

$$\frac{9}{7} \times \frac{9}{7} \times \frac{9}{7} \times \frac{9}{7} = \left(\frac{9}{7}\right)^4$$

Example 3

$$\left(-\frac{4}{11}\right)^3 = \left(-\frac{4}{11}\right) \times \left(-\frac{4}{11}\right) \times \left(-\frac{4}{11}\right)$$

Example 4

$$(-2)^6 = (-2) \times (-2) \times (-2) \times (-2) \times (-2) \times (-2)$$

Example 5

$$3.8^4 = 3.8 \times 3.8 \times 3.8 \times 3.8$$

- ▪ Notice the use of parentheses in Examples 2, 3, and 4. Do you know why?
  - □ *In cases where the base is either fractional or negative, it prevents ambiguity about which portion of the expression is going to be multiplied repeatedly.*
- ▪ Suppose n is a fixed **positive integer**, then 3^n; by definition, is $3^n = \underbrace{(3 \times \cdots \times 3)}_{n \text{ times}}$.
- ▪ Again, if n is a fixed positive integer, then by definition,

$$7^n = \underbrace{(7 \times \cdots \times 7)}_{n \text{ times}},$$

$$\left(\frac{4}{5}\right)^n = \underbrace{\left(\frac{4}{5} \times \cdots \times \frac{4}{5}\right)}_{n \text{ times}},$$

$$(-2.3)^n = \underbrace{\left((-2.3) \times \cdots \times (-2.3)\right)}_{n \text{ times}}.$$

> *Note to Teacher:* If students ask about values of n that are not positive integers, let them know that positive and negative fractional exponents will be introduced in Algebra II and that negative *integer* exponents will be discussed in Lesson 4 of this module.

MP.6

- ▪ In general, for any number x, $x^1 = x$, and for any positive integer $n > 1$, x^n is by definition,

$$x^n = \underbrace{(x \cdot x \cdots x)}_{n \text{ times}}.$$

- ▪ The number x^n is called x **raised to the nth power**, n is the **exponent** of x in x^n and x is the **base** of x^n.
- ▪ x^2 is called the **square** of x, and x^3 is its **cube**.
- ▪ You have seen this kind of notation before when you gave the expanded form of a whole number for powers of 10; it is called **exponential notation**.

Exercises 1–10 (5 minutes)

Students complete independently and check answers before moving on.

Exercise 1

$$\underbrace{4 \times \cdots \times 4}_{7 \text{ times}} = 4^7$$

Exercise 2

$$\underbrace{3.6 \times \cdots \times 3.6}_{\underline{\hspace{1cm}} \text{ times}} = 3.6^{47}$$

47 times

Exercise 3

$$\underbrace{(-11.63) \times \cdots \times (-11.63)}_{34 \text{ times}} = (-11.63)^{34}$$

Exercise 4

$$\underbrace{12 \times \cdots \times 12}_{\underline{\hspace{1cm}} \text{ times}} = 12^{15}$$

15 times

Exercise 5

$$\underbrace{(-5) \times \cdots \times (-5)}_{10 \text{ times}} = (-5)^{10}$$

Exercise 6

$$\underbrace{\frac{7}{2} \times \cdots \times \frac{7}{2}}_{21 \text{ times}} = \left(\frac{7}{2}\right)^{21}$$

Exercise 7

$$\underbrace{(-13) \times \cdots \times (-13)}_{6 \text{ times}} = (-13)^6$$

Exercise 8

$$\underbrace{\left(-\frac{1}{14}\right) \times \cdots \times \left(-\frac{1}{14}\right)}_{10 \text{ times}} = \left(-\frac{1}{14}\right)^{10}$$

Exercise 9

$$\underbrace{x \cdot x \cdots x}_{185 \text{ times}} = x^{185}$$

Exercise 10

$$\underbrace{x \cdot x \cdots x}_{\underline{\hspace{1cm}} \text{ times}} = x^n$$

n times

Exercises 11–14 (15 minutes)

Allow students to complete Exercises 11–14 individually or in a small group.

- When a negative number is raised to an odd power, what is the sign of the result?
- When a negative number is raised to an even power, what is the sign of the result?

Make the point that when a negative number is raised to an odd power, the sign of the answer is negative. Conversely, if a negative number is raised to an even power, the sign of the answer is positive.

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Lesson 1:   Exponential Notation
Date: 5/23/14

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Exercise 11

Will these products be positive or negative? How do you know?

$$\underbrace{(-1) \times (-1) \times \cdots \times (-1)}_{12 \text{ times}} = (-1)^{12}$$

This product will be positive. Students may state that they computed the product and it was positive; if they say that, let them show their work. Students may say that the answer is positive because the exponent is positive; this would not be acceptable in view of the next example.

$$\underbrace{(-1) \times (-1) \times \cdots \times (-1)}_{13 \text{ times}} = (-1)^{13}$$

This product will be negative. Students may state that they computed the product and it was negative; if so, they must show their work. Based on the discussion that occurred during the last problem, you may need to point out that a positive exponent does not always result in a positive product.

The two problems in Exercise 12 force the students to think beyond the computation level. If students have trouble, go back to the previous two problems and have them discuss in small groups what an even number of negative factors yields and what an odd number of negative factors yields.

Exercise 12

Is it necessary to do all of the calculations to determine the sign of the product? Why or why not?

$$\underbrace{(-5) \times (-5) \times \cdots \times (-5)}_{95 \text{ times}} = (-5)^{95}$$

Students should state that an odd number of negative factors yields a negative product.

$$\underbrace{(-1.8) \times (-1.8) \times \cdots \times (-1.8)}_{122 \text{ times}} = (-1.8)^{122}$$

Students should state that an even number of negative factors yields a positive product.

Exercise 13

Fill in the blanks about whether the number is positive or negative.

If n is a positive even number, then $(-55)^n$ is _positive._

If n is a positive odd number, then $(-72.4)^n$ is _negative._

Exercise 14

Josie says that $\underbrace{(-15) \times \cdots \times (-15)}_{6 \text{ times}} = -15^6$. Is she correct? How do you know?

Students should state that Josie is not correct for the following two reasons: (1) They just stated that an even number of factors yields a positive product, and this conflicts with the answer Josie provided, and (2) the notation is used incorrectly because, as is, the answer is the negative of 15^6, instead of the product of 6 copies of -15. The base is (-15). Recalling the discussion at the beginning of the lesson, when the base is negative it should be written clearly through the use of parentheses. Have students write the answer correctly.

Closing (5 minutes)

- Why should we bother with exponential notation? Why not just write out the multiplication?

Engage the class in discussion, but make sure that they get to know at least the following two reasons:

1. Like all good notation, exponential notation saves writing.

2. Exponential notation is used for recording scientific measurements of very large and very small quantities. It is indispensable for the clear indication of the magnitude of a number (see Lessons 10–13).

   - Here is an example of the labor saving aspect of the exponential notation: Suppose a colony of bacteria doubles in size every 8 hours for a few days under tight laboratory conditions. If the initial size is B, what is the size of the colony after 2 days?

     - *In 2 days, there are six 8-hour periods; therefore, the size will be $2^6 B$.*

Give more examples if time allows as a lead in to Lesson 2. Example situations: exponential decay with respect to heat transfer, vibrations, ripples in a pond, or exponential growth with respect to interest on a bank deposit after some years have passed.

Exit Ticket (5 minutes)

Name _____   Date_____

Lesson 1: Exponential Notation

Exit Ticket

1.   a.   Express the following in exponential notation:

$$\underbrace{(-13) \times \cdots \times (-13)}_{35 \text{ times}}.$$

   b.   Will the product be positive or negative?

2.   Fill in the blank:

$$\underbrace{\frac{2}{3} \times \cdots \times \frac{2}{3}}_{\underline{\hspace{1cm}}\text{times}} = \left(\frac{2}{3}\right)^4$$

3.   Arnie wrote:

$$\underbrace{(-3.1) \times \cdots \times (-3.1)}_{4 \text{ times}} = -3.1^4$$

   Is Arnie correct in his notation? Why or why not?

Exit Ticket Sample Solutions

1. a. Express the following in exponential notation:

$$\underbrace{(-13) \times \cdots \times (-13)}_{35 \text{ times}}$$

$$(-13)^{35}$$

b. Will the product be positive or negative?

The product will be negative.

2. Fill in the blank:

$$\underbrace{\frac{2}{3} \times \cdots \times \frac{2}{3}}_{\underline{\quad} \text{times}} = \left(\frac{2}{3}\right)^4$$

4 *times*

3. Arnie wrote:

$$\underbrace{(-3.1) \times \cdots \times (-3.1)}_{4 \text{ times}} = -3.1^4$$

Is Arnie correct in his notation? Why or why not?

Arnie is not correct. The base, -3.1, should be in parentheses to prevent ambiguity; at present the notation is not correct.

Problem Set Sample Solutions

1. Use what you know about exponential notation to complete the expressions below.

$$\underbrace{(-5) \times \cdots \times (-5)}_{17 \text{ times}} = (-5)^{17}$$

$$\underbrace{3.7 \times \cdots \times 3.7}_{\underline{\quad} \text{times}} = 3.7^{19}$$

19 *times*

$$\underbrace{7 \times \cdots \times 7}_{\underline{\quad} \text{times}} = 7^{45}$$

$$\underbrace{6 \times \cdots \times 6}_{4 \text{ times}} = 6^4$$

45 *times*

$$\underbrace{4.3 \times \cdots \times 4.3}_{13 \text{ times}} = 4.3^{13}$$

$$\underbrace{(-1.1) \times \cdots \times (-1.1)}_{9 \text{ times}} = (-1.1)^9$$

$$\underbrace{\left(\frac{2}{3}\right) \times \cdots \times \left(\frac{2}{3}\right)}_{19 \text{ times}} = \left(\frac{2}{3}\right)^{19}$$

$$\underbrace{\left(-\frac{11}{5}\right) \times \cdots \times \left(-\frac{11}{5}\right)}_{\underline{\quad} \text{times}} = \left(-\frac{11}{5}\right)^x$$

***x* times**

$$\underbrace{(-12) \times \cdots \times (-12)}_{\underline{\quad} \text{times}} = (-12)^{15}$$

$$\underbrace{a \times \cdots \times a}_{m \text{ times}} = a^m$$

15 *times*

2.   Write an expression with (-1) as its base that will produce a positive product.

Accept any answer with (-1) to an exponent that is even.

3.   Write an expression with (-1) as its base that will produce a negative product.

Accept any answer with (-1) to an exponent that is odd.

4.   Rewrite each number in exponential notation using 2 as the base.

$8 = 2^3$	$16 = 2^4$	$32 = 2^5$
$64 = 2^6$	$128 = 2^7$	$256 = 2^8$

5.   Tim wrote 16 as $(-2)^4$. Is he correct?

Tim is correct that $16 = (-2)^4$.

6.   Could -2 be used as a base to rewrite 32? 64? Why or why not?

A base of -2 <u>cannot</u> be used to rewrite 32 because $(-2)^5 = -32$. A base of -2 <u>can</u> be used to rewrite 64 because $(-2)^6 = 64$. If the exponent, n, is even, $(-2)^n$ will be positive. If the exponent, n, is odd, $(-2)^n$ cannot be a positive number.

Lesson 2: Multiplication of Numbers in Exponential Form

Student Outcomes

- Students use the definition of exponential notation to make sense of the first law of exponents.
- Students see a rule for simplifying exponential expressions involving division as a consequence of the first law of exponents.
- Students write equivalent numerical and symbolic expressions using the first law of exponents.

Classwork

Discussion (8 minutes)

We have to find out the basic properties of this new concept, "raising a number to a power." There are three simple ones, and we will discuss them in this and the next lesson.

Scaffolding:
- Use concrete numbers for x, m, and n.

- (1) How to multiply different powers of the same number x: if m, n are positive integers, what is $x^m \cdot x^n$?

Let students explore on their own and then in groups: $3^5 \times 3^7$.

- *Answer:* $3^5 \times 3^7 = \underbrace{(3 \times \cdots \times 3)}_{5 \text{ times}} \times \underbrace{(3 \times \cdots \times 3)}_{7 \text{ times}} = \underbrace{(3 \times \cdots \times 3)}_{5+7 \text{ times}} = 3^{5+7}$

In general, if x is any number and m, n are positive integers, then

$$x^m \cdot x^n = x^{m+n}$$

because

$$x^m \times x^n = \underbrace{(x \cdots x)}_{m \text{ times}} \times \underbrace{(x \cdots x)}_{n \text{ times}} = \underbrace{(x \cdots x)}_{m+n \text{ times}} = x^{m+n}.$$

In general, if x is any number and m, n are positive integers, then

$$x^m \cdot x^n = x^{m+n}$$

because

$$x^m \times x^n = \underbrace{(x \cdots x)}_{m \text{ times}} \times \underbrace{(x \cdots x)}_{n \text{ times}} = \underbrace{(x \cdots x)}_{m+n \text{ times}} = x^{m+n}.$$

Examples 1–2

Work through Examples 1 and 2 in the same manner as just shown (supplement with additional examples if needed).

It is preferable to write the answers as an addition of exponents to emphasize the use of the identity. That step should not be left out. That is, $5^2 \times 5^4 = 5^6$ does not have the same instructional value as $5^2 \times 5^4 = 5^{2+4}$.

Example 1

$5^2 \times 5^4 = 5^{2+4}$

Scaffolding:
- Remind students that to remove ambiguity, bases that contain fractions or negative numbers require parentheses.

Example 2

$$\left(-\frac{2}{3}\right)^4 \times \left(-\frac{2}{3}\right)^5 = \left(-\frac{2}{3}\right)^{4+5}$$

- What is the analog of $x^m \cdot x^n = x^{m+n}$ in the context of repeated addition of a number x?

Allow time for a brief discussion.

MP.2 & MP.7

□ *If we add m copies of x and then add to it another n copies of x, we end up adding $m + n$ copies of x By the distributive law:*

$$mx + nx = (m + n)x.$$

This is further confirmation of what we observed at the beginning of Lesson 1: the exponent $m + n$ in x^{m+n} in the context of repeated multiplication corresponds exactly to the $m + n$ in $(m + n)x$ in the context of repeated addition.

Exercises 1–20 (9 minutes)

Students complete Exercises 1–8 independently. Check answers, and then have students complete Exercises 9–20.

Exercise 1

$14^{23} \times 14^8 = 14^{23+8}$

Exercise 2

$(-72)^{10} \times (-72)^{13} = (-72)^{10+13}$

Exercise 3

$5^{94} \times 5^{78} = 5^{94+78}$

Exercise 4

$(-3)^9 \times (-3)^5 = (-3)^{9+5}$

Exercise 5

Let a be a number.

$a^{23} \cdot a^8 = a^{23+8}$

Exercise 6

Let f be a number.

$f^{10} \cdot f^{13} = f^{10+13}$

Exercise 7

Let b be a number.

$b^{94} \cdot b^{78} = b^{94+78}$

Exercise 8

Let x be a positive integer. If $(-3)^9 \times (-3)^x = (-3)^{14}$, what is x?

$x = 5$

In Exercises 9–16, students will need to think about how to rewrite some factors so the bases are the same. Specifically, $2^4 \times 8^2 = 2^4 \times 2^6 = 2^{4+6}$ and $3^7 \times 9 = 3^7 \times 3^2 = 3^{7+2}$. Make clear that these expressions can only be simplified when the bases are the same. Also included is a non-example, $5^4 \times 2^{11}$, that cannot be simplified using this identity. Exercises 17–20 are further applications of the identity.

What would happen if there were more terms with the same base? Write an equivalent expression for each problem.

Exercise 9

$9^4 \times 9^6 \times 9^{13} = 9^{4+6+13}$

Exercise 10

$2^3 \times 2^5 \times 2^7 \times 2^9 = 2^{3+5+7+9}$

Can the following expressions be simplified? If so, write an equivalent expression. If not, explain why not.

Exercise 11

$6^5 \times 4^9 \times 4^3 \times 6^{14} = 4^{9+3} \times 6^{5+14}$

Exercise 14

$2^4 \times 8^2 = 2^4 \times 2^6 = 2^{4+6}$

Exercise 12

$(-4)^2 \cdot 17^5 \cdot (-4)^3 \cdot 17^7 = (-4)^{2+3} \cdot 17^{5+7}$

Exercise 15

$3^7 \times 9 = 3^7 \times 3^2 = 3^{7+2}$

Exercise 13

$15^2 \cdot 7^2 \cdot 15 \cdot 7^4 = 15^{2+1} \cdot 7^{2+4}$

Exercise 16

$5^4 \times 2^{11} =$

Cannot be simplified. Bases are different and cannot be rewritten in the same base.

Exercise 17

Let x be a number. Simplify the expression of the following number:

$(2x^3)(17x^7) = 34x^{10}$

Exercise 18

Let a and b be numbers. Use the distributive law to simplify the expression of the following number:

$a(a + b) = a^2 + ab$

Exercise 19

Let a and b be numbers. Use the distributive law to simplify the expression of the following number:

$b(a + b) = ab + b^2$

Exercise 20

Let a and b be numbers. Use the distributive law to simplify the expression of the following number:

$(a + b)(a + b) = a^2 + ab + ba + b^2 = a^2 + 2ab + b^2$

Discussion (9 minutes)

Now that we know something about multiplication, we actually know a little about how to divide numbers in exponential notation too. This is not a new law of exponents to be memorized but a (good) consequence of knowing the first law of exponents. Make this clear to students.

- (2) We have just learned how to multiply two different positive integer powers of the same number x. It is time to ask how to *divide* different powers of a number x. If m, n are positive integers, what is $\frac{x^m}{x^n}$?

Allow time for a brief discussion.

- What is $\frac{3^7}{3^5}$? (Observe: The power 7 in the numerator is bigger than the power of 5 in the denominator. The general case of arbitrary exponents will be addressed in Lesson 5, so all problems in this lesson will have bigger exponents in the numerator than in the denominator.)

  - *Expect students to write $\frac{3^7}{3^5} = \frac{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}$. However, we should nudge them to see how the formula $x^m \cdot x^n = x^{m+n}$ comes into play.*

  - *Answer:*

$$\frac{3^7}{3^5} = \frac{3^5 \cdot 3^2}{3^5} \qquad by\ x^m x^n = x^{m+n}$$

$$= 3^2 \qquad by\ equivalent\ fractions$$

$$= 3^{7-5}$$

Observe that the exponent 2 in 3^2 *is* the difference of 7 and 5 (see the numerator $3^5 \cdot 3^2$ on the first line).

In general, if x is nonzero and m, n are positive integers, then:

$$\frac{x^m}{x^n} = x^{m-n}, if\ m > n.$$

Since $m > n$, then there is a positive integer l, so that $m = n + l$. Then, we can rewrite the identity as follows:

$$\frac{x^m}{x^n} = \frac{x^{n+l}}{x^n}$$

$$= \frac{x^n \cdot x^l}{x^n} \qquad by\ x^m x^n = x^{m+n}$$

$$= x^l \qquad by\ equivalent\ fractions$$

$$= x^{m-n} \qquad because\ m = n + l\ implies\ l = m - n$$

Therefore, $\frac{x^m}{x^n} = x^{m-n}$, if $m > n$.

In general, if x is nonzero and m, n are positive integers, then

$$\frac{x^m}{x^n} = x^{m-n}, \text{ if } m > n.$$

This formula is as far as we can go. We cannot write down $\frac{3^5}{3^7}$ in terms of exponents because $3^{5-7} = 3^{-2}$ makes no sense at the moment since we have no meaning for a negative exponent. This explains why the formula above requires $m > n$. This also motivates our search for a definition of negative exponent, as we shall do in Lesson 5.

MP.7

- What is the analog of $\frac{x^m}{x^n} = x^{m-n}$, if $m > n$ in the context of repeated addition of a number x?

    □ *Division is to multiplication as subtraction is to addition, so if n copies of a number x is subtracted from m copies of x, and m > n, then $(mx) - (nx) = (m - n)x$ by the distributive law. (Incidentally, observe once more how the exponent $m - n$ in x^{m-n} in the context of repeated multiplication, corresponds exactly to the $m - n$ in $(m - n)x$ in the context of repeated addition.)*

Examples 3–4

Work through Examples 3 and 4 in the same manner as shown (supplement with additional examples if needed).

It is preferable to write the answers as a subtraction of exponents to emphasize the use of the identity.

Example 3

$$\frac{\left(\frac{3}{5}\right)^8}{\left(\frac{3}{5}\right)^6} = \left(\frac{3}{5}\right)^{8-6}$$

Example 4

$$\frac{4^5}{4^2} = 4^{5-2}$$

Exercises 21–32 (10 minutes)

Students complete Exercises 21–24 independently. Check answers, and then have students complete Exercises 25–32 in pairs or small groups.

Exercise 21

$$\frac{7^9}{7^6} = 7^{9-6}$$

Exercise 22

$$\frac{(-5)^{16}}{(-5)^7} = (-5)^{16-7}$$

Exercise 23

$$\frac{\left(\frac{8}{5}\right)^9}{\left(\frac{8}{5}\right)^2} = \left(\frac{8}{5}\right)^{9-2}$$

Exercise 24

$$\frac{13^5}{13^4} = 13^{5-4}$$

Exercise 25

Let a, b be nonzero numbers. What is the following number?

$$\frac{\left(\dfrac{a}{b}\right)^9}{\left(\dfrac{a}{b}\right)^2} = \left(\frac{a}{b}\right)^{9-2}$$

Exercise 26

Let x be a nonzero number. What is the following number?

$$\frac{x^5}{x^4} = x^{5-4}$$

Can the following expressions be simplified? If yes, write an equivalent expression for each problem. If not, explain why not.

Exercise 27

$$\frac{2^7}{4^2} = \frac{2^7}{2^4} = 2^{7-4}$$

Exercise 28

$$\frac{3^{23}}{27} = \frac{3^{23}}{3^3} = 3^{23-3}$$

Exercise 29

$$\frac{3^5 \cdot 2^8}{3^2 \cdot 2^3} = 3^{5-2} \cdot 2^{8-3}$$

Exercise 30

$$\frac{(-2)^7 \cdot 95^5}{(-2)^5 \cdot 95^4} = (-2)^{7-5} \cdot 95^{5-4}$$

Exercise 31

Let x be a number. Simplify the expression of each of the following numbers:

    a. $\dfrac{5}{x^3}(3x^8) = 15x^5$

    b. $\dfrac{5}{x^3}(-4x^6) = -20x^3$

    c. $\dfrac{5}{x^3}(11x^4) = 55x$

Exercise 32

Anne used an online calculator to multiply $2,000,000,000 \times 2,000,000,000,000$. The answer showed up on the calculator as $4e + 21$, as shown below. Is the answer on the calculator correct? How do you know?

$2,000,000,000 \times 2,000,000,000,000 =$
$4,000,000,000,000,000,000,000.$

The answer must mean 4 followed by 21 zeroes. That means that the answer on the calculator is correct.

This problem is hinting at scientific notation; i.e., $(2 \times 10^9)(2 \times 10^{12}) = 4 \times 10^{9+12}$. Accept any reasonable explanation of the answer.

Closing (3 minutes)

Summarize, or have students summarize, the lesson.

- State the two identities and how to write equivalent expressions for each.

Optional Fluency Exercise (2 minutes)

This exercise is not an expectation of the standard, but may prepare students for work with squared numbers in Module 2 with respect to the Pythagorean Theorem. For that reason this is an optional fluency exercise.

Have students chorally respond to numbers squared and cubed that you provide. For example, you say "1 squared" and students respond, "1." Next, "2 squared" and students respond "4." Have students respond to all squares, in order, up to 15. When squares are finished, start with "1 cubed" and students respond "1." Next, "2 cubed" and students respond "8." Have students respond to all cubes, in order, up to 10. If time allows, you can have students respond to random squares and cubes.

Exit Ticket (2 minutes)

Name _____   Date_____

Lesson 2: Multiplication of Numbers in Exponential Form

Exit Ticket

Simplify each of the following numerical expressions as much as possible:

1. Let a and b be positive integers. $23^a \times 23^b =$

2. $5^3 \times 25 =$

3. Let x and y be positive integers and $x > y$. $\dfrac{11^x}{11^y} =$

4. $\dfrac{2^{13}}{8} =$

Exit Ticket Sample Solutions

Note to Teacher: Accept both forms of the answer; in other words, the answer that shows the exponents as a sum or difference and the answer where the numbers were actually added or subtracted.

Simplify each of the following numerical expressions as much as possible:

1. Let a and b be positive integers. $23^a \times 23^b =$

   $$23^a \times 23^b = 23^{a+b}$$

2. $5^3 \times 25 =$

   $$5^3 \times 25 = 5^3 \times 5^2$$
   $$= 5^{3+2}$$
   $$= 5^5$$

3. Let x and y be positive integers and $x > y$. $\dfrac{11^x}{11^y} =$

   $$\frac{11^x}{11^y} = 11^{x-y}$$

4. $\dfrac{2^{13}}{8} =$

   $$\frac{2^{13}}{8} = \frac{2^{13}}{2^3}$$
   $$= 2^{13-3}$$
   $$= 2^{10}$$

Problem Set Sample Solutions

To ensure success, students need to complete at least bounces 1–4 with support in class.

Students may benefit from a simple drawing of the scenario. It will help them see why the factor of 2 is necessary when calculating the distance traveled for each bounce. Make sure to leave the total distance traveled in the format shown so that students can see the pattern that is developing. Simplifying at any step will make it extremely difficult to write the general statement for n number of bounces.

1. A certain ball is dropped from a height of x feet. It always bounces up to $\frac{2}{3}x$ feet. Suppose the ball is dropped from 10 feet and is caught exactly when it touches the ground after the 30th bounce. What is the total distance traveled by the ball? Express your answer in exponential notation.

Bounce	Computation of Distance Traveled in Previous Bounce	Total Distance Traveled (in feet)
1	$2\left(\frac{2}{3}\right)10$	$10 + 2\left(\frac{2}{3}\right)10$
2	$2\left[\frac{2}{3}\left(\frac{2}{3}\right)10\right]$ $= 2\left(\frac{2}{3}\right)^2 10$	$10 + 2\left(\frac{2}{3}\right)10 + 2\left(\frac{2}{3}\right)^2 10$
3	$2\left[\frac{2}{3}\left(\frac{2}{3}\right)^2 10\right]$ $= 2\left(\frac{2}{3}\right)^3 10$	$10 + 2\left(\frac{2}{3}\right)10 + 2\left(\frac{2}{3}\right)^2 10 + 2\left(\frac{2}{3}\right)^3 10$
4	$2\left[\frac{2}{3}\left(\frac{2}{3}\right)^3 10\right]$ $= 2\left(\frac{2}{3}\right)^4 10$	$10 + 2\left(\frac{2}{3}\right)10 + 2\left(\frac{2}{3}\right)^2 10 + 2\left(\frac{2}{3}\right)^3 10 + 2\left(\frac{2}{3}\right)^4 10$
30	$2\left(\frac{2}{3}\right)^{30} 10$	$10 + 2\left(\frac{2}{3}\right)10 + 2\left(\frac{2}{3}\right)^2 10 + 2\left(\frac{2}{3}\right)^3 10 + 2\left(\frac{2}{3}\right)^4 10 + \cdots + 2\left(\frac{2}{3}\right)^{30} 10$
n	$2\left(\frac{2}{3}\right)^n 10$	$10 + 20\left(\frac{2}{3}\right)\left(1 + \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + \cdots + \left(\frac{2}{3}\right)^n\right)$

2. If the same ball is dropped from 10 feet and is caught exactly at the highest point after the 25th bounce, what is the total distance traveled by the ball? Use what you learned from the last problem.

   Based on the last problem we know that each bounce causes the ball to travel $2\left(\frac{2}{3}\right)^n 10$ feet. If the ball is caught at the highest point of the 25th bounce, then the distance traveled on that last bounce is just $\left(\frac{2}{3}\right)^{25} 10$ because it does not make the return trip to the ground. Therefore, the total distance traveled by the ball in this situation is

$$10 + 2\left(\frac{2}{3}\right)10 + 2\left(\frac{2}{3}\right)^2 10 + 2\left(\frac{2}{3}\right)^3 10 + 2\left(\frac{2}{3}\right)^4 10 + \cdots + 2\left(\frac{2}{3}\right)^{23} 10 + 2\left(\frac{2}{3}\right)^{24} 10$$

3. Let a and b be numbers and $b \neq 0$, and let m and n be positive integers. Simplify each of the following expressions as much as possible:

$(-19)^5 \cdot (-19)^{11} = (-19)^{5+11}$	$2.7^5 \times 2.7^3 = 2.7^{5+3}$
$\dfrac{7^{10}}{7^3} = 7^{10-3}$	$\left(\dfrac{1}{5}\right)^2 \cdot \left(\dfrac{1}{5}\right)^{15} = \left(\dfrac{1}{5}\right)^{2+15}$
$\left(-\dfrac{9}{7}\right)^m \cdot \left(-\dfrac{9}{7}\right)^n = \left(-\dfrac{9}{7}\right)^{m+n}$	$\dfrac{ab^3}{b^2} = ab^{3-2}$

4. Let the dimensions of a rectangle be $(4 \times (871209)^5 + 3 \times 49762105)$ ft. by $(7 \times (871209)^3 - (49762105)^4)$ ft. Determine the area of the rectangle. No need to expand all the powers.

$Area = (4 \times (871209)^5 + 3 \times 49762105)(7 \times (871209)^3 - (49762105)^4)$

$= 28 \times (871209)^8 - 4 \times (871209)^5(49762105)^4 + 21 \times (871209)^3(49762105) - 3 \times (49762105)^5 sq.\ ft.$

5. A rectangular area of land is being sold off in smaller pieces. The total area of the land is 2^{15} square miles. The pieces being sold are 8^3 square miles in size. How many smaller pieces of land can be sold at the stated size? Compute the actual number of pieces.

$8^3 = 2^9$ $\dfrac{2^{15}}{2^9} = 2^{15-9} = 2^6 = 64$ *64 pieces of land can be sold.*

Lesson 3: Numbers in Exponential Form Raised to a Power

Student Outcomes

- Students will know how to take powers of powers. Students will know that when a product is raised to a power, each factor of the product is raised to that power.
- Students will write simplified, equivalent numeric, and symbolic expressions using this new knowledge of powers.

Classwork

Discussion (10 minutes)

Suppose we add 4 copies of 3, thereby getting $(3 + 3 + 3 + 3)$, and then add 5 copies of the sum. We get

$$(3 + 3 + 3 + 3) + (3 + 3 + 3 + 3) + (3 + 3 + 3 + 3) + (3 + 3 + 3 + 3) + (3 + 3 + 3 + 3).$$

Now, by the definition of multiplication, adding 4 copies of 3 is denoted by (4×3), and adding 5 copies of this product is then denoted by $5 \times (4 \times 3)$. So,

$$5 \times (4 \times 3) = (3 + 3 + 3 + 3) + (3 + 3 + 3 + 3) + (3 + 3 + 3 + 3) + (3 + 3 + 3 + 3) + (3 + 3 + 3 + 3).$$

A closer examination of the right side of the above equation reveals that we are adding 3 to itself 20 times (i.e., adding 3 to itself (5×4) times). Therefore,

$$5 \times (4 \times 3) = (5 \times 4) \times 3$$

Now, replace repeated addition by repeated multiplication.

$$\left(\text{For example, } \underbrace{(3 \times 3 \times 3 \times 3) \times (3 \times 3 \times 3 \times 3) \cdots \times (3 \times 3 \times 3 \times 3)}_{5 \text{ times}} = \underbrace{3^4 \times 3^4 \cdots \times 3^4}_{5 \text{ times}}.\right)$$

- What is multiplying 4 copies of 3 and then multiplying 5 copies of the product?
  - *Multiplying 4 copies of 3 is 3^4, and multiplying 5 copies of the product is $(3^4)^5$. We wish to say this is equal to 3^x for some positive integer x. By the analogy initiated in Lesson 1, the 5×4 in $(5 \times 4) \times 3$ should correspond to the exponent x in 3^x; therefore, the answer should be*

$$(3^4)^5 = 3^{5 \times 4}.$$

This is correct because

$$(3^4)^5 = (3 \times 3 \times 3 \times 3)^5$$

$$= \underbrace{(3 \times 3 \times 3 \times 3) \times \cdots \times (3 \times 3 \times 3 \times 3)}_{5 \text{ times}}$$

$$= \underbrace{3 \times 3 \times \cdots \times 3}_{5 \times 4 \text{ times}}$$

$$= 3^{5 \times 4}$$

MP.2 & MP.7

Examples 1–2

Work through Examples 1 and 2 in the same manner as just shown (supplement with additional examples if needed).

Example 1

$(7^2)^6 = (7 \times 7)^6$

$= \underbrace{(7 \times 7) \times \cdots \times (7 \times 7)}_{6 \text{ times}}$

$= \underbrace{7 \times \cdots \times 7}_{6 \times 2 \text{ times}}$

$= 7^{6 \times 2}$

Example 2

$(1.3^3)^{10} = (1.3 \times 1.3 \times 1.3)^{10}$

$= \underbrace{(1.3 \times 1.3 \times 1.3) \times \cdots \times (1.3 \times 1.3 \times 1.3)}_{10 \text{ times}}$

$= \underbrace{1.3 \times \cdots \times 1.3}_{10 \times 3 \text{ times}}$

$= 1.3^{10 \times 3}$

In the same way, we have

For any number x and any positive integers m and n,

$$(x^m)^n = x^{mn}$$

because

$$(x^m)^n = \underbrace{(x \cdot x \cdots x)^n}_{m \text{ times}}$$

$$= \underbrace{\underbrace{(x \cdot x \cdots x)}_{m \text{ times}} \times \cdots \times \underbrace{(x \cdot x \cdots x)}_{m \text{ times}}}_{n \text{ times}}$$

$$= x^{mn} \,.$$

Exercises 1–6 (10 minutes)

Students complete Exercises 1–4 independently. Check answers, and then have students complete Exercises 5–6.

Exercise 1

$(15^3)^9 = 15^{9 \times 3}$

Exercise 2

$((-2)^5)^8 = (-2)^{8 \times 5}$

Exercise 3

$(3.4^{17})^4 = 3.4^{4 \times 17}$

Exercise 4

Let s be a number.

$(s^{17})^4 = s^{4 \times 17}$

Exercise 5

Sarah wrote $(3^5)^7 = 3^{12}$. Correct her mistake. Write an exponential expression using a base of 3 and exponents of 5, 7, and 12 that would make her answer correct.

Correct way: $(3^5)^7 = 3^{35}$, *Rewritten Problem:* $3^5 \times 3^7 = 3^{5+7} = 3^{12}$

Exercise 6

A number y satisfies $y^{24} - 256 = 0$. What equation does the number $x = y^4$ satisfy?

Since $x = y^4$, then $(x)^6 = (y^4)^6$. Therefore, $x = y^4$ would satisfy the equation $x^6 - 256 = 0$.

Discussion (10 minutes)

From the point of view of algebra and arithmetic, the most basic question about raising a number to a power has to be the following: How is this operation related to the four arithmetic operations? In other words, for two numbers x, y and a positive integer n,

MP.7

1. How is $(xy)^n$ related to x^n and y^n?

2. How is $\left(\dfrac{x}{y}\right)^n$ related to x^n and y^n, $y \neq 0$?

3. How is $(x + y)^n$ related to x^n and y^n?

4. How is $(x - y)^n$ related to x^n and y^n?

The answers to the last two questions turn out to be complicated; students will learn about this in high school under the heading of the **binomial theorem**. However, they should at least be aware that, in general,

$(x + y)^n \neq x^n + y^n$, *unless $n = 1$. For example, $(2 + 3)^2 \neq 2^2 + 3^2$.*

Allow time for discussion of Question 1. Students can begin by talking in partners or small groups and then share with the class.

Some students may want to simply multiply 5×8, but remind them to focus on the above stated goal which is to relate $(5 \times 8)^{17}$ to 5^{17} and 8^{17}. Therefore, we want to see 17 copies of 5 and 17 copies of 8 on the right side. Multiplying 5×8 would take us in a different direction.

Scaffolding:

- Provide a numeric example for students to work on $(5 \times 8)^{17} = 5^{17} \times 8^{17}$.

$$(5 \times 8)^{17} = \underbrace{(5 \times 8) \times \cdots \times (5 \times 8)}_{17 \text{ times}}$$
$$= \underbrace{(5 \times \cdots \times 5)}_{17 \text{ times}} \times \underbrace{(8 \times \cdots \times 8)}_{17 \text{ times}}$$
$$= 5^{17} \times 8^{17}$$

The following computation is a different way of proving the equality.

$$5^{17} \times 8^{17} = \underbrace{(5 \times \cdots \times 5)}_{17 \text{ times}} \times \underbrace{(8 \times \cdots \times 8)}_{17 \text{ times}}$$
$$= \underbrace{(5 \times 8) \times \cdots \times (5 \times 8)}_{17 \text{ times}}$$
$$= (5 \times 8)^{17}$$

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Lesson 3: Numbers in Exponential Form Raised to a Power
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31

Answer to Question 1:

Because in $(xy)^n$, the factors x and y will be repeatedly multiplied n times, resulting in factors of x^n and y^n:

$$(xy)^n = x^n y^n$$

because

$$(xy)^n = \underbrace{(xy) \cdots (xy)}_{n \text{ times}}$$ By definition of raising a number to the nth power

$$= \underbrace{(x \cdot x \cdots x)}_{n \text{ times}} \cdot \underbrace{(y \cdot y \cdots y)}_{n \text{ times}}$$ By commutative and associative properties

$$= x^n y^n$$ By definition of x^n

For any numbers x and y, and positive integer n,

$$(xy)^n = x^n y^n$$

because

$$(xy)^n = \underbrace{(xy) \cdots (xy)}_{n \text{ times}}$$

$$= \underbrace{(x \cdot x \cdots x)}_{n \text{ times}} \cdot \underbrace{(y \cdot y \cdots y)}_{n \text{ times}}$$

$$= x^n y^n.$$

Exercises 7–13 (10 minutes)

Students complete Exercises 17–12 independently. Check answers.

Exercise 7

$(11 \times 4)^9 = 11^{9 \times 1} \times 4^{9 \times 1}$

Exercise 8

$(3^2 \times 7^4)^5 = 3^{5 \times 2} \times 7^{5 \times 4}$

Exercise 9

Let a, b, and c be numbers.

$(3^2 a^4)^5 = 3^{5 \times 2} a^{5 \times 4}$

Exercise 10

Let x be a number.

$(5x)^7 = 5^{7 \times 1} \cdot x^{7 \times 1}$

Exercise 11

Let x and y be numbers.

$(5xy^2)^7 = 5^{7 \times 1} \cdot x^{7 \times 1} \cdot y^{7 \times 2}$

Exercise 12

Let a, b, and c be numbers.

$(a^2 bc^3)^4 = a^{4 \times 2} \cdot b^{4 \times 1} \cdot c^{4 \times 3}$

Next, have students work in pairs or small groups on Exercise 13 after you present the problem:

Ask students to first explain why we must assume $y \neq 0$. Students should say that if the denominator were zero then the fraction would be undefined.

- The answer to the fourth question is similar to the third: If x, y are any two numbers, such that $y \neq 0$ and n is a positive integer, then

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}.$$

Exercise 13

Let x and y be numbers, $y \neq 0$, and let n be a positive integer. How is $\left(\frac{x}{y}\right)^n$ related to x^n and y^n?

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

Because

$$\left(\frac{x}{y}\right)^n = \underbrace{\frac{x}{y} \times \cdots \times \frac{x}{y}}_{n \text{ times}} \qquad \textbf{\textit{By definition}}$$

$$= \frac{\overbrace{x \cdot x \cdots x}^{n \text{ times}}}{\underbrace{y \cdot y \cdots y}_{n \text{ times}}} \qquad \textbf{\textit{By the product formula}}$$

$$= \frac{x^n}{y^n} \qquad \textbf{\textit{By definition}}$$

Let the students know that this type of reasoning is required to prove facts in mathematics. They should always supply a reason for each step or at least know the reason the facts are connected. Further, it is important to keep in mind what we already know in order to figure out what we do not know. Students are required to write two proofs for the Problem Set that are extensions of the proofs they have done in class.

Closing (2 minutes)

Summarize, or have students summarize the lesson. Students should state that they now know how to take powers of powers.

Exit Ticket (3 minutes)

Name _____   Date_____

Lesson 3: Numbers in Exponential Form Raised to a Power

Exit Ticket

Write each answer as a base raised to a power or as the product of bases raised to powers that is equivalent to the given one.

1. $(9^3)^6 =$

2. $(113^2 \times 37 \times 51^4)^3 =$

3. Let x, y, z be numbers. $(x^2yz^4)^3 =$

4. Let x, y, z be numbers and let m, n, p, q be positive integers. $(x^m y^n z^p)^q =$

5. $\dfrac{4^8}{5^8} =$

Exit Ticket Sample Solutions

Write each answer as a base raised to a power or as the product of bases raised to powers that is equivalent to the given one.

1. $(9^3)^6 =$

   $(9^3)^6 = 9^{6 \times 3} = 9^{18}$

2. $(113^2 \times 37 \times 51^4)^3 =$

   $\begin{aligned} (113^2 \times 37 \times 51^4)^3 &= \big((113^2 \times 37) \times 51^4\big)^3 && \textit{(associative law)} \\ &= (113^2 \times 37)^3 \times (51^4)^3 && \textit{(because } (xy)^n = x^n y^n \textit{ for all numbers } x, y) \\ &= (113^2)^3 \times 37^3 \times (51^4)^3 && \textit{(because } (xy)^n = x^n y^n \textit{ for all numbers } x, y) \\ &= 113^6 \times 37^3 \times 51^{12} && \textit{(because } (x^m)^n = x^{mn} \textit{ for all numbers } x) \end{aligned}$

3. **Let x, y, z be numbers.** $(x^2 y z^4)^3 =$

   $\begin{aligned} (x^2 y z^4)^3 &= \big((x^2 \times y) \times z^4\big)^3 && \textit{(associative law)} \\ &= (x^2 \times y)^3 \times (z^4)^3 && \textit{(because } (xy)^n = x^n y^n \textit{ for all numbers } x, y) \\ &= (x^2)^3 \times y^3 \times (z^4)^3 && \textit{(because } (xy)^n = x^n y^n \textit{ for all numbers } x, y) \\ &= x^6 \times y^3 \times z^{12} && \textit{(because } (x^m)^n = x^{mn} \textit{ for all numbers } x) \\ &= x^6 y^3 z^{12} \end{aligned}$

4. **Let x, y, z be numbers and let m, n, p, q be positive integers.** $(x^m y^n z^p)^q =$

   $\begin{aligned} (x^m y^n z^p)^q &= \big((x^m \times y^n) \times z^p\big)^q && \textit{(associative law)} \\ &= (x^m \times y^n)^q \times (z^p)^q && \textit{(because } (xy)^n = x^n y^n \textit{ for all numbers } x, y) \\ &= (x^m)^q \times (y^n)^q \times (z^p)^q && \textit{(because } (xy)^n = x^n y^n \textit{ for all numbers } x, y) \\ &= x^{mp} \times y^{nq} \times z^{pq} && \textit{(because } (x^m)^n = x^{mn} \textit{ for all numbers } x) \\ &= x^{mq} y^{nq} z^{pq} \end{aligned}$

5. $\dfrac{4^8}{5^8} =$

   $\dfrac{4^8}{5^8} = \left(\dfrac{4}{5}\right)^8$

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Problem Set Sample Solutions

1. Show (prove) in detail why $(2 \cdot 3 \cdot 7)^4 = 2^4 3^4 7^4$.

   $(2 \cdot 3 \cdot 7)^4 = (2 \cdot 3 \cdot 7)(2 \cdot 3 \cdot 7)(2 \cdot 3 \cdot 7)(2 \cdot 3 \cdot 7)$ *By definition*

   $ = (2 \cdot 2 \cdot 2 \cdot 2)(3 \cdot 3 \cdot 3 \cdot 3)(7 \cdot 7 \cdot 7 \cdot 7)$ *By repeated use of the commutative and associative properties*

   $ = 2^4 3^4 7^4$ *By definition*

2. Show (prove) in detail why $(xyz)^4 = x^4 y^4 z^4$ for any numbers x, y, z.

   The left side of the equation, $(xyz)^4$, means $(xyz)\,(xyz)\,(xyz)\,(xyz)$. Using the commutative and associative properties of multiplication, we can write $(xyz)\,(xyz)\,(xyz)\,(xyz)$ as $(xxxx)(\,yyyy\,)\,(zzzz)$, which in turn can be written as $x^4 y^4 z^4$, which is what the right side of the equation states.

3. Show (prove) in detail why $(xyz)^n = x^n y^n z^n$ for any numbers x, y, and z and for any positive integer n.

   Beginning with the left side of the equation, $(xyz)^n$ means $\underbrace{(xyz) \cdot (xyz) \cdots (xyz)}_{n \text{ times}}$. Using the commutative and

   associative properties of multiplication, $\underbrace{(xyz) \cdot (xyz) \cdots (xyz)}_{n \text{ times}}$ can be rewritten as $\underbrace{(x \cdot x \cdots x)}_{n \text{ times}} \underbrace{(y \cdot y \cdots y)}_{n \text{ times}} \underbrace{(z \cdot z \cdots z)}_{n \text{ times}}$

   and finally, $x^n y^n z^n$, which is what the right side of the equation states. We can also prove this equality by a

   different method, as follows. Beginning with the right side, $x^n y^n z^n$ means $\underbrace{(x \cdot x \cdots x)}_{n \text{ times}} \underbrace{(y \cdot y \cdots y)}_{n \text{ times}} \underbrace{(z \cdot z \cdots z)}_{n \text{ times}}$, which

   by the commutative property of multiplication can be rewritten as $\underbrace{(xyz) \cdot (xyz) \cdots (xyz)}_{n \text{ times}}$. Using exponential

   notation, $\underbrace{(xyz) \cdot (xyz) \cdots (xyz)}_{n \text{ times}}$ can be rewritten as $(xyz)^n$, which is what the left side of the equation states.

Lesson 4: Numbers Raised to the Zeroth Power

Student Outcomes

- Students know that a number raised to the zeroth power is equal to one.
- Students recognize the need for the definition to preserve the properties of exponents.

Classwork

Concept Development (5 minutes)

Let us summarize our main conclusions about exponents. For any numbers x, y and any positive integers m, n, the following holds

$$x^m \cdot x^n = x^{m+n} \tag{1}$$

$$(x^m)^n = x^{mn} \tag{2}$$

$$(xy)^n = x^n y^n. \tag{3}$$

For any numbers x, y, and any positive integers m, n, the following holds

$$x^m \cdot x^n = x^{m+n} \tag{1}$$

$$(x^m)^n = x^{mn} \tag{2}$$

$$(xy)^n = x^n y^n. \tag{3}$$

Definition: _____

And if we assume $x > 0$ in equation (4) and $y > 0$ in equation (5) below, then we also have

$$\frac{x^m}{x^n} = x^{m-n}, \ m > n \tag{4}$$

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}. \tag{5}$$

There is an obvious reason why the x in (4) and the y in (5) must be nonzero: we cannot divide by 0. However, the reason for further restricting x and y to be positive is only given when fractional exponents have been defined. This will be done in high school.

We group equations (1)–(3) together because they are the foundation on which all the results about exponents rest. When they are suitably generalized, as will be done below, they will imply (4) and (5). Therefore, we concentrate on (1)–(3).

The most important feature of (1)–(3) is that they are *simple* and they are *formally (symbolically) natural*. Mathematicians want these three identities to continue to hold for *all* exponents m and n, without the restriction that m and n be positive integers because of these two desirable qualities. We will have to do it one step at a time. Our goal in this grade is to extend the validity of (1)–(3) to all *integers* m and n.

Exploratory Challenge (10 minutes)

The first step in this direction is to introduce the definition of the 0th exponent of a positive number and then use it to prove that (1)–(3) remain valid when m and n are not just positive integers but all whole numbers (including 0). Since our goal is to make sure (1)–(3) remain valid even when m and n may be 0, the very definition of the 0th exponent of a number must pose no *obvious* contradiction to (1)–(3). With this in mind, let us consider what it means to raise a positive number x to the zeroth power. For example, what should 3^0 mean?

- *Students will likely respond that 3^0 should equal* 0. When they do, demonstrate why that would contradict our existing understanding of properties of exponents using (1). Specifically, if m is a positive integer and we let $3^0 = 0$, then

$$3^m \cdot 3^0 = 3^{m+0},$$

  but since we let $3^0 = 0$ it means that the left side of the equation would equal zero. That creates a contradiction because

$$0 \neq 3^{m+0}$$

  Therefore, letting $3^0 = 0$ will not help us to extend (1)–(3) to all whole numbers m and n.

- *Next, students may say that we should let $3^0 = 3$.* Show the two problematic issues this would lead to. First, we have already learned that by definition $x^1 = x$ in Lesson 1, and we do not want to have two powers that yield the same result. Second, it would violate the existing rules we have developed: Looking specifically at (1) again, if we let $3^0 = 3$, then
$$3^m \cdot 3^0 = 3^{m+0},$$
  but
$$3^m \cdot 3^0 = \underbrace{3 \times \cdots \times 3}_{m \text{ times}} \cdot 3$$
$$= 3^{m+1}$$

  which is a contradiction again.

If we believe that equation (1) should hold even when $n = 0$, then, for example, $3^{2+0} = 3^2 \times 3^0$, which is the same as $3^2 = 3^2 \times 3^0$; therefore, after multiplying both sides by the number $\frac{1}{3^2}$, we get $1 = 3^0$. In the same way, our belief that (1) should hold when either m or n is 0 would lead us to the conclusion that we should define $x^0 = 1$ for any nonzero x. Therefore, we give the following definition:

MP.6

Definition: For any positive number x, we define $x^0 = 1$.

Students will need to write this definition of x^0 in the lesson summary box on their classwork paper.

Exploratory Challenge 2 (10 minutes)

Now that x^n is defined for all whole numbers n, check carefully that (1)–(3) remain valid for all whole numbers m and n.

MP.3 Have students independently complete Exercise 1; provide correct values for m and n before proceeding. (Development of cases (A)–(C)).

Exercise 1

List all possible cases of whole numbers m and n for identity (1). More precisely, when $m > 0$ and $n > 0$, we already know that (1) is correct. What are the other possible cases of m and n for which (1) is yet to be verified?

Case (A): $m > 0$ and $n = 0$

Case (B): $m = 0$ and $n > 0$

Case (C): $m = n = 0$

Model how to check the validity of a statement using Case (A) with equation (1) as part of Exercise 2. Have students work independently or in pairs to check the validity of (1) in Case (B) and Case (C) to complete Exercise 2. Next, have students check the validity of equations (2) and (3) using Cases (A)–(C) for Exercises 3 and 4.

Exercise 2

Check that equation (1) is correct for each of the cases listed in Exercise 1.

Case (A): $x^m \cdot x^0 = x^m$? *Yes, because $x^m \cdot x^0 = x^m \cdot 1 = x^m$.*

Case (B): $x^0 \cdot x^n = x^n$? *Yes, because $x^0 \cdot x^n = 1 \cdot x^n = x^n$.*

Case (C): $x^0 \cdot x^0 = x^0$? *Yes, because $x^0 \cdot x^0 = 1 \cdot 1 = x^0$.*

Exercise 3

Do the same with equation (2) by checking it case-by-case.

Case (A): $(x^m)^0 = x^{0 \times m}$? Yes, because x^m is a number, and a number raised to a zero power is 1. $1 = x^0 = x^{0 \times m}$.
So, the left side is 1. The right side is also 1 because $x^{0 \times m} = x^0 = 1$.

Case (B): $(x^0)^n = x^{n \times 0}$? Yes, because by definition $x^0 = 1$ and $1^n = 1$, so the left side is equal to 1. The right side is equal to $x^0 = 1$, and so both sides are equal.

Case (C): $(x^0)^0 = x^{0 \times 0}$? Yes, because by definition of the zeroth power of x, both sides are equal to 1.

Exercise 4

Do the same with equation (3) by checking it case-by-case.

Case (A): $(xy)^0 = x^0 y^0$? Yes, because the left side is 1 by the definition of the zeroth power while the right side is $1 \times 1 = 1$.

Case (B): Since $n > 0$, we already know that (3) is valid.

Case (C): This is the same as Case (A), which we have already shown to be valid.

Exploratory Challenge 3 (5 minutes)

Students will practice writing numbers in expanded notation in Exercises 5 and 6. Students will use the definition of x^0, for any positive number x, learned in this lesson.

Clearly state that you want to see the ones digit multiplied by 10^0. That is the important part of the expanded notation. This will lead to the use of negative powers of 10 for decimals in Lesson 5.

Scaffolding:
- You may need to remind students how to write numbers in expanded form with Exercise 5.

Exercise 5

Write the expanded form of $8,374$ using exponential notation.

$$8,374 = (8 \times 10^3) + (3 \times 10^2) + (7 \times 10^1) + (4 \times 10^0)$$

Exercise 6

Write the expanded form of $6,985,062$ using exponential notation.

$$6,985,062 = (6 \times 10^6) + (9 \times 10^5) + (8 \times 10^4) + (5 \times 10^3) + (0 \times 10^2) + (6 \times 10^1) + (2 \times 10^0)$$

Closing (3 minutes)

Summarize, or have students summarize, the lesson.

- The rules of exponents that students have worked on prior to today only work for positive integer exponents; now those same rules have been extended to all whole numbers.
- The next logical step is to attempt to extend these rules to all integer exponents.

Exit Ticket (2 minutes)

Fluency Exercise (10 minutes)

Sprint: Rewrite expressions with the same base for positive exponents only. Make sure to tell the students that all letters within the problems of the sprint are meant to denote numbers. This exercise can be administered at any point during the lesson. Refer to the Sprints and Sprint Delivery Script sections in the Module Overview for directions to administer a Sprint.

COMMON CORE

Lesson 4:	Numbers Raised to the Zeroth Power
Date:	6/17/14

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Name _____   Date_____

Lesson 4: Numbers Raised to the Zeroth Power

Exit Ticket

1. Simplify the following expression as much as possible.

$$\frac{4^{10}}{4^{10}} \cdot 7^0 =$$

2. Let a and b be two numbers. Use the distributive law and then the definition of zeroth power to show that the numbers $(a^0 + b^0)a^0$ and $(a^0 + b^0)b^0$ are equal.

Exit Ticket Sample Solutions

1. Simplify the following expression as much as possible.

$$\frac{4^{10}}{4^{10}} \cdot 7^0 = 4^{10-10} \cdot 1 = 4^0 \cdot 1 = 1 \cdot 1 = 1$$

2. Let a and b be two numbers. Use the distributive law and then the definition of zeroth power to show that the numbers $(a^0 + b^0)a^0$ and $(a^0 + b^0)b^0$ are equal.

$$\begin{aligned}(a^0 + b^0)a^0 &= a^0 \cdot a^0 + b^0 \cdot a^0 \\ &= a^{0+0} + a^0 b^0 \\ &= a^0 + a^0 b^0 \\ &= 1 + 1 \cdot 1 \\ &= 1 + 1 \\ &= 2\end{aligned}$$

$$\begin{aligned}(a^0 + b^0)b^0 &= a^0 \cdot b^0 + b^0 \cdot b^0 \\ &= a^0 b^0 + b^{0+0} \\ &= a^0 b^0 + b^0 \\ &= 1 \cdot 1 + 1 \\ &= 1 + 1 \\ &= 2\end{aligned}$$

Since both numbers are equal to 2, they are equal.

Problem Set Sample Solutions

Let x, y be numbers $(x, y \neq 0)$. Simplify each of the following expressions of numbers.

1.
$$\begin{aligned}\frac{y^{12}}{y^{12}} &= y^{12-12} \\ &= y^0 \\ &= 1\end{aligned}$$

2.
$$\begin{aligned}9^{15} \cdot \frac{1}{9^{15}} &= \frac{9^{15}}{9^{15}} \\ &= 9^{15-15} \\ &= 9^0 \\ &= 1\end{aligned}$$

3.
$$\begin{aligned}(7(123456.789)^4)^0 &= \\ &= 7^0 (123456.789)^{4 \times 0} \\ &= 7^0 (123456.789)^0 \\ &= 1\end{aligned}$$

4.
$$\begin{aligned}2^2 \cdot \frac{1}{2^5} \cdot 2^5 \cdot \frac{1}{2^2} &= \frac{2^2}{2^2} \cdot \frac{2^5}{2^5} \\ &= 2^{2-2} \cdot 2^{5-5} \\ &= 2^0 \cdot 2^0 \\ &= 1\end{aligned}$$

5.
$$\begin{aligned}\frac{x^{41}}{y^{15}} \cdot \frac{y^{15}}{x^{41}} &= \frac{x^{41} \cdot y^{15}}{y^{15} \cdot x^{41}} \\ &= \frac{x^{41}}{x^{41}} \cdot \frac{y^{15}}{y^{15}} \\ &= x^{41-41} \cdot y^{15-15} \\ &= x^0 \cdot y^0 \\ &= 1\end{aligned}$$

Number Correct: _____

Equivalent Expressions in Exponential Notation – Round 1

Directions: Rewrite each item as an equivalent expression in exponential notation. All letters denote numbers.

1.	$2^2 \cdot 2^3 =$		23.	$6^3 \cdot 6^2 =$	
2.	$2^2 \cdot 2^4 =$		24.	$6^2 \cdot 6^3 =$	
3.	$2^2 \cdot 2^5 =$		25.	$(-8)^3 \cdot (-8)^7 =$	
4.	$3^7 \cdot 3^1 =$		26.	$(-8)^7 \cdot (-8)^3 =$	
5.	$3^8 \cdot 3^1 =$		27.	$(0.2)^3 \cdot (0.2)^7 =$	
6.	$3^9 \cdot 3^1 =$		28.	$(0.2)^7 \cdot (0.2)^3 =$	
7.	$7^6 \cdot 7^2 =$		29.	$(-2)^{12} \cdot (-2)^1 =$	
8.	$7^6 \cdot 7^3 =$		30.	$(-2.7)^{12} \cdot (-2.7)^1 =$	
9.	$7^6 \cdot 7^4 =$		31.	$1.1^6 \cdot 1.1^9 =$	
10.	$11^{15} \cdot 11 =$		32.	$57^6 \cdot 57^9 =$	
11.	$11^{16} \cdot 11 =$		33.	$x^6 \cdot x^9 =$	
12.	$2^{12} \cdot 2^2 =$		34.	$2^7 \cdot 4 =$	
13.	$2^{12} \cdot 2^4 =$		35.	$2^7 \cdot 4^2 =$	
14.	$2^{12} \cdot 2^6 =$		36.	$2^7 \cdot 16 =$	
15.	$99^5 \cdot 99^2 =$		37.	$16 \cdot 4^3 =$	
16.	$99^6 \cdot 99^3 =$		38.	$3^2 \cdot 9 =$	
17.	$99^7 \cdot 99^4 =$		39.	$3^2 \cdot 27 =$	
18.	$5^8 \cdot 5^2 =$		40.	$3^2 \cdot 81 =$	
19.	$6^8 \cdot 6^2 =$		41.	$5^4 \cdot 25 =$	
20.	$7^8 \cdot 7^2 =$		42.	$5^4 \cdot 125 =$	
21.	$r^8 \cdot r^2 =$		43.	$8 \cdot 2^9 =$	
22.	$s^8 \cdot s^2 =$		44.	$16 \cdot 2^9 =$	

Equivalent Expressions in Exponential Notation – Round 1 [KEY]

Directions: Rewrite each item as an equivalent expression in exponential notation. All letters denote numbers.

1.	$2^2 \cdot 2^3 =$	$\mathbf{2^5}$	23.	$6^3 \cdot 6^2 =$	$\mathbf{6^5}$
2.	$2^2 \cdot 2^4 =$	$\mathbf{2^6}$	24.	$6^2 \cdot 6^3 =$	$\mathbf{6^5}$
3.	$2^2 \cdot 2^5 =$	$\mathbf{2^7}$	25.	$(-8)^3 \cdot (-8)^7 =$	$\mathbf{(-8)^{10}}$
4.	$3^7 \cdot 3^1 =$	$\mathbf{3^8}$	26.	$(-8)^7 \cdot (-8)^3 =$	$\mathbf{(-8)^{10}}$
5.	$3^8 \cdot 3^1 =$	$\mathbf{3^9}$	27.	$(0.2)^3 \cdot (0.2)^7 =$	$\mathbf{(0.2)^{10}}$
6.	$3^9 \cdot 3^1 =$	$\mathbf{3^{10}}$	28.	$(0.2)^7 \cdot (0.2)^3 =$	$\mathbf{(0.2)^{10}}$
7.	$7^6 \cdot 7^2 =$	$\mathbf{7^8}$	29.	$(-2)^{12} \cdot (-2)^1 =$	$\mathbf{(-2)^{13}}$
8.	$7^6 \cdot 7^3 =$	$\mathbf{7^9}$	30.	$(-2.7)^{12} \cdot (-2.7)^1 =$	$\mathbf{(-2.7)^{13}}$
9.	$7^6 \cdot 7^4 =$	$\mathbf{7^{10}}$	31.	$1.1^6 \cdot 1.1^9 =$	$\mathbf{1.1^{15}}$
10.	$11^{15} \cdot 11 =$	$\mathbf{11^{16}}$	32.	$57^6 \cdot 57^9 =$	$\mathbf{57^{15}}$
11.	$11^{16} \cdot 11 =$	$\mathbf{11^{17}}$	33.	$x^6 \cdot x^9 =$	$\mathbf{x^{15}}$
12.	$2^{12} \cdot 2^2 =$	$\mathbf{2^{14}}$	34.	$2^7 \cdot 4 =$	$\mathbf{2^9}$
13.	$2^{12} \cdot 2^4 =$	$\mathbf{2^{16}}$	35.	$2^7 \cdot 4^2 =$	$\mathbf{2^{11}}$
14.	$2^{12} \cdot 2^6 =$	$\mathbf{2^{18}}$	36.	$2^7 \cdot 16 =$	$\mathbf{2^{11}}$
15.	$99^5 \cdot 99^2 =$	$\mathbf{99^7}$	37.	$16 \cdot 4^3 =$	$\mathbf{4^5}$
16.	$99^6 \cdot 99^3 =$	$\mathbf{99^9}$	38.	$3^2 \cdot 9 =$	$\mathbf{3^4}$
17.	$99^7 \cdot 99^4 =$	$\mathbf{99^{11}}$	39.	$3^2 \cdot 27 =$	$\mathbf{3^5}$
18.	$5^8 \cdot 5^2 =$	$\mathbf{5^{10}}$	40.	$3^2 \cdot 81 =$	$\mathbf{3^6}$
19.	$6^8 \cdot 6^2 =$	$\mathbf{6^{10}}$	41.	$5^4 \cdot 25 =$	$\mathbf{5^6}$
20.	$7^8 \cdot 7^2 =$	$\mathbf{7^{10}}$	42.	$5^4 \cdot 125 =$	$\mathbf{5^7}$
21.	$r^8 \cdot r^2 =$	$\mathbf{r^{10}}$	43.	$8 \cdot 2^9 =$	$\mathbf{2^{12}}$
22.	$s^8 \cdot s^2 =$	$\mathbf{s^{10}}$	44.	$16 \cdot 2^9 =$	$\mathbf{2^{13}}$

Equivalent Expressions in Exponential Notation – Round 2

Directions: Rewrite each item as an equivalent expression in exponential notation. All letters denote numbers.

1.	$5^2 \cdot 5^3 =$		23.	$7^3 \cdot 7^2 =$	
2.	$5^2 \cdot 5^4 =$		24.	$7^2 \cdot 7^3 =$	
3.	$5^2 \cdot 5^5 =$		25.	$(-4)^3 \cdot (-4)^{11} =$	
4.	$2^7 \cdot 2^1 =$		26.	$(-4)^{11} \cdot (-4)^3 =$	
5.	$2^8 \cdot 2^1 =$		27.	$(0.2)^3 \cdot (0.2)^{11} =$	
6.	$2^9 \cdot 2^1 =$		28.	$(0.2)^{11} \cdot (0.2)^3 =$	
7.	$3^6 \cdot 3^2 =$		29.	$(-2)^9 \cdot (-2)^5 =$	
8.	$3^6 \cdot 3^3 =$		30.	$(-2.7)^5 \cdot (-2.7)^9 =$	
9.	$3^6 \cdot 3^4 =$		31.	$3.1^6 \cdot 3.1^6 =$	
10.	$7^{15} \cdot 7 =$		32.	$57^6 \cdot 57^6 =$	
11.	$7^{16} \cdot 7 =$		33.	$z^6 \cdot z^6 =$	
12.	$11^{12} \cdot 11^2 =$		34.	$4 \cdot 2^9 =$	
13.	$11^{12} \cdot 11^4 =$		35.	$4^2 \cdot 2^9 =$	
14.	$11^{12} \cdot 11^6 =$		36.	$16 \cdot 2^9 =$	
15.	$23^5 \cdot 23^2 =$		37.	$16 \cdot 4^3 =$	
16.	$23^6 \cdot 23^3 =$		38.	$9 \cdot 3^5 =$	
17.	$23^7 \cdot 23^4 =$		39.	$3^5 \cdot 9 =$	
18.	$13^7 \cdot 13^3 =$		40.	$3^5 \cdot 27 =$	
19.	$15^7 \cdot 15^3 =$		41.	$5^7 \cdot 25 =$	
20.	$17^7 \cdot 17^3 =$		42.	$5^7 \cdot 125 =$	
21.	$x^7 \cdot x^3 =$		43.	$2^{11} \cdot 4 =$	
22.	$y^7 \cdot y^3 =$		44.	$2^{11} \cdot 16 =$	

Equivalent Expressions in Exponential Notation – Round 2 [KEY]

Directions: Rewrite each item as an equivalent expression in exponential notation. All letters denote numbers.

1.	$5^2 \cdot 5^3 =$	5^5	23.	$7^3 \cdot 7^2 =$	7^5
2.	$5^2 \cdot 5^4 =$	5^6	24.	$7^2 \cdot 7^3 =$	7^5
3.	$5^2 \cdot 5^5 =$	5^7	25.	$(-4)^3 \cdot (-4)^{11} =$	$(-4)^{14}$
4.	$2^7 \cdot 2^1 =$	2^8	26.	$(-4)^{11} \cdot (-4)^3 =$	$(-4)^{14}$
5.	$2^8 \cdot 2^1 =$	2^9	27.	$(0.2)^3 \cdot (0.2)^{11} =$	$(0.2)^{14}$
6.	$2^9 \cdot 2^1 =$	2^{10}	28.	$(0.2)^{11} \cdot (0.2)^3 =$	$(0.2)^{14}$
7.	$3^6 \cdot 3^2 =$	3^8	29.	$(-2)^9 \cdot (-2)^5 =$	$(-2)^{14}$
8.	$3^6 \cdot 3^3 =$	3^9	30.	$(-2.7)^5 \cdot (-2.7)^9 =$	$(-2.7)^{14}$
9.	$3^6 \cdot 3 =$	3^{10}	31.	$3.1^6 \cdot 3.1^6 =$	3.1^{12}
10.	$7^{15} \cdot 7 =$	7^{16}	32.	$57^6 \cdot 57^6 =$	57^{12}
11.	$7^{16} \cdot 7 =$	7^{17}	33.	$z^6 \cdot z^6 =$	z^{12}
12.	$11^{12} \cdot 11^2 =$	11^{14}	34.	$4 \cdot 2^9 =$	2^{11}
13.	$11^{12} \cdot 11^4 =$	11^{16}	35.	$4^2 \cdot 2^9 =$	2^{13}
14.	$11^{12} \cdot 11^6 =$	11^{18}	36.	$16 \cdot 2^9 =$	2^{13}
15.	$23^5 \cdot 23^2 =$	23^7	37.	$16 \cdot 4^3 =$	4^5
16.	$23^6 \cdot 23^3 =$	23^9	38.	$9 \cdot 3^5 =$	3^7
17.	$23^7 \cdot 23^4 =$	23^{11}	39.	$3^5 \cdot 9 =$	3^7
18.	$13^7 \cdot 13^3 =$	13^{10}	40.	$3^5 \cdot 27 =$	3^8
19.	$15^7 \cdot 15^3 =$	15^{10}	41.	$5^7 \cdot 25 =$	5^9
20.	$17^7 \cdot 17^3 =$	17^{10}	42.	$5^7 \cdot 125 =$	5^{10}
21.	$x^7 \cdot x^3 =$	x^{10}	43.	$2^{11} \cdot 4 =$	2^{13}
22.	$y^7 \cdot y^3 =$	y^{10}	44.	$2^{11} \cdot 16 =$	2^{15}

Lesson 5: Negative Exponents and the Laws of Exponents

Student Outcomes

- Students know the definition of a number raised to a negative exponent.
- Students simplify and write equivalent expressions that contain negative exponents.

Classwork

Discussion (10 minutes)

This lesson, and the next, refers to several of the equations used in the previous lessons. It may be helpful if students have some way of referencing these equations quickly (e.g., a poster in the classroom or handout). For your convenience an equation reference sheet has been provided on page 54.

Let x and y be positive numbers throughout this lesson. Recall that we have the following three identities (6)–(8):

For all whole numbers m and n:

$$x^m \cdot x^n = x^{m+n} \tag{6}$$

$$(x^m)^n = x^{mn} \tag{7}$$

$$(xy)^n = x^n y^n \tag{8}$$

Make clear that we want (6)–(8) to remain true even when m and n are *integers*. Before we can say that, we have to first decide what something like 3^{-5} should mean.

Allow time for the class to discuss the question, "What should 3^{-5} mean?" As in Lesson 4, where we introduced the concept of the zeroth power of a number, the overriding idea here is that the negative power of a number should be defined in a way to ensure that (6)–(8) continue to hold when m and n are integers and not just whole numbers. Students will likely say that it should mean -3^5. Tell students that if that is what it meant, that is what we would write.

When they get stuck, ask students this question, "Using equation (6), what *should* $3^5 \cdot 3^{-5}$ equal?" Students should respond that they want to believe that equation (6) is still correct even when m and n are integers, and therefore, they *should* have $3^5 \cdot 3^{-5} = 3^{5+(-5)} = 3^0 = 1$.

Scaffolding:
- Ask students, if x is a number, then what value of x would make the following true: $3^5 \cdot x = 1$?

- What does this say about the value 3^{-5}?
  - *The value 3^{-5} must be a fraction because $3^5 \cdot 3^{-5} = 1$, specifically the reciprocal of 3^5.*

- Then, would it not be reasonable to define 3^{-n}, in general, as $\frac{1}{3^n}$?

MP.6 **Definition:** For any positive number x and for any positive integer n, we define $x^{-n} = \frac{1}{x^n}$.

Note that this definition of negative exponents says x^{-1} is just the reciprocal $\frac{1}{x}$ of x. In particular, x^{-1} would make no sense if $x = 0$. This explains why we must restrict x to be nonzero at this juncture.

The definition has the following consequence:

$$\text{For a positive } x, \quad x^{-b} = \frac{1}{x^b} \qquad \text{for all } integers \ b. \tag{9}$$

Note that (9) contains more information than the definition of negative exponent. For example, it implies that (with $b = -3$ in (9)) $5^3 = \frac{1}{5^{-3}}$.

Proof of (9): There are three possibilities for b: $b > 0$, $b = 0$, and $b < 0$. If the b in (9) is positive, then (9) is just the definition of x^{-b}, and there is nothing to prove. If $b = 0$, then both sides of (9) are seen to be equal to 1 and are, therefore, equal to each other. Again, (9) is correct. Finally, in general, let b be negative. Then $b = -n$ for some *positive* integer n. The left side of (9) is $x^{-b} = x^{-(-n)}$. The right side of (9) is equal to:

$$\frac{1}{x^{-n}} = \frac{1}{\frac{1}{x^n}} = 1 \times \frac{x^n}{1} = x^n$$

where we have made use of invert and multiply to simplify the complex fraction. Hence, the left side of (9) is again equal to the right side. The proof of (9) is complete.

Definition: For any positive number x and for any positive integer n, we define $x^{-n} = \frac{1}{x^n}$.

Note that this definition of negative exponents says x^{-1} is just the reciprocal, $\frac{1}{x}$, of x.

As a consequence of the definition, for a positive x and all *integers b,* we get

$$x^{-b} = \frac{1}{x^b}.$$

Allow time to discuss why we need to understand negative exponents.

- Answer: As we have indicated in Lesson 4, the basic impetus for the consideration of negative (and, in fact, arbitrary) exponents is the fascination with identities (1)–(3) (Lesson 4), which are valid only for positive integer exponents. Such nice looking identities *should be* valid for all exponents. These identities are the starting point for the consideration of all other exponents beyond the positive integers. Even without knowing this aspect of identities (1)–(3), one can see the benefit of having negative exponents by looking at the **complete expanded form of a decimal**. For example, the complete expanded form of 328.5403 is:

  $(3 \times 10^2) + (2 \times 10^1) + (8 \times 10^0) + (5 \times 10^{-1}) + (4 \times 10^{-2}) + (0 \times 10^{-3}) + (3 \times 10^{-4}).$

By writing the place value of the decimal digits in negative powers of 10, one gets a sense of the *naturalness* of the complete expanded form as the sum of whole number multiples of *descending* powers of 10.

Exercises 1–10 (10 minutes)

Students complete Exercise 1 independently or in pairs. Provide the correct solution. Next, have students complete Exercises 2–10 independently.

Exercise 1

Verify the general statement $x^{-b} = \dfrac{1}{x^b}$ for $x = 3$ and $b = -5$.

If b were a positive integer, then we have what the definition states. However, b is a negative integer, specifically $b = -5$, so the general statement in this case reads

$$3^{-(-5)} = \frac{1}{3^{-5}}.$$

The right side of this equation is

$$\frac{1}{3^{-5}} = \frac{1}{\frac{1}{3^5}} = 1 \times \frac{3^5}{1} = 3^5.$$

Since the left side is also 3^5, both sides are equal.

$$3^{-(-5)} = \frac{1}{3^{-5}} = 3^5.$$

Exercise 2

What is the value of (3×10^{-2})?

$$(3 \times 10^{-2}) = 3 \times \frac{1}{10^2} = \frac{3}{10^2} = 0.03$$

Exercise 3

What is the value of (3×10^{-5})?

$$(3 \times 10^{-5}) = 3 \times \frac{1}{10^5} = \frac{3}{10^5} = 0.00003$$

Exercise 4

Write the complete expanded form of the decimal 4.728 in exponential notation.

$$4.728 = (4 \times 10^0) + (7 \times 10^{-1}) + (2 \times 10^{-2}) + (8 \times 10^{-3})$$

For Exercises 5–10, write an equivalent expression, in exponential notation, to the one given and simplify as much as possible.

Exercise 5

$$5^{-3} = \frac{1}{5^3}$$

Exercise 6

$$\frac{1}{8^9} = 8^{-9}$$

Exercise 7

$$3 \cdot 2^{-4} = 3 \cdot \frac{1}{2^4} = \frac{3}{2^4}$$

Exercise 8

Let x be a nonzero number.

$$x^{-3} = \frac{1}{x^3}$$

Exercise 9	Exercise 10
Let x be a nonzero number.	Let x, y be two nonzero numbers.
$\dfrac{1}{x^9} = x^{-9}$	$xy^{-4} = x \cdot \dfrac{1}{y^4} = \dfrac{x}{y^4}$

Discussion (5 minutes)

We now state our main objective: *For any positive number x, y and for all* integers a and b,

$$x^a \cdot x^b = x^{a+b} \tag{10}$$

$$(x^b)^a = x^{ab} \tag{11}$$

$$(xy)^a = x^a y^a. \tag{12}$$

We accept that for positive numbers x, y and all integers a and b,

$$x^a \cdot x^b = x^{a+b}$$

$$(x^b)^a = x^{ab}$$

$$(xy)^a = x^a y^a.$$

We claim

$$\frac{x^a}{x^b} = x^{a-b} \qquad \text{for all integers } a, b.$$

$$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a} \qquad \text{for any integer } a.$$

Identities (10)–(12) are called the **laws of exponents** for integer exponents. They clearly generalize (6)–(8).

The laws of exponents will be proved in the next lesson. For now, we want to use them effectively.

MP.2

In the process, we will get a glimpse of why they are worth learning. We will show that knowing (10)–(12) means also knowing (4) and (5) automatically. Thus, it is enough to know only three facts, (10)–(12), rather than *five* facts, (10)–(12) and (4) and (5). Incidentally, the preceding sentence demonstrates why it is essential to learn how to use symbols because if (10)–(12) were stated in terms of explicit numbers, the preceding sentence would not even make sense.

Note to Teacher: You could mention that (10)–(12) are valid even when a and b are rational numbers (make sure they know "rational numbers" refer to positive and negative fractions). The fact that they are true also for all real numbers can only be proved in college.

We reiterate the following: The discussion below assumes the validity of (10)–(12) for the time being.   We claim

$$\frac{x^a}{x^b} = x^{a-b} \qquad \text{for all integers } a, b \qquad (13)$$

$$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a} \qquad \text{for any integer } a. \qquad (14)$$

Note that identity (13) says much more than (4): Here, a and b can be integers, rather than positive integers and moreover there is no requirement that $a > b$. Similarly unlike (5), the a in (14) is an integer rather than just a positive integer.

> *Note to Teacher:* The need for formulas about complex fractions will be obvious in subsequent lessons and will not be consistently pointed out. This fact should be brought to the attention of students. Ask students why these must be considered complex fractions.

Exercises 11 and 12 (5 minutes)

Students complete Exercises 11 and 12 independently or in pairs in preparation of the proof of (13) in general.

Exercise 11	**Exercise 12**
$\dfrac{19^2}{19^5} = 19^{2-5}$	$\dfrac{17^{16}}{17^{-3}} = 17^{16} \times \dfrac{1}{17^{-3}} = 17^{16} \times 17^3 = 17^{16+3}$

Proof of (13):

$$\frac{x^a}{x^b} = x^a \cdot \frac{1}{x^b} \qquad \text{By the product formula for complex fractions}$$

$$= x^a \cdot x^{-b} \qquad \text{By } x^{-b} = \frac{1}{x^b} \ (9)$$

$$= x^{a+(-b)} \qquad \text{By } x^a \cdot x^b = x^{a+b} \ (10)$$

$$= x^{a-b}$$

Exercises 13 and 14 (10 minutes)

Students complete Exercise 13 in preparation for the proof of (14). Check before continuing to the general proof of (14).

> **Exercise 13**
>
> **If we let $b = -1$ in (11), a be any integer, and y be any positive number, what do we get?**
>
> $$(y^{-1})^a = y^{-a}$$

Exercise 14

Show directly that $\left(\dfrac{7}{5}\right)^{-4} = \dfrac{7^{-4}}{5^{-4}}.$

$\left(\dfrac{7}{5}\right)^{-4} = \left(7 \cdot \dfrac{1}{5}\right)^{-4}$ *By the product formula*

$= (7 \cdot 5^{-1})^{-4}$ *By definition*

$= 7^{-4} \cdot (5^{-1})^{-4}$ *By $(xy)^a = x^a y^a$ (12)*

$= 7^{-4} \cdot 5^4$ *By $(x^b)^a = x^{ab}$ (11)*

$= 7^{-4} \cdot \dfrac{1}{5^{-4}}$ *By $x^{-b} = \dfrac{1}{x^b}$ (9)*

$= \dfrac{7^{-4}}{5^{-4}}$ *By product formula*

Proof of (14):

$\left(\dfrac{x}{y}\right)^a = \left(x \cdot \dfrac{1}{y}\right)^a$ By the product formula for complex fractions

$= (xy^{-1})^a$ By definition

$= x^a (y^{-1})^a$ By $(xy)^a = x^a y^a$ (12)

$= x^a y^{-a}$ By $(x^b)^a = x^{ab}$ (11), also see Exercise 13

$= x^a \cdot \dfrac{1}{y^a}$ By $x^{-b} = \dfrac{1}{x^b}$ (9)

$= \dfrac{x^a}{y^a}$

Students complete Exercise 14 independently. Provide the solution when they are finished.

Closing (5 minutes)

Summarize, or have students summarize, the lesson.

- By assuming (10)–(12) were true for integer exponents, we see that (4) and (5) would also be true.
- (10)–(12) are worth remembering because they are so useful and allow us to limit what we need to memorize.

Exit Ticket (5 minutes)

Name _____   Date_____

Lesson 5: Negative Exponents and the Laws of Exponents

Exit Ticket

Write each answer as a simplified expression that is equivalent to the given one.

1. $76543^{-4} =$

2. Let f be a nonzero number. $f^{-4} =$

3. $671 \times 28796^{-1} =$

4. Let a, b be numbers ($b \neq 0$). $ab^{-1} =$

5. Let g be a nonzero number. $\dfrac{1}{g^{-1}} =$

Exit Ticket Sample Solutions

Write each answer as a simplified expression that is equivalent to the given one.

1. $76543^{-4} = \dfrac{1}{76543^4}$

2. Let f be a nonzero number. $f^{-4} = \dfrac{1}{f^4}$

3. $671 \times 28796^{-1} = 671 \times \dfrac{1}{28796} = \dfrac{671}{28796}$

4. Let a, b be numbers $(b \neq 0)$. $ab^{-1} = a \cdot \dfrac{1}{b} = \dfrac{a}{b}$

5. Let g be a nonzero number. $\dfrac{1}{g^{-1}} = g$

Problem Set Sample Solutions

1. Compute: $3^3 \times 3^2 \times 3^1 \times 3^0 \times 3^{-1} \times 3^{-2} = 3^3 = 27$

   Compute: $5^2 \times 5^{10} \times 5^8 \times 5^0 \times 5^{-10} \times 5^{-8} = 5^2 = 25$

   Compute for a nonzero number, a: $a^m \times a^n \times a^l \times a^{-n} \times a^{-m} \times a^{-l} \times a^0 = a^0 = 1$

2. Without using (10), show directly that $(17.6^{-1})^8 = 17.6^{-8}$.

   $$(17.6^{-1})^8 = \left(\frac{1}{17.6}\right)^8 \qquad \textit{By definition}$$

   $$= \frac{1^8}{17.6^8} \qquad \textit{By } \left(\frac{x}{y}\right)^n = \frac{x^n}{y^n} \textbf{ (5)}$$

   $$= \frac{1}{17.6^8}$$

   $$= 17.6^{-8} \qquad \textit{By definition}$$

3. Without using (10), show (prove) that for any whole number n and any positive number y, $(y^{-1})^n = y^{-n}$.

   $$(y^{-1})^n = \left(\frac{1}{y}\right)^n \qquad \textit{By definition}$$

   $$= \frac{1^n}{y^n} \qquad \textit{By } \left(\frac{x}{y}\right)^n = \frac{x^n}{y^n} \textbf{ (5)}$$

   $$= \frac{1}{y^n}$$

   $$= y^{-n} \qquad \textit{By definition}$$

4. Show directly without using (13) that $\dfrac{2.8^{-5}}{2.8^7} = 2.8^{-12}$.

$$\dfrac{2.8^{-5}}{2.8^7} = 2.8^{-5} \times \dfrac{1}{2.8^7}$$ *By the product formula for complex fractions*

$$= \dfrac{1}{2.8^5} \times \dfrac{1}{2.8^7}$$ *By definition*

$$= \dfrac{1}{2.8^5 \times 2.8^7}$$ *By the product formula for complex fractions*

$$= \dfrac{1}{2.8^{5+7}}$$ *By $x^a \cdot x^b = x^{a+b}$ (10)*

$$= \dfrac{1}{2.8^{12}}$$

$$= 2.8^{-12}$$ *By definition*

Equation Reference Sheet

For any numbers x, y ($x \neq 0$ in (4) and $y \neq 0$ in (5)) and any positive integers m, n, the following holds:

$$x^m \cdot x^n = x^{m+n} \tag{1}$$

$$(x^m)^n = x^{mn} \tag{2}$$

$$(xy)^n = x^n y^n \tag{3}$$

$$\frac{x^m}{x^n} = x^{m-n} \tag{4}$$

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n} \tag{5}$$

For any numbers x, y and for all whole numbers m, n, the following holds:

$$x^m \cdot x^n = x^{m+n} \tag{6}$$

$$(x^m)^n = x^{mn} \tag{7}$$

$$(xy)^n = x^n y^n \tag{8}$$

For any positive number x and all integers b, the following holds:

$$x^{-b} = \frac{1}{x^b} \tag{9}$$

For any numbers x, y and all integers a, b, the following holds:

$$x^a \cdot x^b = x^{a+b} \tag{10}$$

$$(x^b)^a = x^{ab} \tag{11}$$

$$(xy)^a = x^a y^a \tag{12}$$

$$\frac{x^a}{x^b} = x^{a-b} \tag{13}$$

$$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a} \tag{14}$$

Lesson 6: Proofs of Laws of Exponents

Student Outcomes

- Students extend the previous laws of exponents to include all integer exponents.
- Students base symbolic proofs on concrete examples to show that $(x^b)^a = x^{ab}$ is valid for all integer exponents.

Lesson Notes

This lesson is not designed for all students, but for those who would benefit from a lesson that enriches their existing understanding of the laws of exponents. For that reason this is an optional lesson that can be used with students who have demonstrated mastery over concepts in Topic A.

Classwork

Discussion (8 minutes)

The goal of this lesson is to show why the laws of exponents, (10)–(12), are correct for all integers a and b and for all $x, y > 0$. We recall (10)–(12):

For all $x, y > 0$ and for all integers a and b, we have

$$x^a \cdot x^b = x^{a+b} \tag{10}$$

$$\left(x^b\right)^a = x^{ab} \tag{11}$$

$$(xy)^a = x^a y^a. \tag{12}$$

MP.7 & MP.8

This is a tedious process as the proofs for all three are somewhat similar. The proof of (10) is the most complicated of the three, but if one understands the proof of the easier identity (11), one will get a good idea of how all three proofs go. Therefore, we will only prove (11) completely.

We have to first decide on a strategy to prove (11). Ask students what we already know about (11).

Elicit the following from the students

- Equation (7) of Lesson 5 says for any positive x, $(x^m)^n = x^{mn}$ for all whole numbers m and n.

MP.1

How does this help us? It tells us that:

(A) (11) is already known to be true when the integers a and b, in addition, satisfy $a \geq 0, b \geq 0$.

Scaffolding:
- Keep statements (A), (B), and (C) visible throughout the lesson for reference purposes.

- Equation (9) of Lesson 5 says that the following holds:

(B) $x^{-m} = \dfrac{1}{x^m}$ for any whole number m.

How does this help us? As we shall see from an exercise below, (B) is the statement that another special case of (11) is known.

- We also know that if x is positive, then

MP.1

(C) $\left(\dfrac{1}{x}\right)^m = \dfrac{1}{x^m}$ for any whole number m.

This is because if m is a positive integer, (C) is implied by equation (5) of Lesson 4, and if $m = 0$, then both sides of (C) are equal to 1.

How does this help us? We will see from another exercise below that (C) is in fact another special case of (11), which is already known to be true.

The Laws of Exponents

For $x, y > 0$, and all integers a, b, the following holds:

$$x^a \cdot x^b = x^{a+b}$$

$$(x^b)^a = x^{ab}$$

$$(xy)^a = x^a y^a$$

Facts we will use to prove (11):

(A) (11) is already known to be true when the integers a and b satisfy $a \geq 0$, $b \geq 0$.

(B) $x^{-m} = \dfrac{1}{x^m}$ for any whole number m.

(C) $\left(\dfrac{1}{x}\right)^m = \dfrac{1}{x^m}$ for any whole number m.

Exercises 1–3 (6 minutes)

Students complete Exercises 1–3 in small groups.

Exercise 1

Show that (C) is implied by equation (5) of Lesson 4 when $m > 0$, and explain why (C) continues to hold even when $m = 0$.

Equation (5) says for any numbers $x, y, (y \neq 0)$ and any positive integer n, the following holds: $\left(\dfrac{x}{y}\right)^n = \dfrac{x^n}{y^n}$. *So,*

$$\left(\dfrac{1}{x}\right)^m = \dfrac{1^m}{x^m} \qquad\qquad By \left(\dfrac{x}{y}\right)^n = \dfrac{x^n}{y^n} \text{ for positive integer } n \text{ and nonzero } y \text{ (5)}$$

$$= \dfrac{1}{x^m} \qquad\qquad Because \; 1^m = 1$$

If m = 0, then the left side is

$$\left(\frac{1}{x}\right)^m = \left(\frac{1}{x}\right)^0$$

$$= 1 \qquad\qquad \textit{By definition of } x^0$$

and the right side is

$$\frac{1}{x^m} = \frac{1}{x^0}$$

$$= \frac{1}{1} \qquad\qquad \textit{By definition of } x^0$$

$$= 1.$$

Exercise 2

Show that (B) is in fact a special case of (11) by rewriting it as $(x^m)^{-1} = x^{(-1)m}$ for any whole number m, so that if $b = m$ (where m is a whole number) and $a = -1$, (11) becomes (B).

(B) says $x^{-m} = \frac{1}{x^m}$.

The left side of (B), x^{-m} is equal to $x^{(-1)m}$.

The right side of (B), $\frac{1}{x^m}$, is equal to $(x^m)^{-1}$ by the definition of $(x^m)^{-1}$ in Lesson 5.

Therefore, (B) says exactly that $(x^m)^{-1} = x^{(-1)m}$.

Exercise 3

Show that (C) is a special case of (11) by rewriting (C) as $(x^{-1})^m = x^{m(-1)}$ for any whole number m. Thus, (C) is the special case of (11) when $b = -1$ and $a = m$, where m is a whole number.

(C) says $\left(\frac{1}{x}\right)^m = \frac{1}{x^m}$ for any whole number m.

The left side of (C) is equal to

$$\left(\frac{1}{x}\right)^m = (x^{-1})^m \qquad\qquad \textit{By definition of } x^{-1},$$

and the right side of (C) is equal to

$$\frac{1}{x^m} = x^{-m} \qquad\qquad \textit{By definition of } x^{-m},$$

and the latter is equal to $x^{m(-1)}$. Therefore, (C) says $(x^{-1})^m = x^{m(-1)}$ for any whole number m.

Discussion (4 minutes)

MP.3

In view of the fact that the reasoning behind the proof of **(A)** (Lesson 4) clearly cannot be extended to the case when a and/or b is negative, it may be time to consider proving (11) in several separate cases so that, at the end, these cases *together* cover all possibilities. **(A)** suggests that we consider the following four separate cases of identity (11):

(i) $a, b \geq 0$

(ii) $a \geq 0, b < 0$

(iii) $a < 0, b \geq 0$

(iv) $a, b < 0$.

- Why are there are no other possibilities?
- Do we need to prove case (*i*)?
  - *No, because* **(A)** *corresponds to case (i) of (11).*

> **Scaffolding:**
> - Have students think about the four quadrants of the plane.
> - Read aloud the meaning of the four cases as you write them symbolically.

We will prove the three remaining cases in succession.

Discussion (10 minutes)

Case (ii): We have to prove that for any positive x, $(x^b)^a = x^{ab}$, when the integers a and b satisfy $a \geq 0, b < 0$. For example, we have to show that $(5^{-3})^4 = 5^{(-3)4}$, or $(5^{-3})^4 = 5^{-12}$. The following is the proof:

$$(5^{-3})^4 = \left(\frac{1}{5^3}\right)^4 \qquad \text{By definition}$$

$$= \frac{1}{(5^3)^4} \qquad \text{By } \left(\frac{1}{x}\right)^m = \frac{1}{x^m} \text{ for any whole number } m \text{ (C)}$$

$$= \frac{1}{5^{12}} \qquad \text{By } (x^m)^n = x^{mn} \text{ for all whole numbers } m \text{ and } n \text{ (A)}$$

$$= 5^{-12} \qquad \text{By definition}$$

In general, we just imitate this argument. Let $b = -c$, where c is a positive integer. We now show that the left side and the right side of $(x^b)^a = x^{ab}$ are equal. The left side is

$$(x^b)^a = (x^{-c})^a$$

$$= \left(\frac{1}{x^c}\right)^a \qquad \text{By } x^{-m} = \frac{1}{x^m} \text{ for any whole number } m \text{ (B)}$$

$$= \frac{1}{(x^c)^a} \qquad \text{By } \left(\frac{1}{x}\right)^m = \frac{1}{x^m} \text{ for any whole number } m \text{ (C)}$$

$$= \frac{1}{x^{ac}} \qquad \text{By } (x^m)^n = x^{mn} \text{ for all whole numbers } m \text{ and } n \text{ (A)}$$

> **Scaffolding:**
> - Keep the example, done previously with concrete numbers, visible so students can relate the symbolic argument to the work just completed.
> - Remind students that when concrete numbers are used, we can push through computations and show that the left side and right side are the same. For symbolic arguments we must look at each side separately and show that the two sides are equal.

The right side is

$$x^{ab} = x^{a(-c)}$$
$$= x^{-(ac)}$$
$$= \frac{1}{x^{ac}}. \qquad \text{By } x^{-m} = \frac{1}{x^m} \text{ for any whole number } m \text{ (B)}$$

The left side and the right side are equal; thus, *case (ii)* is done.

Case (iii): We have to prove that for any positive x, $(x^b)^a = x^{ab}$, when the integers a and b satisfy $a < 0$ and $b \geq 0$. This is very similar to case *(ii)*, so it will be left as an exercise.

Exercise 4 (4 minutes)

Students complete Exercise 4 independently or in pairs.

Exercise 4

Proof of Case (iii): **Show that when $a < 0$ and $b \geq 0$, $(x^b)^a = x^{ab}$ is still valid. Let $a = -c$ for some positive integer c. Show that the left side and right sides of $(x^b)^a = x^{ab}$ are equal.**

The left side is

$$(x^b)^a = (x^b)^{-c}$$
$$= \frac{1}{(x^b)^c} \qquad \textit{By } x^{-m} = \frac{1}{x^m} \textit{ for any whole number m (B)}$$
$$= \frac{1}{x^{cb}}. \qquad \textit{By } (x^m)^n = x^{mn} \textit{ for all whole numbers m and n (A)}$$

The right side is

$$x^{ab} = x^{(-c)b}$$
$$= x^{-(cb)}$$
$$= \frac{1}{x^{cb}}. \qquad \textit{By } x^{-m} = \frac{1}{x^m} \textit{ for any whole number m (B)}$$

So, the two sides are equal.

Discussion (8 minutes)

The only case remaining in the proof of (11) is *case (iv)*. Thus, we have to prove that for any positive x, $(x^b)^a = x^{ab}$ when the integers a and b satisfy $a < 0$ and $b < 0$. For example, $(7^{-5})^{-8} = 7^{5 \cdot 8}$ because

$$(7^{-5})^{-8} = \frac{1}{(7^{-5})^8} \qquad \text{By } x^{-m} = \frac{1}{x^m} \text{ for any whole number } m \text{ (B)}$$

$$= \frac{1}{7^{-(5 \cdot 8)}} \qquad \text{By case } \textit{(ii)}$$

$$= 7^{5 \cdot 8}. \qquad \text{By } x^{-m} = \frac{1}{x^m} \text{ for any whole number } m \text{ (B)}$$

In general, we can imitate this explicit argument with numbers as we did in case *(ii)*. Let $a = -c$ and $b = -d$, where c and d are positive integers. Then, the left side is

$$(x^b)^a = (x^{-c})^{-d}$$

$$= \frac{1}{(x^{-c})^d} \qquad \text{By } x^{-m} = \frac{1}{x^m} \text{ for any whole number } m \text{ (B)}$$

$$= \frac{1}{x^{-cd}} \qquad \text{By case } (ii)$$

$$= \frac{1}{\frac{1}{x^{cd}}} \qquad \text{By } x^{-m} = \frac{1}{x^m} \text{ for any whole number } m \text{ (B)}$$

$$= x^{cd}. \qquad \text{By invert-and-multiply for division of complex fractions}$$

The right side is

$$x^{ab} = x^{(-c)(-d)}$$

$$= x^{cd}.$$

Scaffolding:

- Students may ask why the exponent c remained negative in the second line while d became positive. Reconcile this through the use of a concrete example or by pointing to the previous problem $(7^{-5})^{-8} = \frac{1}{(7^{-5})^8}$ first line.

The left side is equal to the right side; thus, *case (iv)* is finished. Putting all of the cases together, the proof of (11) is complete. We now know that (11) is true for any positive integer x and any integers a, b.

Closing (2 minutes)

Summarize, or have students summarize, the lesson.

- Students have proven the laws of exponents are valid for any integer exponent.

Exit Ticket (3 minutes)

Name _____   Date_____

Lesson 6: Proofs of Laws of Exponents

Exit Ticket

1. Show directly that for any positive integer x, $x^{-5} \cdot x^{-7} = x^{-12}$.

2. Show directly that for any positive integer x, $(x^{-2})^{-3} = x^6$.

Exit Ticket Sample Solutions

1. Show directly that for any positive integer x, $x^{-5} \cdot x^{-7} = x^{-12}$.

$$x^{-5} \cdot x^{-7} = \frac{1}{x^5} \cdot \frac{1}{x^7} \qquad \text{By } x^{-m} = \frac{1}{x^m} \text{ for any whole number } m \text{ (B)}$$

$$= \frac{1}{x^5 \cdot x^7} \qquad \text{By the Product Formula for Complex Fractions}$$

$$= \frac{1}{x^{5+7}} \qquad \text{By } x^m \cdot x^n = x^{m+n} \text{ for whole numbers } m \text{ and } n \text{ (6)}$$

$$= \frac{1}{x^{12}}$$

$$= x^{-12} \qquad \text{By } x^{-m} = \frac{1}{x^m} \text{ for any whole number } m \text{ (B)}$$

2. Show directly that for any positive integer x, $(x^{-2})^{-3} = x^6$.

$$(x^{-2})^{-3} = \frac{1}{(x^{-2})^3} \qquad \text{By } x^{-m} = \frac{1}{x^m} \text{ for any whole number } m \text{ (B)}$$

$$= \frac{1}{x^{-(2 \cdot 3)}} \qquad \text{By Case (ii) of (11)}$$

$$= \frac{1}{x^{-6}}$$

$$= x^6 \qquad \text{By } x^{-m} = \frac{1}{x^m} \text{ for any whole number } m \text{ (B)}$$

Problem Set Sample Solutions

1. You sent a photo of you and your family on vacation to seven Facebook friends. If each of them sends it to five of their friends, and each of those friends sends it to five of their friends, and those friends send it to five more, how many people (not counting yourself) will see your photo? No friend received the photo twice. Express your answer in exponential notation.

# of New People to View Your Photo	Total # of People to View Your Photo
7	7
5×7	$7 + (5 \times 7)$
$5 \times 5 \times 7$	$7 + (5 \times 7) + (5^2 \times 7)$
$5 \times 5 \times 5 \times 7$	$7 + (5 \times 7) + (5^2 \times 7) + (5^3 \times 7)$

The total number of people who viewed the photo is $(5^0 + 5^1 + 5^2 + 5^3) \times 7$.

2. **Show directly, without using (11), that $(1.27^{-36})^{85} = 1.27^{-36 \cdot 85}$.**

$$(1.27^{-36})^{85} = \left(\frac{1}{1.27^{36}}\right)^{85} \qquad \textit{By definition.}$$

$$= \frac{1}{(1.27^{36})^{85}} \qquad \textit{By } \left(\frac{1}{x}\right)^m = \frac{1}{x^m} \textit{ for any whole number } m \textit{ (C)}$$

$$= \frac{1}{1.27^{36 \cdot 85}} \qquad \textit{By } (x^m)^n = x^{mn} \textit{ for whole numbers } m \textit{ and } n \textit{ (7)}$$

$$= 1.27^{-36 \cdot 85} \qquad \textit{By } x^{-m} = \frac{1}{x^m} \textit{ for any whole number } m \textit{ (B)}$$

3. **Show directly that $\left(\frac{2}{13}\right)^{-127} \cdot \left(\frac{2}{13}\right)^{-56} = \left(\frac{2}{13}\right)^{-183}$.**

$$\left(\frac{2}{13}\right)^{-127} \cdot \left(\frac{2}{13}\right)^{-56} = \frac{1}{\left(\frac{2}{13}\right)^{127}} \cdot \frac{1}{\left(\frac{2}{13}\right)^{56}} \qquad \textit{By definition}$$

$$= \frac{1}{\left(\frac{2}{13}\right)^{127} \cdot \left(\frac{2}{13}\right)^{56}} \qquad \textit{By the product formula for complex fractions}$$

$$= \frac{1}{\left(\frac{2}{13}\right)^{127+56}} \qquad \textit{By } x^m \cdot x^n = x^{m+n} \textit{ for whole numbers } m \textit{ and } n \textit{ (6)}$$

$$= \frac{1}{\left(\frac{2}{13}\right)^{183}}$$

$$= \left(\frac{2}{13}\right)^{-183} \qquad \textit{By } x^{-m} = \frac{1}{x^m} \textit{ for any whole number } m \textit{ (B)}$$

4. **Prove for any positive number x, $x^{-127} \cdot x^{-56} = x^{-183}$**

$$x^{-127} \cdot x^{-56} = \frac{1}{x^{127}} \cdot \frac{1}{x^{56}} \qquad \textit{By definition}$$

$$= \frac{1}{x^{127} \cdot x^{56}} \qquad \textit{By the product formula for complex fractions}$$

$$= \frac{1}{x^{127+56}} \qquad \textit{By } x^m \cdot x^n = x^{m+n} \textit{ for whole numbers } m \textit{ and } n \textit{ (6)}$$

$$= \frac{1}{x^{183}}$$

$$= x^{-183} \qquad \textit{By } x^{-m} = \frac{1}{x^m} \textit{ for any whole number } m \textit{ (B)}$$

5. **Prove for any positive number x, $x^{-m} \cdot x^{-n} = x^{-m-n}$ for positive integers m and n.**

$$x^{-m} \cdot x^{-n} = \frac{1}{x^m} \cdot \frac{1}{x^n} \qquad \textit{By definition}$$

$$= \frac{1}{x^m \cdot x^n} \qquad \textit{By the product formula for complex fractions}$$

$$= \frac{1}{x^{m+n}} \qquad \textit{By } x^m \cdot x^n = x^{m+n} \textit{ for whole numbers } m \textit{ and } n \textit{ (6)}$$

$$= x^{-(m+n)} \qquad \textit{By } x^{-m} = \frac{1}{x^m} \textit{ for any whole number } m \textit{ (B)}$$

$$= x^{-m-n}$$

6. Which of the preceding four problems did you find easiest to do? Explain.

   Students will likely say that $x^{-m} \cdot x^{-n} = x^{-m-n}$ (Problem 5) was the easiest problem to do. It requires the least amount of writing because the symbols are easier to write than decimal or fraction numbers.

7. Use the properties of exponents to write an equivalent expression that is a product of distinct primes, each raised to an integer power.

$$\frac{10^5 \cdot 9^2}{6^4} = \frac{(2 \cdot 5)^5 \cdot (3 \cdot 3)^2}{(2 \cdot 3)^4} = \frac{2^5 \cdot 5^5 \cdot 3^2 \cdot 3^2}{2^4 \cdot 3^4} = 2^{5-4} \cdot 3^{4-4} \cdot 5^5 = 2^1 \cdot 3^0 \cdot 5^5 = 2^1 \cdot 1 \cdot 5^5 = 2 \cdot 5^5$$

Name _____   Date _____

1. The number of users of social media has increased significantly since the year 2001. In fact, the approximate number of users has tripled each year. It was reported that in 2005 there were 3 million users of social media.

   a. Assuming that the number of users continues to triple each year, for the next three years, determine the number of users in 2006, 2007, and 2008.

   b. Assume the trend in the numbers of users tripling each year was true for all years from 2001 to 2009. Complete the table below using 2005 as year 1 with 3 million as the number of users that year.

Year	−3	−2	−1	0	1	2	3	4	5
# of users in millions					3				

   c. Given only the number of users in 2005 and the assumption that the number of users triples each year, how did you determine the number of users for years 2, 3, 4, and 5?

   d. Given only the number of users in 2005 and the assumption that the number of users triples each year, how did you determine the number of users for years 0, −1, −2, and −3?

e.   Write an equation to represent the number of users in millions, N, for year, t, $t \geq -3$.

f.   Using the context of the problem, explain whether or not the formula, $N = 3^t$ would work for finding the number of users in millions in year t, for all $t \leq 0$.

g.   Assume the total number of users continues to triple each year after 2009. Determine the number of users in 2012. Given that the world population at the end of 2011 was approximately 7 billion, is this assumption reasonable? Explain your reasoning.

2. Let m be a whole number.

    a. Use the properties of exponents to write an equivalent expression that is a product of unique primes, each raised to an integer power.

$$\frac{6^{21} \cdot 10^{7}}{30^{7}}$$

    b. Use the properties of exponents to prove the following identity:

$$\frac{6^{3m} \cdot 10^{m}}{30^{m}} = 2^{3m} \cdot 3^{2m}$$

    c. What value of m could be substituted into the identity in part (b) to find the answer to part (a)?

3.

a. Jill writes $2^3 \cdot 4^3 = 8^6$ and the teacher marked it wrong. Explain Jill's error.

b. Find n so that the number sentence below is true:
$2^3 \cdot 4^3 = 2^3 \cdot 2^n = 2^9$

c. Use the definition of exponential notation to demonstrate why $2^3 \cdot 4^3 = 2^9$ is true.

d. You write $7^5 \cdot 7^{-9} = 7^{-4}$. Keisha challenges you, "Prove it!" Show directly why your answer is correct without referencing the laws of exponents for integers; in other words, $x^a \cdot x^b = x^{a+b}$ for positive numbers x and integers a and b.

A Progression Toward Mastery					
Assessment Task Item		STEP 1 Missing or incorrect answer and little evidence of reasoning or application of mathematics to solve the problem.	STEP 2 Missing or incorrect answer but evidence of some reasoning or application of mathematics to solve the problem.	STEP 3 A correct answer with some evidence of reasoning or application of mathematics to solve the problem, or an incorrect answer with substantial evidence of solid reasoning or application of mathematics to solve the problem.	STEP 4 A correct answer supported by substantial evidence of solid reasoning or application of mathematics to solve the problem.
1	a–d 8.EE.A.1	Student answered 0–1 parts of (a)–(d) correctly. Student was able to complete the table for at least values of 0–5 for part (b). Student was unable to respond to questions or left items blank.	Student answered 2–3 parts of (a)–(d) correctly. Student was able to complete the table in part (b) correctly for 5 or more entries, including at least one value on each side of the value given for year 1. Student provided a limited expression of reasoning in parts (c) and (d).	Student answered 3–4 parts of (a)–(d) correctly. Student provided correct answers with some reasoning for making calculations OR Student had a few miscalculations but provided substantial reasoning with proper use of grade-level vocabulary.	Student answered all parts of (a)–(d) correctly. Student provided solid reasoning for making calculations with proper use of grade-level vocabulary.
	e–g 8.EE.A.1	Student answered 0–1 parts of (e)–(g) correctly. Student was unable to relate the pattern in the problem to exponential growth.	Student answered 1–2 parts of (e)–(g) correctly. Student was able to relate the pattern in the problem to exponential growth by writing an equation. Student justifications were incomplete.	Student answered 2–3 parts of (e)–(g) correctly. Equation given was correct, and student was able to answer questions, but justifications were incomplete. OR The equation given related the pattern to exponential growth, but was incomplete or contained a minor error, AND student was able to answer questions using solid reasoning based on the information provided.	Student answered all parts of (e)–(g) correctly. Student justified answers and made accurate conclusions based on the information provided in the problem. Student was able to explain limitations of equation when looking ahead in time and back in time.

2	a 8.EE.A.1	Student answered incorrectly. No evidence of use of properties of exponents.	Student answered incorrectly. Properties of exponents were used incorrectly.	Student answered correctly. Some evidence of use of properties of exponents is shown in calculations.	Student answered correctly. Student provided substantial evidence of the use of properties of exponents to simplify the expression to distinct primes.
	b–c 8.EE.A.1	Student answered parts (b)–(c) incorrectly. No evidence of use of properties of exponents.	Student answered parts (b)–(c) incorrectly. Properties of exponents were used incorrectly.	Student answered part (b) and/or part (c) correctly. Some evidence of use of properties of exponents is shown in calculations.	Student answered both parts (b) and (c) correctly. Student provided substantial evidence of the use of properties of exponents to prove the identity.
3	a 8.EE.A.1	Student stated that Jill's response was correct. <u>OR</u> Student was unable to identify the mistake and provided no additional information.	Student stated that Jill's answer was incorrect. Student was unable to identify the mistake of multiplying unlike bases. Student may have used what he or she knows about exponential notation to multiply numbers to show the answer was incorrect.	Student identified Jill's error as "multiplied unlike bases."	Student identified Jill's error as "multiplied unlike bases." Student provided a thorough explanation as to how unlike bases can be rewritten so that properties of exponents can be used properly.
	b 8.EE.A.1	Student was unable to identify the correct value for n.	Student correctly answered $n = 6$. No explanation was provided as to why the answer is correct.	Student correctly answered $n = 6$. Student stated that $4^3 = 2^6$ with little or no explanation or work shown.	Student correctly answered $n = 6$. Student *clearly showed* that 4^3 is equivalent to 2^6.
	c 8.EE.A.1	Student used the definition of exponential notation to rewrite 4^3 as $4 \times 4 \times 4$. Student was unable to complete the problem.	Student multiplied 4^3 to get 64 and was able to rewrite it as a number with a base of 2 but had the wrong exponent.	Student correctly rewrote 4^3 as 2^6, and then used the first property of exponents to show that the answer was correct.	Student correctly rewrote 4^3 as 2^6. Student used definition of exponential notation to rewrite each number as repeated multiplication. Student clearly showed how/why the exponents are added to simplify such expressions.

| d

8.EE.A.1 | Student may have been able to rewrite 7^{-9} as a fraction but was unable to operate with fractions. | Student was unable to show why part (d) was correct but may have used a property of exponents to state that the given answer was correct. | Student answered part (d) but misused or left out definitions in explanations and proofs. | Student answered part (d) correctly and used definitions and properties to thoroughly explain and prove the answer. Answer showed strong evidence that student understands exponential notation and can use the properties of exponents proficiently. |

Name _____ Date _____

1. The number of users of social media has increased significantly since the year 2001. In fact, the approximate number of users has tripled each year. It was reported that in 2005 there were 3 million users of social media.

   a. Assuming that the number of users continues to triple each year, for the next three years, determine the number of users in 2006, 2007, and 2008.

   2006 – 9 MILLION
   2007 – 27 MILLION
   2008 – 81 MILLION

   b. Assume the trend in the numbers of users tripling each year was true for all years from 2001 to 2009. Complete the table below using 2005 as year 1 with 3 million as the number of users that year.

Year	-3	-2	-1	0	1	2	3	4	5
# of users in millions	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27	81	243

   c. Given only the number of users in 2005 and the assumption that the number of users triples each year, how did you determine the number of users for years 2, 3, 4, and 5?

   I MULTIPLIED THE PRECEDING YEAR'S NUMBER OF USERS BY 3.

   d. Given only the number of users in 2005 and the assumption that the number of users triples each year, how did you determine the number of users for years 0, −1, −2, and −3?

   I DIVIDED THE NEXT YEAR'S NUMBER OF USERS BY 3.

e.   Write an equation to represent the number of users in millions, N, for year, t, $t \geq -3$.

$$N = 3^t$$

f.   Using the context of the problem, explain whether or not the formula, $N = 3^t$ would work for finding the number of users in millions in year t, for all $t \leq 0$.

WE ONLY KNOW THAT THE NUMBER OF USERS HAS TRIPLED EACH YEAR IN THE TIME FRAME OF 2001 TO 2009. FOR THAT REASON, WE CANNOT RELY ON THE FORMULA, $N=3^t$, TO WORK FOR ALL $t \leq 0$, JUST TO $t=-3$, WHICH IS THE YEAR 2001.

g.   Assume the total number of users continues to triple each year after 2009. Determine the number of users in 2012. Given that the world population at the end of 2011 was approximately 7 billion, is this assumption reasonable? Explain your reasoning.

2012 IS $t=8$, SO WHEN $t=8$ IN $N=3^t$, $N=6,561,000,000$. THE NUMBER OF USERS IN 2012, 6,561,000,000 DOES NOT EXCEED THE WORLD POPULATION OF 7 BILLION, THEREFORE IT IS POSSIBLE TO HAVE THAT NUMBER OF USERS. BUT 6,561,000,000 IS APPROXIMATELY 94% OF THE WORLD'S POPULATION. THE NUMBER OF USERS IS LIKELY LESS THAN THAT DUE TO POVERTY, ILLNESS, INFANCY, ETC. THE ASSUMPTION IS POSSIBLE, BUT NOT REASONABLE.

2. Let m be a whole number.

   a. Use the properties of exponents to write an equivalent expression that is a product of unique primes, each raised to an integer power.

$$\frac{6^{21} \cdot 10^7}{30^7} = \frac{(3 \cdot 2)^{21} \cdot 10^7}{(3 \cdot 10)^7} = \frac{3^{21} \cdot 2^{21} \cdot 10^7}{3^7 \cdot 10^7}$$

$$= 3^{21-7} \cdot 2^{21} \cdot 10^{7-7}$$

$$= 3^{14} \cdot 2^{21} \cdot 10^0$$

$$= 3^{14} \cdot 2^{21}$$

   b. Use the properties of exponents to prove the following identity:

$$\frac{6^{3m} \cdot 10^m}{30^m} = 2^{3m} \cdot 3^{2m}$$

$$\frac{6^{3m} \cdot 10^m}{30^m} = \frac{(3 \cdot 2)^{3m} \cdot 10^m}{(3 \cdot 10)^m}$$

$$= \frac{3^{3m} \cdot 2^{3m} \cdot 10^m}{3^m \cdot 10^m}$$

$$= 3^{3m-m} \cdot 2^{3m} \cdot 10^{m-m} = 3^{2m} \cdot 2^{3m} = 2^{3m} \cdot 3^{2m}$$

   c. What value of m could be substituted into the identity in part (b) to find the answer to part (a)?

$$2^{3m} \cdot 3^{2m} = 2^{21} \cdot 3^{14}$$

$$3m = 21 \qquad 2m = 14$$
$$m = 7 \qquad\quad m = 7$$

THEREFORE, $m = 7$.

3.

    a.   Jill writes $2^3 \cdot 4^3 = 8^6$ and the teacher marked it wrong. Explain Jill's error.

> JILL MULTIPLIED THE BASES, 2 AND 4, AND ADDED THE
> EXPONENTS. YOU CAN ONLY ADD THE EXPONENTS WHEN
> THE BASES BEING MULTIPLIED ARE THE SAME.

    b.   Find n so that the number sentence below is true:
$2^3 \cdot 4^3 = 2^3 \cdot 2^n = 2^9$

> $4^3 = 4 \cdot 4 \cdot 4$
> $= (2 \cdot 2)(2 \cdot 2)(2 \cdot 2)$
> $= 2^6$
>
> THEREFORE:
> $2^3 \cdot 4^3 = 2^3 \cdot 2^6 = 2^9$
> SO $n = 6$

    c.   Use the definition of exponential notation to demonstrate why $2^3 \cdot 4^3 = 2^9$ is true.

> $4^3 = 2^6$, SO $2^3 \cdot 4^3 = 2^9$ IS EQUIVALENT TO $2^3 \cdot 2^6 = 2^9$.
> BY DEFINITION OF EXPONENTIAL NOTATION:
> $2^3 \cdot 2^6 = \underbrace{(2 \times \cdots \times 2)}_{3 \text{ times}} \times \underbrace{(2 \times \cdots \times 2)}_{6 \text{ times}} = \underbrace{(2 \times \cdots \times 2)}_{3+6 \text{ times}} = 2^{3+6} = 2^9$

    d.   You write $7^5 \cdot 7^{-9} = 7^{-4}$. Keisha challenges you, "Prove it!" Show directly why your answer is correct without referencing the laws of exponents for integers; in other words, $x^a \cdot x^b = x^{a+b}$ for positive numbers x and integers a and b.

> $7^5 \cdot 7^{-9} = 7^5 \cdot \dfrac{1}{7^9}$ BY DEFINITION
>
> $= \dfrac{7^5}{7^9}$ BY PRODUCT FORMULA
>
> $= \dfrac{7^5}{7^5 \cdot 7^4}$ BY $x^m \cdot x^n = x^{m+n}$ for $x > 0$, $m, n \geq 0$
>
> $= \dfrac{1}{7^4}$ BY EQUIVALENT FRACTIONS
>
> $= 7^{-4}$ BY DEFINITION.

Mathematics Curriculum

Topic B:

Magnitude and Scientific Notation

8.EE.A.3, 8.EE.A.4

Focus Standard:	8.EE.A.3	Use numbers expressed in the form of a single digit times an integer power of 10 to estimate very large or very small quantities, and to express how many times as much one is than the other. *For example, estimate the population of the United States as 3 x 10^8 and the population of the world as 7 x 10^9, and determine that the world population is more than 20 times larger.*
	8.EE.A.4	Perform operations with numbers expressed in scientific notation, including problems where both decimal and scientific notation are used. Use scientific notation and choose units of appropriate size for measurements of very large or very small quantities (e.g., use millimeters per year for seafloor spreading). Interpret scientific notation that has been generated by technology.
Instructional Days:	7	
Lesson 7:	Magnitude (P)[1]	
Lesson 8:	Estimating Quantities (P)	
Lesson 9:	Scientific Notation (P)	
Lesson 10:	Operations with Numbers in Scientific Notation (P)	
Lesson 11:	Efficacy of Scientific Notation (S)	
Lesson 12:	Choice of Unit (E)	
Lesson 13:	Comparison of Numbers Written in Scientific Notation and Interpreting Scientific Notation Using Technology (E)	

In Topic B, students' understanding of integer exponents is expanded to include the concept of magnitude as a measurement. Students learn to estimate how big or how small a number is using magnitude. In Lesson 7, students learn that positive powers of ten are large numbers and negative powers of 10 are very small numbers. In Lesson 8, students will express large numbers in the form of a single digit times a positive power of 10 and express how many times as much one of these numbers is compared to another. Students estimate and compare national to household debt and use estimates of the number of stars in the universe to compare with the number of stars an average human can see.

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[1] Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson.

Lessons 9–13 immerse students in scientific notation. Each lesson demonstrates the need for such a notation and then how to compare and compute with numbers in scientific notation. In Lesson 9, students learn how to write numbers in scientific notation and the importance of the exponent with respect to magnitude. The number line is used to illustrate different magnitudes of 10, and students estimate where a particular number, written in scientific notation, belongs on the number line. Also, in this set of lessons, students will use what they know about exponential notation, properties of exponents, and scientific notation to interpret results that have been generated by technology.

Continuing with magnitude, Lesson 10 shows students how to operate with numbers in scientific notation by making numbers have the same magnitude. In Lessons 11–13, students reason quantitatively with scientific notation to understand several instances of how the notation is used in science. For example, students compare masses of protons and electrons written in scientific notation, and then compute how many times heavier one is than the other by using their knowledge of ratio and properties of exponents. Students use the population of California and their knowledge of proportions to estimate the population of the U.S. assuming population density is the same. Students calculate the average lifetime of subatomic particles and rewrite very small quantities (e.g., 1.6×10^{-27} kg) in a power-of-ten unit of kilograms that supports easier comparisons of the mass.

It is the direct relationship to science in Lesson 12 that provides an opportunity for students to understand why certain units were developed, like the gigaelectronvolt. Given a list of very large numbers, students will choose a unit of appropriate size and then rewrite numbers in the new unit to make comparisons easier. In Lesson 13, students combine all the skills of Module 1 as they compare numbers written in scientific notation by rewriting the given numbers as numbers with the same magnitude, using the properties of exponents. By the end of this topic, students are able to compare and perform operations on numbers given in both decimal and scientific notation.

Lesson 7: Magnitude

Student Outcomes

- Students know that positive powers of 10 are very large numbers, and negative powers of 10 are very small numbers.
- Students know that the exponent of an expression provides information about the magnitude of a number.

Classwork

Discussion (5 minutes)

In addition to applications within mathematics, exponential notation is indispensable in science. It is used to clearly display the **magnitude** of a measurement (e.g., *How big? How small?*). We will explore this aspect of exponential notation in the next seven lessons.

Understanding magnitude demands an understanding of the *integer* powers of 10. Therefore, we begin with two fundamental facts about the integer powers of 10. What does it mean to say that 10^n for large positive integers n are *big* numbers. What does it mean to say that 10^{-n} for large positive integers n are *small* numbers?

> *Scaffolding:*
> - Remind students that special cases are cases when concrete numbers are used. They provide a way to become familiar with the mathematics before moving to the general case.

MP.6

Fact 1: The numbers 10^n for arbitrarily large positive integers n are big numbers; given a number M (no matter how big it is), there is a power of 10 that exceeds M.

MP.8

Fact 2: The numbers 10^{-n} for arbitrarily large positive integers n are small numbers; given a positive number S (no matter how small it is), there is a (negative) power of 10 that is smaller than S.

Fact 2 is a consequence of *Fact 1*. We address *Fact 1* first. The following two special cases illustrate why this is true.

> **Fact 1:** *The number 10^n, for arbitrarily large positive integers n, is a* big number in the sense that given a number M (no matter how big it is) there is a power of 10 that exceeds M.
>
> **Fact 2:** *The number 10^{-n}, for arbitrarily large positive integers n, is a* small number in the sense that given a positive number S (no matter how small it is), there is a (negative) power of 10 that is smaller than S.

Examples 1–2 (5 minutes)

Example 1: Let M be the world population as of March 23, 2013. Approximately, $M = 7{,}073{,}981{,}143$. It has 10 digits and is, therefore, smaller than any whole number with 11 digits, such as $10{,}000{,}000{,}000$. But $10{,}000{,}000{,}000 = 10^{10}$, so $M < 10^{10}$ (i.e., the 10th power of 10 exceeds this M).

Example 2: Let M be the U.S. national debt on March 23, 2013. $M = 16{,}755{,}133{,}009{,}522$ to the nearest dollar. It has 14 digits. The largest 14-digit number is $99{,}999{,}999{,}999{,}999$. Therefore,

$$M < 99{,}999{,}999{,}999{,}999 < 100{,}000{,}000{,}000{,}000 = 10^{14}$$

That is, the 14th power of 10 exceeds M.

Exercises 1 and 2 (5 minutes)

Students complete Exercises 1 and 2 independently.

Exercise 1

Let $M = 993{,}456{,}789{,}098{,}765$. **Find the smallest power of 10 that will exceed M.**

$M = 993{,}456{,}789{,}098{,}765 < 999{,}999{,}999{,}999{,}999 < 1{,}000{,}000{,}000{,}000{,}000 = 10^{15}$. *Because M has 15 digits, 10^{15} will exceed it.*

Exercise 2

Let $= 78{,}491\frac{899}{987}$. **Find the smallest power of 10 that will exceed M.**

$M = 78{,}491\frac{899}{987} < 78{,}492 < 99{,}999 < 100{,}000 = 10^5$
Therefore, 10^5 will exceed M.

Example 3 (8 minutes)

This example set is for the general case of *Fact 1*.

<u>Case 1</u>: Let M be a positive integer with n digits.

As we know, the integer $99\cdots99$ (with n 9s) is $\geq M$.

Therefore, $100\cdots00$ (with n 0s) exceeds M.

Since $10^n = 100\cdots00$ (with n 0s), we have $10^n > M$.

Symbolically,

$$M < \underbrace{99\cdots9}_{n} < 1\underbrace{00\cdots0}_{n} = 10^n$$

Therefore, for an n-digit positive integer M, the nth power of 10 (i.e., 10^n) always exceeds M.

<u>Case 2</u>: In Case 1, M was a positive integer. For this case, let M be a non-integer. We know M must lie between two consecutive points on a number line (i.e., there is some *integer N* so that M lies between N and $N + 1$). In other words, $N < M < N + 1$. (e.g., The number $45{,}572.384$ is between $45{,}572$ and $45{,}473$.)
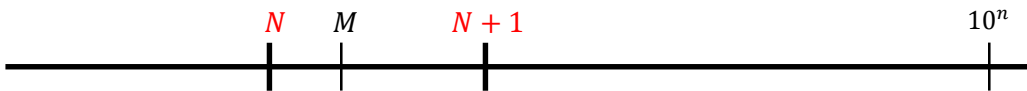
Consider the positive integer $N + 1$: According to the reasoning above, there is a positive integer n so that $10^n > N + 1$. Since $N + 1 > M$, we have $10^n > M$ again. Consequently, for this number M, 10^n exceeds it.

We have now shown why *Fact 1* is correct.

Exercise 3 (2 minutes)

Students discuss Exercise 3 and record their explanations with a partner.

Exercise 3

Let M be a positive integer. Explain how to find the smallest power of 10 that exceeds it.

If M is a positive integer, then the power of 10 that exceeds it will be equal to the number of digits in M. For example, if M were a 10-digit number, then 10^{10} would exceed M. If M is a positive number, but not an integer, then the power of 10 that would exceed it would be the same power of 10 that would exceed the integer to the right of M on a number line. For example, if $M = 5678.9$, the integer to the right of M is $5,679$. Then based on the first explanation, 10^4 exceeds both this integer and M; this is because $M = 5,678.9 < 5,679 < 10,000 = 10^4$.

Example 4 (5 minutes)

▪ The average ant weighs about 0.0003 grams.

Observe that this number is less than 1 and is very small. We want to express this number as a power of 10. We know that $10^0 = 1$, so we must need a power of 10 that is less than zero. Based on our knowledge of decimals and fractions, we can rewrite the weight of the ant as $\frac{3}{10,000}$, which is the same as $\frac{3}{10^4}$. Our work with the laws of exponents taught us that $\frac{3}{10^4} = 3 \times \frac{1}{10^4} = 3 \times 10^{-4}$. Therefore, we can express the weight of an average ant as 3×10^{-4} grams.

▪ The mass of a neutron is 0.000 000 000 000 000 000 000 001 674 9 kilograms.

Let's look at an approximated version of the mass of a neutron, 0.000 000 000 000 000 000 000 000 001 kilograms. We already know that 10^{-4} takes us to the place value four digits to the right of the decimal point (ten-thousandths). We need a power of 10 that would take us 27 places to the right of the decimal, which means that we can represent the simplified mass of a neutron as 10^{-27}.

In general, numbers with a value less than 1 but greater than 0 can be expressed using a negative power of 10. The closer a number is to zero, the smaller the power of 10 that will be needed to express it.

Exercises 4–6 (5 minutes)

Students complete Exercises 4–6 independently.

> **Exercise 4**
>
> The chance of you having the same DNA as another person (other than an identical twin) is approximately 1 in 10 trillion (one trillion is a 1 followed by 12 zeros). Given the fraction, express this very small number using a negative power of 10.
>
> $$\frac{1}{10,000,000,000,000} = \frac{1}{10^{13}} = 10^{-13}$$
>
> **Exercise 5**
>
> The chance of winning a big lottery prize is about 10^{-8}, and the chance of being struck by lightning in the U.S. in any given year is about 0.000001. Which do you have a greater chance of experiencing? Explain.
>
> $$0.000001 = 10^{-6}$$
>
> *There is a greater chance of experiencing a lightning strike. On a number line, 10^{-8} is to the left of 10^{-6}. Both numbers are less than one (one signifies 100% probability of occurring). Therefore, the probability of the event that is greater is 10^{-6} – that is, getting struck by lightning.*
>
> **Exercise 6**
>
> There are about 100 million smartphones in the U.S. Your teacher has one smartphone. What share of U.S. smartphones does your teacher have? Express your answer using a negative power of 10.
>
> $$\frac{1}{100,000,000} = \frac{1}{10^{8}} = 10^{-8}$$

Closing (2 minutes)

Summarize the lesson for the students.

- No matter what number is given, we can find the smallest power of 10 that exceeds that number.
- Very large numbers have a positive power of 10.
- We can use negative powers of 10 to represent very small numbers that are less than one but greater than zero.

Exit Ticket (5 minutes)

Name _____   Date_____

Lesson 7: Magnitude

Exit Ticket

1. Let $M = 118{,}526.65902$. Find the smallest power of 10 that will exceed M.

2. Scott said that 0.09 was a bigger number than 0.1. Use powers of 10 to show that he is wrong.

Exit Ticket Sample Solutions

> 1. Let $M = 118,526.65902$. Find the smallest power of 10 that will exceed M.
>
>   *Since* $M = 118,527 < 1,000,000 < 10^6$, *then* 10^6 *will exceed M.*
>
> 2. Scott said that 0.09 was a bigger number than 0.1. Use powers of 10 to show that he is wrong.
>
>   *We can rewrite* 0.09 *as* $\dfrac{9}{10^2} = 9 \times 10^{-2}$ *and rewrite* 0.1 *as* $\dfrac{1}{10^1} = 1 \times 10^{-1}$. *Because* 0.09 *has a smaller power of 10,* 0.09 *is closer to zero and smaller than* 0.1.

Problem Set Sample Solutions

> 1. What is the smallest power of 10 that would exceed $987,654,321,098,765,432$?
>
> $987,654,321,098,765,432 < 999,999,999,999,999,999 < 1,000,000,000,000,000,000 = 10^{18}$
>
> 2. What is the smallest power of 10 that would exceed $999,999,999,991$?
>
> $999,999,999,991 < 999,999,999,999 < 1,000,000,000,000 = 10^{12}$
>
> 3. Which number is equivalent to 0.0000001: 10^7 or 10^{-7}? How do you know?
>
> $0.0000001 = 10^{-7}$ *Negative powers of* 10 *denote numbers greater than zero but less than* 1. *Also, the decimal* 0.0000001 *is equal to the fraction* $\dfrac{1}{10^7}$, *which is equivalent to* 10^{-7}.
>
> 4. Sarah said that 0.00001 is bigger than 0.001 because the first number has more digits to the right of the decimal point. Is Sarah correct? Explain your thinking using negative powers of 10 and the number line.
>
> $0.00001 = \dfrac{1}{100000} = 10^{-5}$ *and* $0.001 = \dfrac{1}{1000} = 10^{-3}$. *On a number line,* 10^{-5} *is closer to zero than* 10^{-3}; *therefore,* 10^{-5} *is the <u>smaller</u> number, and Sarah is incorrect.*
>
> 5. Place each of the following numbers on a number line in its approximate location:
>
> $$10^5 \qquad 10^{-99} \qquad 10^{-17} \qquad 10^{14} \qquad 10^{-5} \qquad 10^{30}$$
>
>
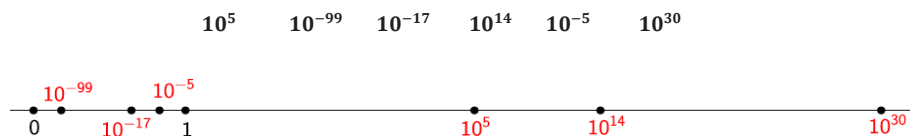
Lesson 8: Estimating Quantities

Student Outcomes

- Students compare and estimate quantities in the form of a single digit times a power of 10.
- Students use their knowledge of ratios, fractions, and laws of exponents to simplify expressions.

Classwork

Discussion (1 minute)

Now that we know about positive and negative powers of 10, we can compare numbers and estimate how many times greater one quantity is compared to another.

With our knowledge of the laws of integer exponents, we can also do other computations to estimate quantities.

Example 1 (4 minutes)

In 1723, the population of New York City was approximately 7,248. By 1870, almost 150 years later, the population had grown to 942,292. We want to determine approximately how many times greater the population was in 1870, compared to 1723.

The word "approximately" in the question lets us know that we do not need to find a precise answer, so we will approximate both populations as powers of 10.

- Population in 1723: $7{,}248 < 9{,}999 < 10{,}000 = 10^4$
- Population in 1870: $942{,}292 < 999{,}999 < 1{,}000{,}000 = 10^6$

We want to compare the population in 1870 to the population in 1723:

$$\frac{10^6}{10^4}$$

Now we can use what we know about the laws of exponents to simplify the expression and answer the question:

$$\frac{10^6}{10^4} = 10^2$$

Therefore, there were approximately 100 *times more people in New York City in* 1870 *compared to* 1723.

Exercise 1 (3 minutes)

Students complete Exercise 1 independently.

Exercise 1

The Federal Reserve states that the average household in January of 2013 had $7,122$ in credit card debt. About how many times greater is the U.S. national debt, which is $16,755,133,009,522$? Rewrite each number to the nearest power of 10 that exceeds it, and then compare.

Household debt $= 7,122 < 9,999 < 10,000 = 10^4$.

U.S. debt $= 16,755,133,009,522 < 99,999,999,999,999 < 100,000,000,000,000 = 10^{14}$.

$\frac{10^{14}}{10^4} = 10^{14-4} = 10^{10}$. *The U.S. national debt is 10^{10} times greater than the average household's credit card debt.*

Discussion (3 minutes)

If our calculations were more precise in the last example, we would have seen that the population of New York City actually increased by about 130 times from what it was in 1723.

In order to be more precise, we need to use estimations of our original numbers that are more precise than just powers of 10.

For example, instead of estimating the population of New York City in 1723 (7,248 people) to be 10^4, we can use a more precise estimation: 7×10^3. Using a single-digit integer times a power of ten is more precise because we are rounding the population to the nearest thousand. Conversely, using only a power of ten, we are rounding the population to the nearest ten thousand.

Consider that the actual population is 7,248.

- $10^4 = 10,000$
- $7 \times 10^3 = 7 \times 1,000 = 7,000$

Which of these two estimations is closer to the actual population?

Clearly, 7×10^3 is a more precise estimation.

Scaffolding:

- Make it clear to students that more precise estimations lead to more precise answers.
- In a later lesson, we will learn how to be even more precise than in Example 2.

Example 2 (4 minutes)

Let's compare the population of New York City to the population of New York State. Specifically, let's find out how many times greater the population of New York State is compared to that of New York City.

The population of New York City is 8,336,697. Let's round this number to the nearest million; this gives us 8,000,000. Written as single-digit integer times a power of 10:

$$8,000,000 = 8 \times 10^6$$

The population of New York State is 19,570,261. Rounding to the nearest million gives us 20,000,000. Written as a single-digit integer times a power of 10:

$$20,000,000 = 2 \times 10^7$$

To estimate the difference in size we compare state population to city population:

$$\frac{2 \times 10^7}{8 \times 10^6}$$

Now we simplify the expression to find the answer:

$$\frac{2 \times 10^7}{8 \times 10^6} = \frac{2}{8} \times \frac{10^7}{10^6}$$ By the product formula

$$= \frac{1}{4} \times 10$$ By equivalent fractions and the first law of exponents

$$= 0.25 \times 10$$

$$= 2.5$$

Therefore, the population of the state is 2.5 *times that of the city.*

Example 3 (4 minutes)

There are about 9 billion devices connected to the Internet. If a wireless router can support 300 devices, about how many wireless routers are necessary to connect all 9 billion devices wirelessly?

Because 9 billion is a very large number, we should express it as a single-digit integer times a power of 10.

$$9,000,000,000 = 9 \times 10^9$$

The laws of exponents tells us that our calculations will be easier if we also express 300 as a single-digit integer times a power of 10, even though 300 is much smaller.

$$300 = 3 \times 10^2$$

We want to know how many wireless routers are necessary to support 9 billion devices, so we must divide

$$\frac{9 \times 10^9}{3 \times 10^2}.$$

Now, we can simplify the expression to find the answer:

$$\frac{9 \times 10^9}{3 \times 10^2} = \frac{9}{3} \times \frac{10^9}{10^2}$$ By the product formula

$$= 3 \times 10^7$$ By equivalent fractions and the first law of exponents

$$= 30,000,000$$

About 30 *million routers are necessary to connect all devices wirelessly.*

Exercises 2–4 (5 minutes)

Students complete Exercises 2–4 independently or in pairs.

Exercise 2

There are about $3,000,000$ students attending school, kindergarten through 12th grade, in New York. Express the number of students as a single-digit integer times a power of 10.

$$3,000,000 = 3 \times 10^6$$

The average number of students attending a middle school in New York is 8×10^2. How many times greater is the overall number of K-12 students compared to the number of middle school students?

$$\frac{3 \times 10^6}{8 \times 10^2} = \frac{3}{8} \times \frac{10^6}{10^2}$$
$$= \frac{3}{8} \times 10^4$$
$$= 0.375 \times 10^4$$
$$= 3,750$$

There are about $3,750$ times more students in K-12 compared to the number of students in middle school.

Exercise 3

A conservative estimate of the number of stars in the universe is 6×10^{22}. The average human can see about $3,000$ stars at night with his naked eye. About how many times more stars are there in the universe, compared to the stars a human can actually see?

$$\frac{6 \times 10^{22}}{3 \times 10^3} = \frac{6}{3} \times \frac{10^{22}}{10^3} = 2 \times 10^{22-3} = 2 \times 10^{19}$$

There are about 2×10^{19} times more stars in the universe compared to the number we can actually see.

Exercise 4

The estimated world population in 2011 was 7×10^9. Of the total population, 682 million of those people were left-handed. Approximately what percentage of the world population is left-handed according to the 2011 estimation?

$$682,000,000 \approx 700,000,000 = 7 \times 10^8$$
$$\frac{7 \times 10^8}{7 \times 10^9} = \frac{7}{7} \times \frac{10^8}{10^9}$$
$$= 1 \times \frac{1}{10}$$
$$= \frac{1}{10}$$

About one-tenth of the population is left-handed, which is equal to 10%.

Lesson 8:   Estimating Quantities
Date: 6/17/14

Example 4 (3 minutes)

The average American household spends about \$40,000 each year. If there are about 1×10^8 households, what is the total amount of money spent by American households in one year?

Let's express \$40,000 as a single-digit integer times a power of 10.

$$40,000 = 4 \times 10^4$$

The question asks us how much money *all* American households spend in one year, which means that we need to multiply the amount spent by one household by the total number of households:

$$(4 \times 10^4)(1 \times 10^8) = (4 \times 1)(10^4 \times 10^8) \quad \text{By repeated use of associative and commutative properties}$$
$$= 4 \times 10^{12} \quad \text{By the first law of exponents}$$

Therefore, American households spend about \$4,000,000,000,000 *each year altogether!*

Exercise 5 (2 minutes)

Students complete Exercise 5 independently.

Exercise 5

The average person takes about $30,000$ breaths per day. Express this number as a single-digit integer times a power of 10.

$$30,000 = 3 \times 10^4$$

If the average American lives about 80 years (or about $30,000$ days), how many total breaths will a person take in her lifetime?

$$(3 \times 10^4) \times (3 \times 10^4) = 9 \times 10^8$$

The average American takes about $900,000,000$ *breaths in a lifetime.*

Closing (2 minutes)

Summarize the lesson for the students.

- In general, close approximation of quantities will lead to more precise answers.
- We can multiply and divide numbers that are written in the form of a single-digit integer times a power of 10.

Exit Ticket (4 minutes)

Fluency Exercise (10 minutes)

Sprint: Practice the laws of exponents. Instruct students to write answers using positive exponents only. This exercise can be administered at any point during the lesson. Refer to the Sprints and Sprint Delivery Script sections in the Module Overview for directions to administer a Sprint.

Name _____   Date_____

Lesson 8: Estimating Quantities

Exit Ticket

Most English-speaking countries use the short-scale naming system, in which a trillion is expressed as 1,000,000,000,000. Some other countries use the long-scale naming system, in which a trillion is expressed as 1,000,000,000,000,000,000,000. Express each number as a single-digit integer times a power of ten. How many times greater is the long-scale naming system than the short-scale?

Exit Ticket Sample Solution

> Most English-speaking countries use the short-scale naming system, in which a trillion is expressed as $1,000,000,000,000$. Some other countries use the long-scale naming system, in which a trillion is expressed as $1,000,000,000,000,000,000,000$. Express each number as a single-digit integer times a power of ten. How many times greater is the long-scale naming system than the short-scale?
>
> $1,000,000,000,000 = 10^{12}$
>
> $1,000,000,000,000,000,000,000 = 10^{21}$
>
> $\frac{10^{21}}{10^{12}} = 10^9$ *The long-scale is about 10^9 times greater than the short-scale.*

Problem Set Sample Solutions

Students practice estimating size of quantities and performing operations on numbers written in the form of a single-digit integer times a power of 10.

> 1. The Atlantic Ocean region contains approximately 2×10^{16} gallons of water. Lake Ontario has approximately $8,000,000,000,000$ gallons of water. How many Lake Ontarios would it take to fill the Atlantic Ocean region in terms of gallons of water?
>
> $$8,000,000,000,000 = 8 \times 10^{12}$$
>
> $$\frac{2 \times 10^{16}}{8 \times 10^{12}} = \frac{2}{8} \times \frac{10^{16}}{10^{12}}$$
> $$= \frac{1}{4} \times 10^4$$
> $$= 0.25 \times 10^4$$
> $$= 2,500$$
>
> $2,500$ *Lake Ontario's would be needed to fill the Atlantic Ocean region.*
>
> 2. U.S. national forests cover approximately $300,000$ square miles. Conservationists want the total square footage of forests to be $300,000^2$ square miles. When Ivanna used her phone to do the calculation, her screen showed the following:
>
> a. What does the answer on her screen mean? Explain how you know.
>
>   *The answer means 9×10^{10}. This is because:*
> $$(300,000)^2 = (3 \times 10^5)^2$$
> $$= 3^2 \times (10^5)^2$$
> $$= 9 \times 10^{10}$$
>
> b. Given that the U.S. has approximately 4 million square miles of land, is this a reasonable goal for conservationists? Explain.
>
> $4,000,000 = 4 \times 10^6$. *It is unreasonable for conservationists to think the current square mileage of forests could increase that much because that number is greater than the number that represents the total number of square miles in the U.S,* $9 \times 10^{10} > 4 \times 10^6$.

3. The average American is responsible for about $20,000$ kilograms of carbon emission pollution each year. Express this number as a single-digit integer times a power of 10.

$$20,000 = 2 \times 10^4$$

4. The United Kingdom is responsible for about 1×10^4 kilograms. Which country is responsible for greater carbon emission pollution each year? By how much?

$$2 \times 10^4 > 1 \times 10^4$$

America is responsible for greater carbon emission pollution each year. America produces twice the amount of the U.K. pollution.

Number Correct: _____

Properties of Integer Exponents to Generate Equivalent Expressions – Round 1

Directions: Simplify each item as much as possible. Answers should have only positive exponents. All letters denote numbers.

1.	$4^5 \cdot 4^{-4} =$		23.	$\left(\dfrac{1}{2}\right)^6 =$	
2.	$4^5 \cdot 4^{-3} =$		24.	$(3x)^5 =$	
3.	$4^5 \cdot 4^{-2} =$		25.	$(3x)^7 =$	
4.	$7^{-4} \cdot 7^{11} =$		26.	$(3x)^9 =$	
5.	$7^{-4} \cdot 7^{10} =$		27.	$(8^{-2})^3 =$	
6.	$7^{-4} \cdot 7^9 =$		28.	$(8^{-3})^3 =$	
7.	$9^{-4} \cdot 9^{-3} =$		29.	$(8^{-4})^3 =$	
8.	$9^{-4} \cdot 9^{-2} =$		30.	$(22^0)^{50} =$	
9.	$9^{-4} \cdot 9^{-1} =$		31.	$(22^0)^{55} =$	
10.	$9^{-4} \cdot 9^0 =$		32.	$(22^0)^{60} =$	
11.	$5^0 \cdot 5^1 =$		33.	$\left(\dfrac{1}{11}\right)^{-5} =$	
12.	$5^0 \cdot 5^2 =$		34.	$\left(\dfrac{1}{11}\right)^{-6} =$	
13.	$5^0 \cdot 5^3 =$		35.	$\left(\dfrac{1}{11}\right)^{-7} =$	
14.	$(12^3)^9 =$		36.	$\dfrac{56^{-23}}{56^{-34}} =$	
15.	$(12^3)^{10} =$		37.	$\dfrac{87^{-12}}{87^{-34}} =$	
16.	$(12^3)^{11} =$		38.	$\dfrac{23^{-15}}{23^{-17}} =$	
17.	$(7^{-3})^{-8} =$		39.	$(-2)^{-12} \cdot (-2)^1 =$	
18.	$(7^{-3})^{-9} =$		40.	$\dfrac{2y}{y^3} =$	
19.	$(7^{-3})^{-10} =$		41.	$\dfrac{5xy^7}{15x^7y} =$	
20.	$\left(\dfrac{1}{2}\right)^9 =$		42.	$\dfrac{16x^6y^9}{8x^{-5}y^{-11}} =$	
21.	$\left(\dfrac{1}{2}\right)^8 =$		43.	$(2^3 \cdot 4)^{-5} =$	
22.	$\left(\dfrac{1}{2}\right)^7 =$		44.	$(9^{-8})(27^{-2}) =$	

Properties of Integer Exponents to Generate Equivalent Expressions – Round 1 [KEY]

Directions: Simplify each item as much as possible. Answers should have only positive exponents. All letters denote numbers.

1.	$4^5 \cdot 4^{-4} =$	4^1	23.	$\left(\frac{1}{2}\right)^6 =$	$\frac{1}{2^6}$
2.	$4^5 \cdot 4^{-3} =$	4^2	24.	$(3x)^5 =$	$3^5 x^5$
3.	$4^5 \cdot 4^{-2} =$	4^3	25.	$(3x)^7 =$	$3^7 x^7$
4.	$7^{-4} \cdot 7^{11} =$	7^7	26.	$(3x)^9 =$	$3^9 x^9$
5.	$7^{-4} \cdot 7^{10} =$	7^6	27.	$(8^{-2})^3 =$	$\frac{1}{8^6}$
6.	$7^{-4} \cdot 7^9 =$	7^5	28.	$(8^{-3})^3 =$	$\frac{1}{8^9}$
7.	$9^{-4} \cdot 9^{-3} =$	$\frac{1}{9^7}$	29.	$(8^{-4})^3 =$	$\frac{1}{8^{12}}$
8.	$9^{-4} \cdot 9^{-2} =$	$\frac{1}{9^6}$	30.	$(22^0)^{50} =$	1
9.	$9^{-4} \cdot 9^{-1} =$	$\frac{1}{9^5}$	31.	$(22^0)^{55} =$	1
10.	$9^{-4} \cdot 9^0 =$	$\frac{1}{9^4}$	32.	$(22^0)^{60} =$	1
11.	$5^0 \cdot 5^1 =$	5^1	33.	$\left(\frac{1}{11}\right)^{-5} =$	11^5
12.	$5^0 \cdot 5^2 =$	5^2	34.	$\left(\frac{1}{11}\right)^{-6} =$	11^6
13.	$5^0 \cdot 5^3 =$	5^3	35.	$\left(\frac{1}{11}\right)^{-7} =$	11^7
14.	$(12^3)^9 =$	12^{27}	36.	$\frac{56^{-23}}{56^{-34}} =$	56^{11}
15.	$(12^3)^{10} =$	12^{30}	37.	$\frac{87^{-12}}{87^{-34}} =$	87^{22}
16.	$(12^3)^{11} =$	12^{33}	38.	$\frac{23^{-15}}{23^{-17}} =$	23^2
17.	$(7^{-3})^{-8} =$	7^{24}	39.	$(-2)^{-12} \cdot (-2)^1 =$	$\frac{1}{(-2)^{11}}$
18.	$(7^{-3})^{-9} =$	7^{27}	40.	$\frac{2y}{y^3} =$	$\frac{2}{y^2}$
19.	$(7^{-3})^{-10} =$	7^{30}	41.	$\frac{5xy^7}{15x^7y} =$	$\frac{y^6}{3x^6}$
20.	$\left(\frac{1}{2}\right)^9 =$	$\frac{1}{2^9}$	42.	$\frac{16x^6y^9}{8x^{-5}y^{-11}} =$	$2x^{11}y^{20}$
21.	$\left(\frac{1}{2}\right)^8 =$	$\frac{1}{2^8}$	43.	$(2^3 \cdot 4)^{-5} =$	$\frac{1}{2^{25}}$
22.	$\left(\frac{1}{2}\right)^7 =$	$\frac{1}{2^7}$	44.	$(9^{-8})(27^{-2}) =$	$\frac{1}{3^{22}}$

Number Correct: _____

Improvement: _____

Properties of Integer Exponents to Generate Equivalent Expressions – Round 2

Directions: Simplify each item as much as possible. Answers should have only positive exponents. All letters denote numbers.

1.	$11^5 \cdot 11^{-4} =$		23.	$\left(\dfrac{3}{7}\right)^5 =$	
2.	$11^5 \cdot 11^{-3} =$		24.	$(18xy)^5 =$	
3.	$11^5 \cdot 11^{-2} =$		25.	$(18xy)^7 =$	
4.	$7^{-7} \cdot 7^9 =$		26.	$(18xy)^9 =$	
5.	$7^{-8} \cdot 7^9 =$		27.	$(5.2^{-2})^3 =$	
6.	$7^{-9} \cdot 7^9 =$		28.	$(5.2^{-3})^3 =$	
7.	$(-6)^{-4} \cdot (-6)^{-3} =$		29.	$(5.2^{-4})^3 =$	
8.	$(-6)^{-4} \cdot (-6)^{-2} =$		30.	$(22^6)^0 =$	
9.	$(-6)^{-4} \cdot (-6)^{-1} =$		31.	$(22^{12})^0 =$	
10.	$(-6)^{-4} \cdot (-6)^0 =$		32.	$(22^{18})^0 =$	
11.	$x^0 \cdot x^1 =$		33.	$\left(\dfrac{4}{5}\right)^{-5} =$	
12.	$x^0 \cdot x^2 =$		34.	$\left(\dfrac{4}{5}\right)^{-6} =$	
13.	$x^0 \cdot x^3 =$		35.	$\left(\dfrac{4}{5}\right)^{-7} =$	
14.	$(12^5)^9 =$		36.	$\left(\dfrac{6^{-2}}{7^5}\right)^{-11} =$	
15.	$(12^6)^9 =$		37.	$\left(\dfrac{6^{-2}}{7^5}\right)^{-12} =$	
16.	$(12^7)^9 =$		38.	$\left(\dfrac{6^{-2}}{7^5}\right)^{-13} =$	
17.	$(7^{-3})^{-4} =$		39.	$\left(\dfrac{6^{-2}}{7^5}\right)^{-15} =$	
18.	$(7^{-4})^{-4} =$		40.	$\dfrac{42ab^{10}}{14a^{-9}b} =$	
19.	$(7^{-5})^{-4} =$		41.	$\dfrac{5xy^7}{25x^7y} =$	
20.	$\left(\dfrac{3}{7}\right)^8 =$		42.	$\dfrac{22a^{15}b^{32}}{121ab^{-5}} =$	
21.	$\left(\dfrac{3}{7}\right)^7 =$		43.	$(7^{-8} \cdot 49)^{-5} =$	
22.	$\left(\dfrac{3}{7}\right)^6 =$		44.	$(36^9)(216^{-2}) =$	

Properties of Integer Exponents to Generate Equivalent Expressions – Round 2 [KEY]

Directions: Simplify each item as much as possible. Answers should have only positive exponents. All letters denote numbers.

1.	$11^5 \cdot 11^{-4} =$	$\mathbf{11^1}$	23.	$\left(\dfrac{3}{7}\right)^5 =$	$\mathbf{\dfrac{3^5}{7^5}}$
2.	$11^5 \cdot 11^{-3} =$	$\mathbf{11^2}$	24.	$(18xy)^5 =$	$\mathbf{18^5 x^5 y^5}$
3.	$11^5 \cdot 11^{-2} =$	$\mathbf{11^3}$	25.	$(18xy)^7 =$	$\mathbf{18^7 x^7 y^7}$
4.	$7^{-7} \cdot 7^9 =$	$\mathbf{7^2}$	26.	$(18xy)^9 =$	$\mathbf{18^9 x^9 y^9}$
5.	$7^{-8} \cdot 7^9 =$	$\mathbf{7^1}$	27.	$(5.2^{-2})^3 =$	$\mathbf{\dfrac{1}{(5.2)^6}}$
6.	$7^{-9} \cdot 7^9 =$	$\mathbf{1}$	28.	$(5.2^{-3})^3 =$	$\mathbf{\dfrac{1}{(5.2)^9}}$
7.	$(-6)^{-4} \cdot (-6)^{-3} =$	$\mathbf{\dfrac{1}{(-6)^7}}$	29.	$(5.2^{-4})^3 =$	$\mathbf{\dfrac{1}{(5.2)^{12}}}$
8.	$(-6)^{-4} \cdot (-6)^{-2} =$	$\mathbf{\dfrac{1}{(-6)^6}}$	30.	$(22^6)^0 =$	$\mathbf{1}$
9.	$(-6)^{-4} \cdot (-6)^{-1} =$	$\mathbf{\dfrac{1}{(-6)^5}}$	31.	$(22^{12})^0 =$	$\mathbf{1}$
10.	$(-6)^{-4} \cdot (-6)^0 =$	$\mathbf{\dfrac{1}{(-6)^4}}$	32.	$(22^{18})^0 =$	$\mathbf{1}$
11.	$x^0 \cdot x^1 =$	$\mathbf{x^1}$	33.	$\left(\dfrac{4}{5}\right)^{-5} =$	$\mathbf{\dfrac{5^5}{4^5}}$
12.	$x^0 \cdot x^2 =$	$\mathbf{x^2}$	34.	$\left(\dfrac{4}{5}\right)^{-6} =$	$\mathbf{\dfrac{5^6}{4^6}}$
13.	$x^0 \cdot x^3 =$	$\mathbf{x^3}$	35.	$\left(\dfrac{4}{5}\right)^{-7} =$	$\mathbf{\dfrac{5^7}{4^7}}$
14.	$(12^5)^9 =$	$\mathbf{12^{45}}$	36.	$\left(\dfrac{6^{-2}}{7^5}\right)^{-11} =$	$\mathbf{6^{22} 7^{55}}$
15.	$(12^6)^9 =$	$\mathbf{12^{54}}$	37.	$\left(\dfrac{6^{-2}}{7^5}\right)^{-12} =$	$\mathbf{6^{24} 7^{60}}$
16.	$(12^7)^9 =$	$\mathbf{12^{63}}$	38.	$\left(\dfrac{6^{-2}}{7^5}\right)^{-13} =$	$\mathbf{6^{26} 7^{65}}$
17.	$(7^{-3})^{-4} =$	$\mathbf{7^{12}}$	39.	$\left(\dfrac{6^{-2}}{7^5}\right)^{-15} =$	$\mathbf{6^{30} 7^{75}}$
18.	$(7^{-4})^{-4} =$	$\mathbf{7^{16}}$	40.	$\dfrac{42ab^{10}}{14a^{-9}b} =$	$\mathbf{3a^{10} b^9}$
19.	$(7^{-5})^{-4} =$	$\mathbf{7^{20}}$	41.	$\dfrac{5xy^7}{25x^7 y} =$	$\mathbf{\dfrac{y^6}{5x^6}}$
20.	$\left(\dfrac{3}{7}\right)^8 =$	$\mathbf{\dfrac{3^8}{7^8}}$	42.	$\dfrac{22a^{15}b^{32}}{121ab^{-5}} =$	$\mathbf{\dfrac{2a^{14}b^{37}}{11}}$
21.	$\left(\dfrac{3}{7}\right)^7 =$	$\mathbf{\dfrac{3^7}{7^7}}$	43.	$(7^{-8} \cdot 49)^{-5} =$	$\mathbf{7^{30}}$
22.	$\left(\dfrac{3}{7}\right)^6 =$	$\mathbf{\dfrac{3^6}{7^6}}$	44.	$(36^9)(216^{-2}) =$	$\mathbf{6^{12}}$

Date: | 5/23/14

Lesson 9: Scientific Notation

Student Outcomes

- Students write, add, and subtract numbers in scientific notation and understand what is meant by the term leading digit.

Classwork

Discussion (5 minutes)

Our knowledge of the integer powers of 10 (i.e., Fact 1 and Fact 2 in Lesson 7) will enable us to understand the next concept, *scientific* notation.

Consider the estimated number of stars in the universe: 6×10^{22}. This is a 23-digit *whole number* with the **leading digit** (the leftmost digit) 6 followed by 22 zeros. When it is written in the form 6×10^{22}, it is said to be expressed in *scientific* notation.

A positive, finite decimal[1] s is said to be written in **scientific notation** if it is expressed as a product $d \times 10^n$, where d is a finite decimal ≥ 1 and < 10 (i.e., $1 \leq d < 10$), and n is an integer (i.e., d is a finite decimal with only a single, nonzero digit to the left of the decimal point). The integer n is called the **order of magnitude** of the decimal $d \times 10^n$.[2]

A positive, finite decimal s is said to be written in scientific notation if it is expressed as a product $d \times 10^n$, where d is a finite decimal so that $1 \leq d < 10$, and n is an integer.

The integer n is called the order of magnitude of the decimal $d \times 10^n$.

Example 1 (2 minutes)

The finite decimal 234.567 is equal to every one of the following:

2.34567×10^2	0.234567×10^3	23.4567×10
234.567×10^0	234567×10^{-3}	234567000×10^{-6}

However, only the first is a representation of 234.567 in scientific notation. Ask students to explain why the first representation of 234.567 is the only example of scientific notation.

---

[1] Recall that every whole number is a finite decimal.
[2] Sometimes the place value, 10^n, of the leading digit of $d \times 10^n$ is called the **order of magnitude**. There is little chance of confusion.

Exercises 1–6 (3 minutes)

Students complete Exercises 1–6 independently.

Are the following numbers written in scientific notation? If not, state the reason.

Exercise 1

1.908×10^{17} *yes*

Exercise 2

0.325×10^{-2} *no, d < 1*

Exercise 3

7.99×10^{32} *yes*

Exercise 4

$4.0701 + 10^7$ *no, it must be a product*

Exercise 5

18.432×5^8 *no, d > 10 and it is × 5 instead of × 10*

Exercise 6

8×10^{-11} *yes*

Discussion (2 minutes)

Exponent n is called the *order of magnitude* of the positive number $s = d \times 10^n$ (in scientific notation) because the following inequalities hold:

$$10^n \leq s \qquad \text{and} \qquad s < 10^{n+1}. \qquad\qquad (18)$$

Thus, the exponent n serves to give an approximate location of s on the number line. That is, n gives the approximate magnitude of s.
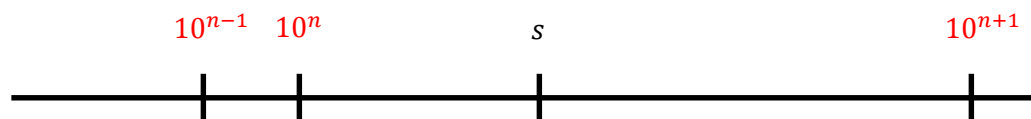
The inequalities in (18) above can be written as $10^n \leq s < 10^{n+1}$, and the number line shows that the number s is between 10^n and 10^{n+1}.

Examples 2 and 3 (10 minutes)

In the previous lesson, we approximated numbers by writing them as a single-digit integer times a power of 10. The guidelines provided by scientific notation allow us to continue to approximate numbers, but now with more precision. For example, we use a finite decimal times a power of 10 instead of using only a single digit times a power of 10.

This allows us to compute with greater accuracy while still enjoying the benefits of completing basic computations with numbers and using the laws of exponents with powers of 10.

Example 2: Let's say we need to determine the difference in the populations of Texas and North Dakota. In 2012, Texas had a population of about 26 million people, and North Dakota had a population of about 6.9×10^4.

We begin by writing each number in scientific notation:

- Texas: $26,000,000 = 2.6 \times 10^7$
- North Dakota: $69,000 = 6.9 \times 10^4$.

To find the difference, we subtract:

$$2.6 \times 10^7 - 6.9 \times 10^4.$$

To compute this easily, we need to make the order of magnitude of each number equal. That is, each number must have the same order of magnitude and the same base. When numbers have the same order of magnitude and the same base, we can use the distributive property to perform operations because each number has a common factor. Specifically for this problem, we can rewrite the population of Texas so that it is an integer multiplied by 10^4, and then subtract.

$$
\begin{aligned}
2.6 \times 10^7 - 6.9 \times 10^4 &= (2.6 \times 10^3) \times 10^4 - 6.9 \times 10^4 &&\text{By the first law of exponents} \\
&= 2600 \times 10^4 - 6.9 \times 10^4 \\
&= (2600 - 6.9) \times 10^4 &&\text{By the distributive property} \\
&= 2593.1 \times 10^4 \\
&= 25,931,000
\end{aligned}
$$

Example 3: Let's say that we need to find the combined mass of two hydrogen atoms and one oxygen atom, which is normally written as H_2O or otherwise known as water. To appreciate the value of scientific notation, the mass of each atom will be given in standard notation:

- One hydrogen atom is approximately 0.000 000 000 000 000 000 000 000 001 7 kilograms.
- One oxygen atom is approximately 0.000 000 000 000 000 000 000 000 027 kilograms.

To determine the combined mass of water, we need the mass of 2 hydrogen atoms plus one oxygen atom. First, we should write each atom in scientific notation.

- Hydrogen: 1.7×10^{-27}
- Oxygen: 2.7×10^{-26}
- 2 Hydrogen atoms $= 2(1.7 \times 10^{-27}) = 3.4 \times 10^{-27}$
- 2 Hydrogen atoms $+1$ Oxygen atom $= 3.4 \times 10^{-27} + 2.7 \times 10^{-26}$

As in the previous example, we must have the same order of magnitude for each number. Thus, changing them both to 10^{-26}:

$$
\begin{aligned}
3.4 \times 10^{-27} + 2.7 \times 10^{-26} &= (3.4 \times 10^{-1}) \times 10^{-26} + 2.7 \times 10^{-26} &&\text{By the first law of exponents} \\
&= 0.34 \times 10^{-26} + 2.7 \times 10^{-26} \\
&= (0.34 + 2.7) \times 10^{-26} &&\text{By the distributive property} \\
&= 3.04 \times 10^{-26}
\end{aligned}
$$

We can also choose to do this problem a different way, by making both numbers have 10^{-27} as the common order of magnitude:

$$3.4 \times 10^{-27} + 2.7 \times 10^{-26} = 3.4 \times 10^{-27} + (2.7 \times 10) \times 10^{-27} \qquad \text{By the first law of exponents}$$
$$= 3.4 \times 10^{-27} + 27 \times 10^{-27}$$
$$= (3.4 + 27) \times 10^{-27} \qquad \text{By the distributive property}$$
$$= 30.4 \times 10^{-27}$$
$$= 3.04 \times 10^{-26}.$$

Exercises 7–9 (10 minutes)

Students complete Exercises 7–9 independently.

Use the table below to complete Exercises 7 and 8.

The table below shows the debt of the three most populous states and the three least populous states.

State	Debt (in dollars)	Population (2012)
California	$407,000,000,000$	$38,000,000$
New York	$337,000,000,000$	$19,000,000$
Texas	$276,000,000,000$	$26,000,000$
North Dakota	$4,000,000,000$	$69,000$
Vermont	$4,000,000,000$	$62,600$
Wyoming	$2,000,000,000$	$57,600$

Exercise 7

a. **What is the sum of the debts for the three most populous states? Express your answer in scientific notation.**

$$(4.07 \times 10^{11}) + (3.37 \times 10^{11}) + (2.76 \times 10^{11}) = (4.07 + 3.37 + 2.76) \times 10^{11}$$
$$= 10.2 \times 10^{11}$$
$$= (1.02 \times 10) \times 10^{11}$$
$$= 1.02 \times 10^{12}$$

b. **What is the sum of the debt for the three least populous states? Express your answer in scientific notation.**

$$(4 \times 10^{9}) + (4 \times 10^{9}) + (2 \times 10^{9}) = (4 + 4 + 2) \times 10^{9}$$
$$= 10 \times 10^{9}$$
$$= (1 \times 10) \times 10^{9}$$
$$= 1 \times 10^{10}$$

c. **How much larger is the combined debt of the three most populous states than that of the three least populous states? Express your answer in scientific notation.**

$$(1.02 \times 10^{12}) - (1 \times 10^{10}) = (1.02 \times 10^{2} \times 10^{10}) - (1 \times 10^{10})$$
$$= (102 \times 10^{10}) - (1 \times 10^{10})$$
$$= (102 - 1) \times 10^{10}$$
$$= 101 \times 10^{10}$$
$$= (1.01 \times 10^{2}) \times 10^{10}$$
$$= 1.01 \times 10^{12}$$

Exercise 8

a. What is the sum of the population of the three most populous states? Express your answer in scientific notation.

$$(3.8 \times 10^7) + (1.9 \times 10^7) + (2.6 \times 10^7) = (3.8 + 1.9 + 2.6) \times 10^7$$
$$= 8.3 \times 10^7$$

b. What is the sum of the population of the three least populous states? Express your answer in scientific notation.

$$(6.9 \times 10^4) + (6.26 \times 10^4) + (5.76 \times 10^4) = (6.9 + 6.26 + 5.76) \times 10^4$$
$$= 18.92 \times 10^4$$
$$= (1.892 \times 10) \times 10^4$$
$$= 1.892 \times 10^5$$

c. Approximately how many times greater is the total population of California, New York, and Texas compared to the total population of North Dakota, Vermont, and Wyoming?

$$\frac{8.3 \times 10^7}{1.892 \times 10^5} = \frac{8.3}{1.892} \times \frac{10^7}{10^5}$$
$$\approx 4.39 \times 10^2$$
$$= 439$$

The combined population of California, New York, and Texas is about 439 times greater than the combined population of North Dakota, Vermont, and Wyoming.

Exercise 9

All planets revolve around the sun in elliptical orbits. Uranus's furthest distance from the sun is approximately 3.004×10^9 km, and its closest distance is approximately 2.749×10^9 km. Using this information, what is the average distance of Uranus from the sun?

$$\textit{average distance} = \frac{(3.004 \times 10^9) + (2.749 \times 10^9)}{2}$$
$$= \frac{(3.004 + 2.749) \times 10^9}{2}$$
$$= \frac{5.753 \times 10^9}{2}$$
$$= 2.8765 \times 10^9$$

On average, Uranus is 2.8765×10^9 km from the sun.

Discussion (5 minutes)

▪ Why are we interested in writing numbers in scientific notation?

▫ *It is essential that we express very large and very small numbers in scientific notation. For example, consider once again the estimated number of stars in the universe. The advantage of presenting it as 6×10^{22}, rather than as $60,000,000,000,000,000,000,000$ (in the standard notation), is perhaps too obvious for discussion. In the standard form, we cannot keep track of the number of zeros!*

- There is another deeper advantage of scientific notation. When faced with very large numbers, one's natural first question is *roughly* how big? Is it near a few hundred billion (a number with 11 digits) or even a few trillion (a number with 12 digits)? The exponent 22 in the scientific notation 6×10^{22} lets us know immediately that there is a 23-digit number and, therefore, far larger than "a few trillion."
- We should elaborate on the last statement. Observe that the number 6234.5×10^{22} does *not* have 23 digits but 26 digits because it is the number $62,345,000,000,000,000,000,000,000$, which equals 6.2345×10^{25}.

Have students check to see that this number actually has 26 digits.

Ask them to think about why it has 26 digits when the exponent is 22. If they need help, point out what we started with: 6×10^{22} and 6234.5×10^{22}. Ask students what makes these numbers different. They should see that the first number is written in proper scientific notation, so the exponent of 22 tells us that this number will have $(22 + 1)$ digits. The second number has a value of d that is in the thousands (recall: $s = d \times 10^n$ and $1 \leq d < 10$). So, we are confident that 6×10^{22} has only 23 digits because 6 is greater than 1 and less than 10.

- Therefore, by normalizing (i.e., standardizing) the d in $d \times 10^n$ to satisfy $1 \leq d < 10$, we can rely on the exponent n to give us a sense of a number's "order of magnitude" of $d \times 10^n$.

Closing (3 minutes)

Summarize, or have students summarize, the lesson.

- Knowing how to write numbers in scientific notation allows us to determine the order of magnitude of a finite decimal.
- We now know how to compute with numbers expressed in scientific notation.

Exit Ticket (5 minutes)

Name _____   Date_____

Lesson 9: Scientific Notation

Exit Ticket

1. The approximate total surface area of Earth is 5.1×10^8 km². All the salt water on Earth has an approximate surface area of 352,000,000 km², and all the freshwater on Earth has an approximate surface area of 9×10^6 km². How much of Earth's surface is covered by water, including both salt and fresh water? Write your answer in scientific notation.

2. How much of Earth's surface is covered by land? Write your answer in scientific notation.

3. Approximately how many times greater is the amount of Earth's surface that is covered by water compared to the amount of Earth's surface that is covered by land?

Exit Ticket Sample Solutions

1. The approximate total surface area of Earth is 5.1×10^8 km². All the salt water on Earth has an approximate surface area of $352{,}000{,}000$ km², and all the freshwater on Earth has an approximate surface area of 9×10^6 km². How much of Earth's surface is covered by water, including both salt and fresh water? Write your answer in scientific notation.

$$(3.52 \times 10^8) + (9 \times 10^6) = (3.52 \times 10^2 \times 10^6) + (9 \times 10^6)$$
$$= (352 \times 10^6) + (9 \times 10^6)$$
$$= (352 + 9) \times 10^6$$
$$= 361 \times 10^6$$
$$= 3.61 \times 10^8 \ km^2$$

2. How much of Earth's surface is covered by land? Write your answer in scientific notation.

$$(5.1 \times 10^8) - (3.61 \times 10^8) = (5.1 - 3.61) \times 10^8$$
$$= 1.49 \times 10^8 \ km^2$$

3. Approximately how many times greater is the amount of Earth's surface that is covered by water compared to the amount of Earth's surface that is covered by land?

$$\frac{3.61 \times 10^8}{1.49 \times 10^8} \approx 2.4$$

About 2.4 times more of the Earth's surface is covered by water than by land.

Problem Set Sample Solutions

Students practice working with numbers written in scientific notation.

1. Write the number $68{,}127{,}000{,}000{,}000{,}000$ in scientific notation. Which of the two representations of this number do you prefer? Explain.

$$68{,}127{,}000{,}000{,}000{,}000 = 6.8127 \times 10^{16}$$

Most likely, students will say that they like the scientific notation better because it allows them to write less. However, they should also take note of the fact that counting the number of zeros in $68{,}127{,}000{,}000{,}000{,}000$ is a nightmare. A strong reason for using scientific notation is to circumvent this difficulty: right away, the exponent 16 shows that this is a 17-digit number.

2. Here are the masses of the so-called inner planets of the Solar System.

Mercury: 3.3022×10^{23} kg

Earth: 5.9722×10^{24} kg

Venus: 4.8685×10^{24} kg

Mars: 6.4185×10^{23} kg

What is the average mass of all four inner planets? Write your answer in scientific notation.

$$average\ mass = \frac{(3.3022 \times 10^{23}) + (4.8685 \times 10^{24}) + (5.9722 \times 10^{24}) + (6.4185 \times 10^{23})}{4}$$

$$= \frac{(3.3022 \times 10^{23}) + (48.685 \times 10^{23}) + (59.722 \times 10^{23}) + (6.4185 \times 10^{23})}{4}$$

$$= \frac{(3.3022 + 48.685 + 59.722 + 6.4185) \times 10^{23}}{4}$$

$$= \frac{118.1277 \times 10^{23}}{4}$$

$$= 29.531925 \times 10^{23}$$

$$= 2.9531925 \times 10^{24}$$

The average mass of the inner planets is $= 2.9531925 \times 10^{24}$ *kg.*

 Lesson 10: Operations with Numbers in Scientific Notation

Student Outcomes

- Students practice operations with numbers expressed in scientific notation and standard notation.

Classwork

Examples 1 and 2 (8 minutes)

Example 1: The world population is about 7 billion. There are 4.6×10^7 ants for every human on the planet. About how many ants are there in the world?

First, write 7 billion in scientific notation: (7×10^9).

To find the number of ants in the world, we need to multiply the world population by the known number of ants for each person: $(7 \times 10^9)(4.6 \times 10^7)$.

$(7 \times 10^9)(4.6 \times 10^7) = (7 \times 4.6)(10^9 \times 10^7)$ By repeated use of the associative and commutative properties

$= 32.2 \times 10^{16}$ By the first law of exponents

$= 3.22 \times 10 \times 10^{16}$

$= 3.22 \times 10^{17}$ By the first law of exponents

There are about 3.22×10^{17} *ants in the world!*

Example 2: A certain social media company processes about 990 billion "likes" per year. If the company has approximately 8.9×10^8 users of the social media, about how many "likes" is each user responsible for per year? Write your answer in scientific and standard notation.

First, write 990 billion in scientific notation: 9.9×10^{11}.

To find the number of "likes" per person, divide the total number of "likes" by the total number of users: $\dfrac{9.9 \times 10^{11}}{8.9 \times 10^8}$

$\dfrac{9.9 \times 10^{11}}{8.9 \times 10^8} = \dfrac{9.9}{8.9} \times \dfrac{10^{11}}{10^8}$ By the product formula

$= 1.11235 \dots \times 10^3$ By the first law of exponents

$\approx 1.1 \times 10^3$

≈ 1100

Each user is responsible for about 1.1×10^3*, or 1,100, "likes" per year.*

Exercises 1 and 2 (10 minutes)

Students complete Exercises 1 and 2 independently.

Exercise 1

The speed of light is $300,000,000$ meters per second. The sun is approximately 1.5×10^{11} meters from Earth. How many seconds does it take for sunlight to reach Earth?

$300,000,000 = 3 \times 10^8$

$$\frac{1.5 \times 10^{11}}{3 \times 10^8} = \frac{1.5}{3} \times \frac{10^{11}}{10^8}$$
$$= 0.5 \times 10^3$$
$$= 0.5 \times 10 \times 10^2$$
$$= 5 \times 10^2$$

It takes 500 seconds for sunlight to reach Earth.

Exercise 2

The mass of the moon is about 7.3×10^{22} kg. It would take approximately $26,000,000$ moons to equal the mass of the sun. Determine the mass of the sun.

$$26,000,000 = 2.6 \times 10^7$$
$$(2.6 \times 10^7)(7.3 \times 10^{22}) = (2.6 \times 7.3)(10^7 \times 10^{22})$$
$$= 18.98 \times 10^{29}$$
$$= 1.898 \times 10 \times 10^{29}$$
$$= 1.898 \times 10^{30}$$

The mass of the sun is 1.898×10^{30} kg.

Example 3 (8 minutes)

In 2010, Americans generated 2.5×10^8 tons of garbage. There are about 2,000 landfills in the United States. Assuming that each landfill is the same size and that trash is divided equally among them, determine how many tons of garbage were sent to each landfill in 2010.

First, write 2,000 in scientific notation: 2×10^3

To find the number of tons of garbage sent to each landfill, divide the total weight of the garbage by the number of landfills: $\frac{2.5 \times 10^8}{2 \times 10^3}$.

$$\frac{2.5 \times 10^8}{2 \times 10^3} = \frac{2.5}{2} \times \frac{10^8}{10^3} \qquad \text{By the product formula}$$
$$= 1.25 \times 10^5 \qquad \text{By the first law of exponents}$$

Each landfill received 1.25×10^5 tons of garbage in 2010.

Actually, not all garbage went to landfills. Some of it was recycled and composted. The amount of recycled and composted material accounted for about 85 million tons of the 2.5×10^8 tons of garbage. Given this new information, how much garbage was actually sent to each landfill?

First, write 85 million in scientific notation: 8.5×10^7.

Next, subtract the amount of recycled and composted material from the garbage: $2.5 \times 10^8 - 8.5 \times 10^7$. To subtract, we must give each number the same order of magnitude, and then use the distributive property.

$$2.5 \times 10^8 - 8.5 \times 10^7 = (2.5 \times 10) \times 10^7 - 8.5 \times 10^7 \qquad \text{By the first law of exponents}$$
$$= (2.5 \times 10) - 8.5)) \times 10^7 \qquad \text{By the distributive property}$$
$$= (25 - 8.5) \times 10^7$$
$$= 16.5 \times 10^7$$
$$= 1.65 \times 10 \times 10^7$$
$$= 1.65 \times 10^8 \qquad \text{By the first law of exponents}$$

Now, divide the new amount of garbage by the number of landfills: $\dfrac{1.65 \times 10^8}{2 \times 10^3}$.

$$\frac{1.65 \times 10^8}{2 \times 10^3} = \frac{1.65}{2} \times \frac{10^8}{10^3} \qquad \text{By the product formula}$$
$$= 0.825 \times 10^5 \qquad \text{By the first law of exponents}$$
$$= 0.825 \times 10 \times 10^4 \qquad \text{By the first law of exponents}$$
$$= 8.25 \times 10^4$$

Each landfill actually received 8.25×10^4 *tons of garbage in* 2010.

Exercises 3–5 (10 minutes)

Students complete Exercises 3–5 independently.

Exercise 3

The mass of Earth is 5.9×10^{24} kg. The mass of Pluto is $13,000,000,000,000,000,000,000$ kg. Compared to Pluto, how much greater is Earth's mass than Pluto's mass?

$$13,000,000,000,000,000,000,000 = 1.3 \times 10^{22}$$
$$5.9 \times 10^{24} - 1.3 \times 10^{22} = (5.9 \times 10^2) \times 10^{22} - 1.3 \times 10^{22}$$
$$= (590 - 1.3) \times 10^{22}$$
$$= 588.7 \times 10^{22}$$
$$= 5.887 \times 10^2 \times 10^{22}$$
$$= 5.887 \times 10^{24}$$

The mass of Earth is 5.887×10^{24} *kg greater than the mass of Pluto.*

Exercise 4

Using the information in Exercises 2 and 3, find the combined mass of the moon, Earth, and Pluto.

$$7.3 \times 10^{22} + 1.3 \times 10^{22} + 5.9 \times 10^{24} = (7.3 \times 10^{22} + 1.3 \times 10^{22}) + 5.9 \times 10^{24}$$
$$= 8.6 \times 10^{22} + 5.9 \times 10^{24}$$
$$= (8.6 + 590) \times 10^{22}$$
$$= 598.6 \times 10^{22}$$
$$= 5.986 \times 10^{2} \times 10^{22}$$
$$= 5.986 \times 10^{24}$$

The combined mass of the moon, Earth, and Pluto is 5.986×10^{24} *kg.*

Exercise 5

How many combined moon, Earth, and Pluto masses (i.e., the answer to Exercise 4) are needed to equal the mass of the sun (i.e., the answer to Exercise 2)?

$$\frac{1.898 \times 10^{30}}{5.986 \times 10^{24}} = \frac{1.898}{5.986} \times \frac{10^{30}}{10^{24}}$$
$$= 0.3170 \ldots \times 10^{6}$$
$$\approx 0.32 \times 10^{6}$$
$$= 0.32 \times 10 \times 10^{5}$$
$$= 3.2 \times 10^{5}$$

It would take 3.2×10^{5} *combined masses of the moon, Earth, and Pluto to equal the mass of the sun.*

Closing (4 minutes)

Summarize, or have students summarize, the lesson.

- We can perform all operations for numbers expressed in scientific notation or standard notation.

Exit Ticket (5 minutes)

Name _____   Date_____

Lesson 10: Operations with Numbers in Scientific Notation

Exit Ticket

1. The speed of light is 3×10^8 meters per second. The sun is approximately 230,000,000,000 meters from Mars. How many seconds does it take for sunlight to reach Mars?

2. If the sun is approximately 1.5×10^{11} meters from Earth, what is the approximate distance from Earth to Mars?

Exit Ticket Sample Solutions

1. **The speed of light is 3×10^8 meters per second. The sun is approximately $230,000,000,000$ meters from Mars. How many seconds does it take for sunlight to reach Mars?**

$$230,000,000,000 = 2.3 \times 10^{11}$$

$$\frac{2.3 \times 10^{11}}{3 \times 10^8} = \frac{2.3}{3} \times \frac{10^{11}}{10^8}$$

$$= 0.76\ldots \times 10^3$$

$$\approx 0.76 \times 10 \times 10^2$$

$$\approx 7.6 \times 10^2$$

It takes approximately 760 seconds for sunlight to reach Mars.

2. **If the sun is approximately 1.5×10^{11} meters from Earth, what is the approximate distance from Earth to Mars?**

$$(2.3 \times 10^{11}) - (1.5 \times 10^{11}) = (2.3 - 1.5) \times 10^{11}$$

$$= 0.8 \times 10^{11}$$

$$= 0.8 \times 10 \times 10^{10}$$

$$= 8 \times 10^{10}$$

The distance from Earth to Mars is 8×10^{10} meters.

Problem Set Sample Solutions

Students practice operations with numbers written in scientific notation and standard notation.

1. **The sun produces 3.8×10^{27} joules of energy per second. How much energy is produced in a year? (Note: a year is approximately $31,000,000$ seconds).**

$$31,000,000 = 3.1 \times 10^7$$

$$(3.8 \times 10^{27})(3.1 \times 10^7) = (3.8 \times 3.1)(10^{27} \times 10^7)$$

$$= 11.78 \times 10^{34}$$

$$= 1.178 \times 10 \times 10^{34}$$

$$= 1.178 \times 10^{35}$$

The sun produces 1.178×10^{35} joules of energy in a year.

2. On average, Mercury is about $57,000,000$ km from the sun, whereas Neptune is about 4.5×10^9 km from the sun. What is the difference between Mercury's and Neptune's distances from the sun?

$$57,000,000 = 5.7 \times 10^7$$

$$4.5 \times 10^9 - 5.7 \times 10^7 = (4.5 \times 10^2) \times 10^7 - 5.7 \times 10^7$$

$$= 450 \times 10^7 - 5.7 \times 10^7$$

$$= (450 - 5.7) \times 10^7$$

$$= 444.3 \times 10^7$$

$$= 4.443 \times 10^2 \times 10^7$$

$$= 4.443 \times 10^9$$

The difference in the distance of Mercury and Neptune from the sun is 4.443×10^9 km.

3. The mass of Earth is approximately 5.9×10^{24} kg, and the mass of Venus is approximately 4.9×10^{24} kg.

   a. Find their combined mass.

$$5.9 \times 10^{24} + 4.9 \times 10^{24} = (5.9 + 4.9) \times 10^{24}$$

$$= 10.8 \times 10^{24}$$

$$= 1.08 \times 10 \times 10^{24}$$

$$= 1.08 \times 10^{25}$$

The combined mass of Earth and Venus is 1.08×10^{25} kg.

   b. Given that the mass of the sun is approximately 1.9×10^{30} kg, how many Venuses and Earths would it take to equal the mass of the sun?

$$\frac{1.9 \times 10^{30}}{1.08 \times 10^{25}} = \frac{1.9}{1.08} \times \frac{10^{30}}{10^{25}}$$

$$= 1.75925\ldots \times 10^5$$

$$\approx 1.8 \times 10^5$$

It would take approximately 1.8×10^5 Venuses and Earths to equal the mass of the sun.

Lesson 11: Efficacy of Scientific Notation

Student Outcomes

- Students continue to practice working with very small and very large numbers expressed in scientific notation.
- Students read, write, and perform operations on numbers expressed in scientific notation.

Lesson Notes

Powers of Ten, a video that demonstrates positive and negative powers of 10, is available online. The video should pique students' interest in why exponential notation is necessary. A link to the video is provided below (9 minutes).

http://www.youtube.com/watch?v=0fKBhvDjuy0

Classwork

Exercises 1 and 2 (3 minutes)

Exercise 1

The mass of a proton is

$$0.0000000000000000000000000001672622 \text{ kg.}$$

In scientific notation it is

$$1.672622 \times 10^{-27} \text{ kg.}$$

Exercise 2

The mass of an electron is

$$0.00000000000000000000000000000091093829 \text{ kg.}$$

In scientific notation it is

$$9.10938291 \times 10^{-31} \text{ kg.}$$

Discussion (3 minutes)

We continue to discuss why it is important to express numbers using scientific notation.

Consider the mass of the proton

$$0.0000000000000000000000000001672622 \text{ kg.}$$

MP.6 It is more informative to write it in scientific notation

$$1.672622 \times 10^{-27} \text{ kg.}$$

The exponent -27 is used because the first nonzero digit (i.e., 1) of this number occurs in the 27th digit after the decimal point.

Similarly, the mass of the electron is

$$0.00000000000000000000000000000910938291 \text{ kg.}$$

It is much easier to read this number in scientific notation

$$9.10938291 \times 10^{-31} \text{ kg.}$$

In this case, the exponent -31 is used because the first nonzero digit (i.e., 9) of this number occurs in the 31st digit to the right of the decimal point.

Exercise 3 (3 minutes)

Before students write the ratio that compares the mass of a proton to the mass of an electron, ask them whether they would rather write the ratio in standard (i.e., decimal) or scientific notation. If necessary, help them understand why scientific notation is more advantageous.

> **Exercise 3**
>
> **Write the ratio that compares the mass of a proton to the mass of an electron.**
>
> *Ratio:* $(\mathbf{1.672622 \times 10^{-27}}) : (\mathbf{9.10938291 \times 10^{-31}})$

Discussion (10 minutes)

This discussion includes Example 1, Exercise 4, and Example 2.

Example 1

The advantage of the scientific notation becomes even more pronounced when we have to compute how many times heavier a proton is than an electron. Instead of writing the ratio, r, as

$$r = \frac{0.000000000000000000000000001672622}{0.00000000000000000000000000000910938291}$$

we express it as

$$r = \frac{1.672622 \times 10^{-27}}{9.10938291 \times 10^{-31}}.$$

- Should we eliminate the power of 10 in the numerator or denominator? Explain.

  □ *Using the theorem on generalized equivalent fractions, we can eliminate the negative power of 10 in the numerator and denominator to see what we are doing more clearly. Anticipating that $10^{-31} \times 10^{31} = 1$, we can multiply the numerator and denominator of the (complex) fraction by 10^{31}*

$$r = \frac{1.672622 \times 10^{-27}}{9.10938291 \times 10^{-31}} \times \frac{10^{31}}{10^{31}}.$$

Scaffold:

- When students ask why we eliminated the negative power of 10 in the denominator (instead of the numerator), explain that positive powers of 10 are easier to interpret.

Using the first law of exponents (10) presented in Lesson 5, we get

$$r = \frac{1.672622 \times 10^4}{9.10938291} = \frac{1.672622}{9.10938291} \times 10^4.$$

Note that since we are using scientific notation, we can interpret an approximate value of r right away. For example, we see

$$\frac{1.672622}{9.10938291} \approx \frac{1.7}{9.1} = \frac{17}{91} \approx \frac{1}{5}$$

so that r is approximately $\frac{1}{5} \times 10,000$, which is 2,000. Thus, we expect a proton to be about two thousand times heavier than an electron.

MP.2

Exercise 4

Students find a more precise answer for Example 1. Allow students to use a calculator to divide 1.672622 by 9.10938291. When they finish, have students compare the approximate answer (2,000) to their more precise answer (1,836).

Exercise 4

Compute how many times heavier a proton is than an electron (that is, find the value of the ratio). Round your final answer to the nearest one.

Let r = the value of the ratio, then:

$$r = \frac{1.672622 \times 10^{-27}}{9.10938291 \times 10^{-31}}$$

$$= \frac{1.672622 \times 10^{-27} \times 10^{31}}{9.10938291 \times 10^{-31} \times 10^{31}}$$

$$= \frac{1.672622 \times 10^4}{9.10938291}$$

$$= \frac{1.672622}{9.10938291} \times 10^4$$

$$= \frac{1.672622 \times 10^8}{9.10938291 \times 10^8} \times 10^4$$

$$= \frac{167,262,200}{910,938,291} \times 10^4$$

$$= 0.183615291675 \times 10^4$$

$$= 1836.15291675$$

$$\approx 1836$$

Example 2

As of March 23, 2013, the U.S. national debt was $16,755,133,009,522 (rounded to the nearest dollar). According to the 2012 U.S. census, there are about 313,914,040 American citizens. What is each citizen's approximate share of the debt?

- How precise should we make our answer? Do we want to know the exact amount, to the nearest dollar, or is a less precise answer alright?
    - *The most precise answer will use the exact numbers listed in the problem. The more the numbers are rounded, the precision of the answer decreases. We should aim for the most precise answer when necessary, but the following problem does not require it since we are finding the "approximate share of the debt."*

Let's round off the debt to the nearest *billion* (10^9). It is $16,755,000,000,000, which is 1.6755×10^{13} dollars. Let's also round off the population to the nearest *million* (10^6), making it 314,000,000, which is 3.14×10^8. Therefore, using the product formula and equation (13) from Lesson 5, we see that each citizen's share of the debt, in dollars, is

$$\frac{1.6755 \times 10^{13}}{3.14 \times 10^8} = \frac{1.6755}{3.14} \times \frac{10^{13}}{10^8}$$

$$= \frac{1.6755}{3.14} \times 10^5.$$

Once again, we note the advantages of computing numbers expressed in scientific notation. Immediately, we can approximate the answer, about half of 10^5, or a hundred thousand dollars, (i.e., about $50,000), because

$$\frac{1.6755}{3.14} \approx \frac{1.7}{3.1} = \frac{17}{31} \approx \frac{1}{2}.$$

More precisely, with the help of a calculator,

$$\frac{1.6755}{3.14} = \frac{16755}{31410} = 0.533598\ldots \approx 0.5336.$$

Therefore, each citizen's share of the U.S. national debt is about $53,360.

Example 2

The U.S. national debt as of March 23, 2013, rounded to the nearest dollar, is $16,755,133,009,522. **According to the 2012 U.S. census, there are about** $313,914,040$ **U.S. citizens. What is each citizen's approximate share of the debt?**

$$\frac{1.6755 \times 10^{13}}{3.14 \times 10^8} = \frac{1.6755}{3.14} \times \frac{10^{13}}{10^8}$$

$$= \frac{1.6755}{3.14} \times 10^5$$

$$= 0.533598\ldots \times 10^5$$

$$\approx 0.5336 \times 10^5$$

$$= 53360$$

Each U.S. citizen's share of the national debt is about $53,360.

Exercises 5 and 6 (7 minutes)

Students work on Exercises 5 and 6 independently.

Exercise 5

The geographic area of California is $163,696$ sq. mi, and the geographic area of the U.S. is $3,794,101$ sq. mi. Let's round off these figures to 1.637×10^5 and 3.794×10^6. In terms of area, roughly estimate how many Californias would make up one U.S. Then compute the answer to the nearest ones.

$$\frac{3.794 \times 10^6}{1.637 \times 10^5} = \frac{3.794}{1.637} \times \frac{10^6}{10^5}$$

$$= \frac{3.794}{1.637} \times 10$$

$$= 2.3176\ldots \times 10$$

$$\approx 2.318 \times 10$$

$$= 23.18$$

It would take about 23 Californias to make up one U.S.

Exercise 6

The average distance from Earth to the moon is about 3.84×10^5 km, and the distance from Earth to Mars is approximately 9.24×10^7 km in year 2014. On this simplistic level, how much further is traveling from Earth to Mars than from Earth to the moon?

$$9.24 \times 10^7 - 3.84 \times 10^5 = 924 \times 10^5 - 3.84 \times 10^5$$

$$= (924 - 3.84) \times 10^5$$

$$= 920.16 \times 10^5$$

$$= 92,016,000$$

It is $92,016,000$ km further to travel from Earth to Mars than from Earth to the moon.

Closing (3 minutes)

Summarize, or have students summarize, the lesson.

- ▪ We can read, write, and operate with numbers expressed in scientific notation.

Exit Ticket (5 minutes)

Name _____   Date_____

Lesson 11: Efficacy of the Scientific Notation

Exit Ticket

1. The two largest mammals on earth are the blue whale and the African elephant. An adult male blue whale weighs about 170 tonnes or long tons. (1 tonne $= 1000$ kg)

    Show that the weight of an adult blue whale is 1.7×10^5 kg.

2. An adult male African elephant weighs about 9.07×10^3 kg.

    Compute how many times heavier an adult male blue whale is than an adult male African elephant (that is, find the value of the ratio). Round your final answer to the nearest one.

Exit Ticket Sample Solutions

1. The two largest mammals on earth are the blue whale and the elephant. An adult male blue whale weighs about 170 tonnes or long tons. (1 tonne = 1000 kg)

   Show that the weight of an adult blue whale is 1.7×10^5 kg.

   Let x (or any other symbol) represent the number of kilograms an adult blue whale weighs.

   $$170 \times 1000 = x$$
   $$1.7 \times 10^5 = x$$

2. An adult male elephant weighs about 9.07×10^3 kg.

   Compute how many times heavier an adult male blue whale is than an adult male elephant (that is, find the value of the ratio). Round your final answer to the nearest one.

   Let r = the ratio

   $$r = \frac{1.7 \times 10^5}{9.07 \times 10^3}$$
   $$= \frac{1.7}{9.07} \times 10^2$$
   $$= 0.18743 \times 10^2$$
   $$= 18.743$$
   $$\approx 19$$

   The blue whale is 19 times heavier than the elephant.

Problem Set Sample Solutions

1. There are approximately 7.5×10^{18} grains of sand on Earth. There are approximately 7×10^{27} atoms in an average human body. Are there more grains of sand on Earth or atoms in an average human body? How do you know?

   *There are more atoms in the average human body. When comparing the order of magnitude of each number, $27 >
   18$; therefore, $7 \times 10^{27} > 7.5 \times 10^{18}$.*

2. About how many times more atoms are in a human body compared to grains of sand on Earth?

   $$\frac{7 \times 10^{27}}{7.5 \times 10^{18}} = \frac{7}{7.5} \times \frac{10^{27}}{10^{18}}$$
   $$\approx 1 \times 10^{27-18}$$
   $$\approx 1 \times 10^9$$
   $$\approx 10^9$$

   There are about $1,000,000,000$ times more atoms in the human body compared to grains of sand on Earth.

3. Suppose the geographic areas of California and the U.S. are 1.637×10^5 and 3.794×10^6 sq. mi., respectively. California's population (as of 2012) is approximately 3.804×10^7 people. If population were proportional to area, what would be the U.S. population?

We already know from the Exercise 5 that it would take about 23 Californias to make up one U.S. Then the population of the U.S. would be 23 times the population of California, which is

$$23 \times 3.804 \times 10^7 = 87.492 \times 10^7$$
$$= 8.7492 \times 10^8$$
$$= 874,920,000.$$

4. The actual population of the U.S. (as of 2012) is approximately 3.14×10^8. How does the population density of California (i.e., the number of people per sq. mi.) compare with the population density of the U.S?

Population density of California per sq. mi.:

$$\frac{3.804 \times 10^7}{1.637 \times 10^5} = \frac{3.804}{1.637} \times \frac{10^7}{10^5}$$
$$= 2.32376\ldots \times 10^2$$
$$\approx 2.32 \times 10^2$$
$$= 232.$$

Population density of the U.S. per sq. mi.:

$$\frac{3.14 \times 10^8}{3.794 \times 10^6} = \frac{3.14}{3.794} \times \frac{10^8}{10^6}$$
$$= 0.8276\ldots \times 10^2$$
$$\approx 0.83 \times 10^2$$
$$= 83.$$

Population density of California compared to the population density of the U.S.:

$$\frac{232}{83} = 2.7951\ldots$$
$$\approx 2.8.$$

California is about 3 times as dense as the U.S. in terms of population.

Lesson 12: Choice of Unit

Student Outcomes

- Students understand how choice of unit determines how easy or difficult it is to understand an expression of measurement.
- Students determine appropriate units for various measurements and rewrite measurements based on new units.

Lesson Notes

This lesson focuses on choosing appropriate units. It is important for students to see the simple example (i.e., dining table measurements), as well as more sophisticated examples from physics and astronomy. We want students to understand the *necessity of learning to read, write, and operate in scientific notation*. For this very reason, we provide real data and explanations for why scientific notation is important and necessary in advanced sciences. This is a challenging, but crucial, lesson and should not be skipped.

Classwork

MP.4

Concept Development (2 minutes): The main reason for using scientific notation is to sensibly and efficiently record and convey the results of measurements. When we use scientific notation, the question of what unit to use naturally arises. In everyday context, this issue is easy to understand. For example, suppose we want to measure the horizontal dimensions of a dining table. In this case, measurements of 42×60 sq. in, or for that matter, $3\frac{1}{2} \times 5$ sq. ft. are commonly accepted. However, when the same measurement is presented as

$$\frac{0.7}{1056} \times \frac{1}{1056} \text{ sq. mi.}$$

it is confusing because we cannot relate a unit as long as a mile to a space as small as a dining room (*recall:* 1 mile is 5,280 feet), not to mention that the numbers $\frac{0.7}{1056}$ and $\frac{1}{1056}$ are unmanageable.

Exercises 1–3 (5 minutes)

Students complete Exercises 1–3 in small groups.

Exercise 1

A certain brand of MP3 player will display how long it will take to play through its entire music library. If the maximum number of songs the MP3 player can hold is $1,000$ (and the average song length is 4 minutes), would you want the time displayed in terms of seconds-, days-, or years-worth of music? Explain.

It makes the most sense to have the time displayed in days because numbers such as $240,000$ seconds worth of music and $\frac{5}{657}$ of a year are more difficult to understand than about 2.8 days.

Exercise 2

You have been asked to make frosted cupcakes to sell at a school fundraiser. Each frosted cupcake contains about 20 grams of sugar. Bake sale coordinators expect 500 people will attend the event. Assume everyone who attends will buy a cupcake; does it make sense to buy sugar in grams, pounds, or tons? Explain.

Because each cupcake contains about 20 *grams of sugar, we will need* 500×20 *grams of sugar. Therefore, grams are too small of a measurement, while tons are too large. Therefore, the sugar should be purchased in pounds.*

Exercise 3

The seafloor spreads at a rate of approximately 10 cm per year. If you were to collect data on the spread of the seafloor each week, which unit should you use to record your data? Explain.

The seafloor spreads 10 *cm per year, which is less than* 1 *cm per month. Data will be collected each week, so it makes the most sense to measure the spread with a unit like millimeters.*

Example 1 (3 minutes)

Now let's look at the field of particle physics or the study of subatomic particles, such as protons, electrons, neutrons, and mesons. In the previous lesson we worked with the masses of protons and electrons, which are

$$1.672622 \times 10^{-27} \text{ and } 9.10938291 \times 10^{-31} \text{ kg, respectively.}$$

The factors 10^{-27} and 10^{-31} suggest that we are dealing with very small quantities; therefore, the use of a unit other than kilograms may be necessary. Should we use gram instead of kilogram? At first glance, yes, but when we do, we get the numbers 1.672622×10^{-24} g and $9.10938291 \times 10^{-28}$ g. One cannot claim that these are much easier to deal with.

> *Scaffold:*
> - Remind students that a kilogram is 1000 grams, so we can quickly write the masses in the new unit using our knowledge of powers of 10.

- Is it easier to visualize something that is 10^{-24} compared to 10^{-27}?
  - *Of course not. That is why a better unit, the gigaelectronvolt, is used.*

For this and other reasons, particle physicists use the **gigaelectronvolt**, $\frac{\textbf{GeV}}{\textbf{c}^2}$ as a unit of mass:

$$1\frac{\text{GeV}}{c^2} = 1.783 \times 10^{-27} \text{ kg.}$$

MP.2

> **The gigaelectronvolt, $\frac{\text{GeV}}{c^2}$, is what particle physicists use for a unit of mass.**
>
> **1 gigaelectronvolt = 1.783×10^{-27} kg**
>
> **Mass of 1 proton = $1.672\,622 \times 10^{-27}$ kg**

The very name of the unit gives a hint that it was created for a purpose, but we do not need explore that at this time. The important piece of information is to understand that "1.783×10^{-27} kg" is a unit, and it represents "1 gigaelectronvolt". Thus, the mass of a proton is $0.938\frac{\text{GeV}}{c^2}$ rounded to the nearest 10^{-3}, and the mass of an electron is $0.000511\frac{\text{GeV}}{c^2}$ rounded to the nearest 10^{-6}. A justification[1] for this new unit is that the masses of a large class of subatomic particles have the same order of magnitude as 1 gigaelectronvolt.

[1] There are other reasons coming from considerations within physics.

Exercise 4 (3 minutes)

Students complete Exercise 4 independently.

Exercise 4

Show that the mass of a proton is $0.938 \frac{GeV}{c^2}$.

Let x represent the number of gigaelectronvolts equal to the mass of a proton.

$$x\left(\frac{GeV}{c^2}\right) = mass\ of\ proton$$

$$x(1.783 \times 10^{-27}) = 1.672622 \times 10^{-27}$$

$$x = \frac{1.672622 \times 10^{-27}}{1.783 \times 10^{-27}}$$

$$= \frac{1.672622}{1.783}$$

$$\approx 0.938$$

Example 2 (4 minutes)

Choosing proper units is also essential for work with very large numbers, such as those involved in astronomy (e.g., astronomical distances). The distance from the sun to the nearest star (Proxima Centauri) is approximately

$4.013\ 336\ 473 \times 10^{13}$ km.

In 1838, F.W. Bessel[2] was the first to measure the distance to a star, 61 Cygni, and its distance from the sun was

$1.078\ 807 \times 10^{14}$ km.

For numbers of this magnitude, we need to use a unit other than kilometers.

MP.2

In popular science writing, a commonly used unit is the light-year, or the <u>distance</u> light travels in one year (note: one year is defined as 365.25 days).

$$1\ light-year = 9,460,730,472,580.800\ km \approx 9.46073 \times 10^{12}\ km$$

One light-year is approximately 9.46073×10^{12} km. Currently, the distance of Proxima Centauri to the sun is approximately 4.2421 light-years, and the distance of 61 Cygni to the sun is approximately 11.403 light-years. When you ignore that 10^{12} is an enormous number, it is easier to think of the distances as 4.2 light-years and 11.4 light years for these stars. For example, we immediately see that 61 Cygni is almost 3 times further from the sun than Proxima Centauri.

To measure the distance of stars in our galaxy, the light-year is a logical unit to use. Since launching the powerful Hubble Space Telescope in 1990, galaxies billions of light-years from the sun have been discovered. For these galaxies, the **gigalight-year** (or 10^9 light-years) is often used.

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[2] Students will discover *Bessel functions* if they pursue STEM subjects at universities. We now know that 61 Cygni is actually a binary system consisting of *two* stars orbiting each other around a point called their *center of gravity*, but Bessel did not have a sufficiently powerful telescope to resolve the binary system.

Exercise 5 (3 minutes)

Students work on Exercise 5 independently.

Exercise 5

The distance of the nearest star (*Proxima Centauri*) to the sun is approximately $4.013\,336\,473 \times 10^{13}$ km. Show that *Proxima Centauri* is 4.2421 light-years from the sun.

Let x represent the number of light-years Proxima Centauri is from the sun.

$$x(9.46073 \times 10^{12}) = 4.013336473 \times 10^{13}$$

$$x = \frac{4.013336473 \times 10^{13}}{9.46073 \times 10^{12}}$$

$$= \frac{4.013336473}{9.46073} \times 10$$

$$= 0.424210021 \times 10$$

$$\approx 4.2421$$

Exploratory Challenge 1 (8 minutes)

Finally, let us look at an example involving the masses of the planets in our solar system. They range from Mercury's 3.3022×10^{23} kg to Jupiter's 1.8986×10^{27} kg. However, Earth's mass is the fourth heaviest among the eight planets[3], and it seems reasonable to use it as the point of reference for discussions among planets. Therefore, a new unit is M_E, the mass of the Earth, or 5.97219×10^{24} kg.

Suggested white-board activity: Show students the table below, leaving the masses for Mercury and Jupiter blank. Demonstrate how to rewrite the mass of Mercury in terms of the new unit, M_E. Then, have students rewrite the mass of Jupiter using the new unit. Finally, complete the chart with the rewritten masses.

<u>Mercury</u>: Let x represent the mass of Mercury in the unit M_E. We want to determine what number times the new unit is equal to the mass of Mercury in kg. Since $M_E = 5.97219 \times 10^{24}$, then:

$$(5.97219 \times 10^{24})x = 3.3022 \times 10^{23}$$

$$x = \frac{3.3022 \times 10^{23}}{5.97219 \times 10^{24}}$$

$$= \frac{3.3022}{5.97219} \times \frac{10^{23}}{10^{24}}$$

$$\approx 0.553 \times 10^{-1}$$

$$= 0.0553.$$

Mercury's mass is $0.0553\ M_E$.

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[3] Since 2006, Pluto is no longer classified as a planet. If Pluto was still considered a planet, then Earth would be the fifth-heaviest planet, right in the middle, which would further boost Earth's claim to be the point of reference.

COMMON CORE

Lesson 12: Choice of Unit
Date: 5/23/14

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Jupiter: Let x represent the mass of Jupiter in the unit M_E. We want to determine what number times the new unit is equal to the mass of Jupiter in kg. Since $M_E = 5.97219 \times 10^{24}$, then:

$$(5.97219 \times 10^{24})x = 1.8986 \times 10^{27}$$

$$x = \frac{1.8986 \times 10^{27}}{5.97219 \times 10^{24}}$$

$$= \frac{1.8986}{5.97219} \times \frac{10^{27}}{10^{24}}$$

$$\approx 0.318 \times 10^{3}$$

$$= 318.$$

Jupiter's mass is $318 \, M_E$.

Mercury	**$0.0553 \, M_E$**	Jupiter	**$318 \, M_E$**
Venus	$0.815 \, M_E$	Saturn	$95.2 \, M_E$
Earth	$1 \, M_E$	Uranus	$14.5 \, M_E$
Mars	$0.107 \, M_E$	Neptune	$17.2 \, M_E$

Exploratory Challenge 2/Exercises 6–8 (10 minutes)

Students complete Exercises 6–8 independently or in small groups. Allow time for groups to discuss their choice of unit and the reasoning for choosing it.

Exploratory Challenge 2

Suppose you are researching atomic diameters and find that credible sources provided the diameters of five different atoms as shown in the table below. All measurements are in cm.

1×10^{-8}	1×10^{-12}	5×10^{-8}	5×10^{-10}	5.29×10^{-11}

Exercise 6

What new unit might you introduce in order to discuss the differences in diameter measurements?

There are several answers that students could give for their choice of unit. Accept any reasonable answer, provided the explanation is clear and correct. Some students may choose 10^{-12} as their unit because all measurements could then be expressed without exponential notation. Other students may decide that 10^{-8} should be the unit because two measurements are already of that order of magnitude. Still, other students may choose 10^{-10} because that is the average of the powers.

Exercise 7

Name your unit and explain why you chose it.

Students can name their unit anything reasonable, as long as they clearly state what their unit is and how it will be written. For example, if a student chooses a unit of 10^{-10}, then he or she should state that the unit will be represented with a letter. For example, Y, then $Y = 10^{-10}$.

Exercise 8

Using the unit you have defined, rewrite the five diameter measurements.

Using the unit $Y = 10^{-10}$, then:

$1 \times 10^{-8} = 100\ Y$	$1 \times 10^{-12} = 0.01\ Y$	$5 \times 10^{-8} = 500\ Y$	$5 \times 10^{-10} = 5\ Y$	$5.29 \times 10^{-11} = 0.529\ Y$

Closing (2 minutes)

Summarize the lesson:

- Choosing an appropriate unit allows us to determine the size of the numbers we are dealing with. For example, the dining table measurement:

$$42 \times 60 \text{ sq. in.} = 3\frac{1}{2} \times 5 \text{ sq. ft.} = \frac{0.7}{1056} \times \frac{1}{1056} \text{ sq. mi.}$$

- Students have reinforced their ability to read, write, and operate with numbers in scientific notation.

Exit Ticket (5 minutes)

Name _____   Date_____

Lesson 12: Choice of Unit

Exit Ticket

1. The table below shows an approximation of the national debt at the beginning of each decade over the last century. Choose a unit that would make a discussion of the increase in the national debt easier. Name your unit and explain your choice.

Year	Debt in Dollars
1900	2.1×10^9
1910	2.7×10^9
1920	2.6×10^{10}
1930	1.6×10^{10}
1940	4.3×10^{10}
1950	2.6×10^{11}
1960	2.9×10^{11}
1970	3.7×10^{11}
1980	9.1×10^{11}
1990	3.2×10^{12}
2000	5.7×10^{12}

2. Using the new unit you have defined, rewrite the debt for years 1900, 1930, 1960, and 2000.

Exit Ticket Sample Solutions

1. The table below shows an approximation of the national debt at the beginning of each decade over the last century. Choose a unit that would make a discussion of the increase in the national debt easier. Name your unit and explain your choice.

Year	Debt in Dollars
1900	2.1×10^9
1910	2.7×10^9
1920	2.6×10^{10}
1930	1.6×10^{10}
1940	4.3×10^{10}
1950	2.6×10^{11}
1960	2.9×10^{11}
1970	3.7×10^{11}
1980	9.1×10^{11}
1990	3.2×10^{12}
2000	5.7×10^{12}

Students will likely choose 10^{11} as their unit because the majority of the data is of that magnitude. Accept any reasonable answer that students provide. Verify that they have named their unit.

2. Using the new unit you have defined, rewrite the debt for years 1900, 1930, 1960, and 2000.

   Let D represent the unit 10^{11}. Then, the debt in 1900 is $0.021D$, in 1930 it is $0.16D$, in 1960 it is $2.9D$, and $57D$ in 2000.

Problem Set Sample Solution

1. Verify the claim that, in terms of gigaelectronvolts, the mass of an electron is 0.000511.

   Let x represent the number of gigaelectronvolts equal to the mass of an electron.

   $$x\left(\frac{GeV}{c^2}\right) = \text{Mass of electron}$$

   $$x(1.783 \times 10^{-27}) = 9.10938291 \times 10^{-31}$$

   $$x = \frac{9.10938291 \times 10^{-31} \times 10^{31}}{1.783 \times 10^{-27} \times 10^{31}}$$

   $$= \frac{9.10938291}{1.783 \times 10^4}$$

   $$= \frac{9.10938291}{17830}$$

   $$\approx 0.000511$$

2. The maximum distance between Earth and the sun is 1.52098232×10^8 km and the minimum distance is 1.47098290×10^8 km.[4] What is the average distance between Earth, and the sun in scientific notation?

$$average\ distance = \frac{1.52098232 \times 10^8 + 1.47098290 \times 10^8}{2}$$

$$= \frac{(1.52098232 + 1.47098290) \times 10^8}{2}$$

$$= \frac{2.99196522 \times 10^8}{2}$$

$$= \frac{2.99196522}{2} \times 10^8$$

$$= 1.49598261 \times 10^8\ km$$

3. Suppose you measure the following masses in terms of kilograms:

2.6×10^{21}	9.04×10^{23}
8.82×10^{23}	2.3×10^{18}
1.8×10^{12}	2.103×10^{22}
8.1×10^{20}	6.23×10^{18}
6.723×10^{19}	1.15×10^{20}
7.07×10^{21}	7.210×10^{29}
5.11×10^{25}	7.35×10^{24}
7.8×10^{19}	5.82×10^{26}

What new unit might you introduce in order to aid discussion of the masses in this problem? Name your unit and express it using some power of 10. Rewrite each number using your newly defined unit.

A very motivated student may search the Internet and find that units exist that convert large masses to reasonable numbers, such as petagrams (10^{12} kg), exagrams (10^{15} kg), or zetagrams (10^{18} kg). More likely, students will decide that something near 10^{20} should be used as a unit because many of the numbers are near that magnitude. There is one value, 1.8×10^{12}, that serves as an outlier and should be ignored because it is much smaller than the majority of the data. Students can name their unit anything reasonable. The answers provided are suggestions, but any reasonable answers should be accepted.

Let U be defined as the unit 10^{20}.

$2.6 \times 10^{21} = 26U$	$9.04 \times 10^{23} = 9040U$
$8.82 \times 10^{23} = 8820U$	$2.3 \times 10^{18} = 0.023U$
$1.8 \times 10^{12} = 0.000000018U$	$2.103 \times 10^{22} = 210.3U$
$8.1 \times 10^0 = 8.1U$	$6.23 \times 10^{18} = 0.0623U$
$6.723 \times 10^{19} = 0.6723U$	$1.15 \times 10^{20} = 1.15U$
$7.07 \times 10^{21} = 70.7U$	$7.210 \times 10^{29} = 7,210,000,000U$
$5.11 \times 10^{25} = 511,000U$	$7.35 \times 10^{24} = 73,500U$
$7.8 \times 10^{19} = 0.78U$	$5.82 \times 10^{26} = 5,820,000U$

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[4] Note: Earth's orbit is elliptical, not circular.

Lesson 13: Comparison of Numbers Written in Scientific Notation and Interpreting Scientific Notation Using Technology

Student Outcomes

- Students compare numbers expressed in scientific notation.
- Students apply the laws of exponents to interpret data and use technology to compute with very large numbers.

Classwork

Examples 1–2/ Exercises 1–2(10 minutes)

Concept Development: We have learned why scientific notation is indispensable in science. This means that we have to learn how to compute and compare numbers in scientific notation. We have already done some computations, so we are ready to take a closer look at comparing the size of different numbers.

There is a general principle that underlies the comparison of two numbers in scientific notation: *Reduce everything to whole numbers if possible.* To this end, we recall two basic facts.

1. Inequality (A) in Lesson 7: Let x and y be numbers and let $z > 0$. Then $x < y$ if and only if $xz < yz$.

2. Comparison of whole numbers:
    a. If two whole numbers have different numbers of digits, then the one with more digits is greater.
    b. Suppose two whole numbers p and q have the same number of digits and, moreover, they agree digit-by-digit (starting from the left) until the nth place. If the digit of p in the $(n+1)^{\text{th}}$ place is greater than the corresponding digit in q, then $p > q$.

Example 1

Among the galaxies closest to Earth, *M82* is about 1.15×10^7 light-years away, and Leo I Dwarf is about 8.2×10^5 light-years away. Which is closer?

MP.8

- *First solution:* This is the down-to-earth, quick, and direct solution. The number 8.2×10^5 equals the 6-digit number 820,000. On the other hand, 1.15×10^7 equals the 8-digit number 11,500,000. By (2a), above, $8.2 \times 10^5 < 1.15 \times 10^7$. Therefore, *Leo I Dwarf* is closer.

Lesson 13:   Comparison of Numbers Written in Scientific Notation and Interpreting
                Scientific Notation Using Technology
Date: 5/23/14

131

- *Second Solution:* This solution is for the long haul, that is, the solution that works every time no matter how large (or small) the numbers become. First, we express both numbers as a product with the same power of 10. Since $10^7 = 10^2 \times 10^5$, we see that the distance to *M82* is

MP.8

$$1.15 \times 10^2 \times 10^5 = 115 \times 10^5.$$

The distance to *Leo I Dwarf* is 8.2×10^5. By (1) above, comparing 1.15×10^7 and 8.2×10^5 is equivalent to comparing 115 and 8.2. Since $8.2 < 115$, we see that $8.2 \times 10^5 < 1.15 \times 10^7$. Thus, the *Leo I Dwarf* is closer.

Scaffold:
- Display the second solution.
- Guide students through solution.

Exercise 1

Students complete Exercise 1 independently, using the logic modeled in the second solution.

Exercise 1

The Fornax Dwarf galaxy is 4.6×10^5 light-years away from Earth, while Andromeda I is 2.430×10^6 light-years away from Earth. Which is closer to Earth?

$$2.430 \times 10^6 = 2.430 \times 10 \times 10^5 = 24.30 \times 10^5$$

Because $4.6 < 24.30$, then $4.6 \times 10^5 < 24.30 \times 10^5$, and since $24.30 \times 10^5 = 2.430 \times 10^6$, we know that $4.6 \times 10^5 < 2.430 \times 10^6$. Therefore, Fornax Dwarf is closer to Earth.

Example 2

Background information for teacher: The next example brings us back to the world of subatomic particles. In the early 20th century, the picture of elementary particles was straightforward: electrons, protons, neutrons, and photons were the fundamental constituents of matter. But in the 1930s, positrons, mesons, and neutrinos were discovered, and subsequent developments rapidly increased the number of subatomic particle types observed. Many of these newly observed particle types are extremely short-lived (see Example 2 below and Exercise 2). The so-called Standard Model developed during the latter part of the last century finally restored some order, and it is now theorized that different kinds of quarks and leptons are the basic constituents of matter.

Many subatomic particles are unstable: charged pions have an average lifetime of 2.603×10^{-8} seconds, while muons have an average lifetime of 2.197×10^{-6} seconds. Which has a longer average lifetime?

We follow the same method as the second solution in Example 1. We have

Theorem Scaffold:
- Remind students about order of magnitude.
- Remind them that if $m < n$, then there is a positive integer k so that $n = k + m$. Therefore, by the first law of exponents (10),

$b \times 10^n = b \times 10^k \times 10^m = (b \times 10^k) \times 10^m$.

- Point out that we just spent time on forcing numbers that were expressed in scientific notation to have the same power of 10, which allowed us to easily compare the numbers. This proof is no different. We just wrote an equivalent expression $(b \times 10^k) \times 10^m$ for $b \times 10^n$, so that we could look at and compare two numbers that both have a magnitude of m.

$$2.197 \times 10^{-6} = 2.197 \times 10^2 \times 10^{-8} = 219.7 \times 10^{-8}.$$

Therefore, comparing 2.603×10^{-8} with 2.197×10^{-6} is equivalent to comparing 2.603 with 219.7 (by (1) above). Since $2.603 < 219.7$, we see that $2.603 \times 10^{-8} < 2.197 \times 10^{-6}$. Thus, muons have a longer lifetime.

COMMON CORE

Lesson 13:   Comparison of Numbers Written in Scientific Notation and Interpreting Scientific Notation Using Technology
Date: 5/23/14

engage^ny

132

Exercise 2 (3 minutes)

Students complete Exercise 2 independently.

Exercise 2

The average lifetime of the tau lepton is 2.906×10^{-13} seconds, and the average lifetime of the neutral pion is 8.4×10^{-17} seconds. Explain which subatomic particle has a longer average lifetime.

$$2.906 \times 10^{-13} = 2.906 \times 10^{4} \times 10^{-17} = 29,060 \times 10^{-17}$$

Since $8.4 < 29,060$, then $8.4 \times 10^{-17} < 29,060 \times 10^{-17}$, and since $29,060 \times 10^{-17} = 2.906 \times 10^{-13}$, we know that $8.4 \times 10^{-17} < 2.906 \times 10^{-13}$. Therefore, tau lepton has a longer average lifetime.

This problem, as well as others, can be solved using an alternate method. Our goal is to make the magnitude of the numbers we are comparing the same, which will allow us to reduce the comparison to that of whole numbers.

Here is an alternate solution:

$$8.4 \times 10^{-17} = 8.4 \times 10^{-4} \times 10^{-13} = 0.00084 \times 10^{-13}.$$

Since $0.00084 < 2.906$, then $0.00084 \times 10^{-13} < 2.906 \times 10^{-13}$, and since $0.00084 \times 10^{-13} = 8.4 \times 10^{-17}$, we know that $8.4 \times 10^{-17} < 2.906 \times 10^{-13}$. Therefore, tau lepton has a longer average lifetime.

Exploratory Challenge 1/Exercise 3 (8 minutes)

Examples 1 and 2 illustrate the following general fact:

MP.8

Theorem. *Given two numbers in scientific notation, $a \times 10^{m}$ and $b \times 10^{n}$. If $m < n$, then $a \times 10^{m} < b \times 10^{n}$.* Allow time for students to discuss, in small groups, how to prove the theorem.

Exploratory Challenge 1/Exercise 3

<u>**Theorem:**</u> Given two numbers in scientific notation, $a \times 10^{m}$ and $b \times 10^{n}$, if $m < n$, then $a \times 10^{m} < b \times 10^{n}$.

Prove the theorem.

If $m < n$, then there is a positive integer k so that $n = k + m$.

By the first law of exponents (10) in Lesson 5, $b \times 10^{n} = b \times 10^{k} \times 10^{m} = (b \times 10^{k}) \times 10^{m}$. Because we are comparing with $a \times 10^{m}$, we know by (1) that we only need to prove $a < (b \times 10^{k})$. By the definition of scientific notation, $a < 10$ and also $(b \times 10^{k}) \geq 10$ because $k \geq 1$ and $b \geq 1$, so that $(b \times 10^{k}) \geq 1 \times 10 = 10$. This proves $a < (b \times 10^{k})$, and therefore, $a \times 10^{m} < b \times 10^{n}$.

Explain to students that we know that $a < 10$ because of the statement given that $a \times 10^{m}$ is a number expressed in scientific notation. That is not enough information to convince students that $a < b \times 10^{k}$; therefore, we need to say something about the right side of the inequality. We know that $k \geq 1$ because k is a positive integer, so that $n = k + m$. We also know that $b \geq 1$ because of the definition of scientific notation. That means that the minimum possible value of $b \times 10^{k}$ is 10 because $1 \times 10^{1} = 10$. Therefore, we can be certain that $a < b \times 10^{k}$.

Therefore, by (1), $a \times 10^{m} < (b \times 10^{k}) \times 10^{m}$. Since $n = k + m$, we can rewrite the right side of the inequality as $b \times 10^{n}$, and finally $a \times 10^{m} < b \times 10^{n}$.

COMMON CORE

Lesson 13: Comparison of Numbers Written in Scientific Notation and Interpreting Scientific Notation Using Technology
Date: 5/23/14

engage^ny

133

Example 3 (2 minutes)

Compare 1.815×10^{14} with 1.82×10^{14}.

By (1), we only have to compare 1.815 with 1.82, and for the same reason, we only need to compare 1.815×10^3 with 1.82×10^3.

Thus, we compare $1,815$ and $1,820$: clearly $1,815 < 1,820$ (use (2b) if you like). Therefore, using (1) repeatedly,

$$1,815 < 1,820 \rightarrow 1.815 < 1.82 \rightarrow 1.815 \times 10^{14} < 1.82 \times 10^{14}.$$

> *Scaffold:*
> - Remind students that it is easier to compare whole numbers; that's why we are multiplying each number by 10^3.

Exercises 4 and 5 (2 minutes)

Students complete Exercises 4 and 5 independently.

Exercise 4

Compare 9.3×10^{28} and 9.2879×10^{28}.

We only need to compare 9.3 and 9.2879. $9.3 \times 10^4 = 93,000$ and $9.2879 \times 10^4 = 92,879$, so we see that $93,000 > 92,879$. Therefore, $9.3 \times 10^{28} > 9.2879 \times 10^{28}$.

Exercise 5

Chris said that $5.3 \times 10^{41} < 5.301 \times 10^{41}$ because 5.3 has fewer digits than 5.301. Show that even though his answer is correct, his reasoning is flawed. Show him an example to illustrate that his reasoning would result in an incorrect answer. Explain.

Chris is correct that $5.3 \times 10^{41} < 5.301 \times 10^{41}$, but that is because when we compare 5.3 and 5.301, we only need to compare 5.3×10^3 and 5.301×10^3 (by (1) above). But, $5.3 \times 10^3 < 5.301 \times 10^3$ or rather $5,300 < 5,301$, and this is the reason that $5.3 \times 10^{41} < 5.301 \times 10^{41}$. However, Chris's reasoning would lead to an incorrect answer for a problem that compares 5.9×10^{41} and 5.199×10^{41}. His reasoning would lead him to conclude that $5.9 \times 10^{41} < 5.199 \times 10^{41}$, but $5,900 > 5,199$, which is equivalent to $5.9 \times 10^3 > 5.199 \times 10^3$. By (1) again, $5.9 > 5.199$, meaning that $5.9 \times 10^{41} > 5.199 \times 10^{41}$.

Exploratory Challenge 2/Exercise 6 (10 minutes)

Students use snapshots of technology displays to determine the exact product of two numbers.

Exploratory Challenge 2/Exercise 6

You have been asked to determine the exact number of Google searches that are made each year. The only information you are provided is that there are $35,939,938,877$ searches performed each week. Assuming the exact same number of searches is performed each week for the 52 weeks in a year, how many total searches will have been performed in one year? Your calculator does not display enough digits to get the exact answer. Therefore, you must break down the problem into smaller parts. Remember, you cannot approximate an answer because you need to find an exact answer. Use the screen shots below to help you reach your answer.

Lesson 13: Comparison of Numbers Written in Scientific Notation and Interpreting Scientific Notation Using Technology

Date: 5/23/14

134

35 939 x 52 =

1868828

(	)	%	AC
7	8	9	÷
4	5	6	×
1	2	3	-
0	.	=	+

938877 x 52 =

48821604

(	)	%	AC
7	8	9	÷
4	5	6	×
1	2	3	-
0	.	=	+

First, I need to rewrite the number of searches for each week using numbers that can be computed using my calculator.

$$35,939,938,877 = 35,939,000,000 + 938,877$$

$$= 35939 \times 10^6 + 938,877$$

Next, I need to multiply each term of the sum by 52, using the distributive law.

$$(35,939 \times 10^6 + 938,877) \times 52 = (35,939 \times 10^6) \times 52 + (938,877 \times 52).$$

By repeated use of the commutative and associative properties, I can rewrite the problem as

$$(35,939 \times 52) \times 10^6 + (938,877 \times 52).$$

According to the screen shots, I get

$$1868828 \times 10^6 + 48821604 = 1,868,828,000,000 + 48,821,604$$

$$= 1,868,876,821,604.$$

Therefore, $1,868,876,821,604$ Google searches are performed each year.

Yahoo is another popular search engine. Yahoo receives requests for $1,792,671,335$ searches each month. Assuming the same number of searches is performed each month, how many searches are performed on Yahoo each year? Use the screen shots below to help determine the answer.

1792 x 12 =

21504

(	)	%	AC
7	8	9	÷
4	5	6	×
1	2	3	-
0	.	=	+

671 335 x 12 =

8056020

(	)	%	AC
7	8	9	÷
4	5	6	×
1	2	3	-
0	.	=	+

First, I need to rewrite the number of searches for each month using numbers that can be computed with my calculator.

$$1,792,671,335 = 1,792,000,000 + 671,335$$

$$= 1792 \times 10^6 + 671,335.$$

Next, I need to multiply each term of the sum by 12, using the distributive law.

$$(1,792 \times 10^6 + 671,335) \times 12 = (1,792 \times 10^6) \times 12 + (671,335 \times 12).$$

By repeated use of the commutative and associative properties, I can rewrite the problem as

$$(1,792 \times 12) \times 10^6 + (671,335 \times 12).$$

According to the screen shots, I get

$$21504 \times 10^6 + 8,056,020 = 21,504,000,000 + 8,056,020$$

$$= 21,512,056,020.$$

Therefore, $21,512,056,020$ Yahoo searches are performed each year.

Closing (2 minutes)

Summarize the lesson and Module 1:

- Students have completed the lessons on exponential notation, the properties of integer exponents, magnitude, and scientific notation.
- Students can read, write, and operate with numbers expressed in scientific notation, which is the language of many sciences. Additionally, they can interpret data using technology.

Exit Ticket (3 minutes)

Fluency Exercise (5 minutes)

Rapid White Board Exchange: Have students respond to your prompts for practice with operations with numbers expressed in scientific notation using white boards (or other display options as available). This exercise can be conducted at any point throughout the lesson. The prompts are listed at the end of the lesson. Refer to the Rapid White Board Exchanges sections in the Module Overview for directions to administer a Rapid White Board Exchange.

COMMON CORE

Lesson 13:   Comparison of Numbers Written in Scientific Notation and Interpreting
Scientific Notation Using Technology
Date: 5/23/14

engage^ny

136

Name _____   Date_____

Lesson 13: Comparison of Numbers Written in Scientific Notation and Interpreting Scientific Notation Using Technology

Exit Ticket

1. Compare 2.01×10^{15} and 2.8×10^{13}. Which number is larger?

2. The wavelength of the color red is about 6.5×10^{-9} m. The wavelength of the color blue is about 4.75×10^{-9} m. Show that the wavelength of red is longer than the wavelength of blue.

COMMON CORE

Lesson 13:   Comparison of Numbers Written in Scientific Notation and Interpreting
                Scientific Notation Using Technology
Date: 5/23/14

engage^ny

137

Exit Ticket Sample Solutions

1. Compare 2.01×10^{15} and 2.8×10^{13}. Which number is larger?

   $2.01 \times 10^{15} = 2.01 \times 10^2 \times 10^{13} = 201 \times 10^{13}$

   Since $201 > 2.8$, we have $201 \times 10^{13} > 2.8 \times 10^{13}$, and since $201 \times 10^{13} = 2.01 \times 10^{15}$, we conclude $2.01 \times 10^{15} > 2.8 \times 10^{13}$.

2. The wavelength of the color red is about 6.5×10^{-9} m. The wavelength of the color blue is about 4.75×10^{-9} m. Show that the wavelength of red is longer than the wavelength of blue.

   We only need to compare 6.5 and 4.75:

   $6.5 \times 10^{-9} = 650 \times 10^{-7}$ *and* $4.75 \times 10^{-9} = 475 \times 10^{-7}$*, so we see that $650 > 475$.
   Therefore, $6.5 \times 10^{-9} > 4.75 \times 10^{-9}$.*

Problem Set Sample Solutions

1. Write out a detailed proof of the fact that, given two numbers in scientific notation, $a \times 10^n$ and $b \times 10^n$, $a < b$, if and only if $a \times 10^n < b \times 10^n$.

   Because $10^n > 0$, we can use inequality (A) (i.e., (1) above) twice to draw the necessary conclusions. First, if $a < b$, then by inequality (A), $a \times 10^n < b \times 10^n$. Second, given $a \times 10^n < b \times 10^n$, we can use inequality (A) again to show $a < b$ by multiplying each side of $a \times 10^n < b \times 10^n$ by 10^{-n}.

   a. Let A and B be two positive numbers, with no restrictions on their size. Is it true that $A \times 10^{-5} < B \times 10^5$?

      No, it is not true that $A \times 10^{-5} < B \times 10^5$. Using inequality (A), we can write $A \times 10^{-5} \times 10^5 < B \times 10^5 \times 10^5$, which is the same as $A < B \times 10^{10}$. To disprove the statement, all we would need to do is find a value of A that exceeds $B \times 10^{10}$.

   b. Now, if $A \times 10^{-5}$ and $B \times 10^5$ are written in scientific notation, is it true that $A \times 10^{-5} < B \times 10^5$? Explain.

      Yes, since the numbers are written in scientific notation, we know that the restrictions for A and B are $1 \leq A < 10$ and $1 \leq B < 10$. The maximum value for A, when multiplied by 10^{-5}, will still be less than 1. The minimum value of B will produce a number at least 10^5 in size.

2. The mass of a neutron is approximately 1.674927×10^{-27} kg. Recall that the mass of a proton is 1.672622×10^{-27} kg. Explain which is heavier.

   Since both numbers have a factor of 10^{-27}, we only need to look at 1.674927 and 1.672622. When we multiply each number by 10^6, we get

   1.674927×10^6 *and* 1.672622×10^6,

   which is the same as

   $1,674,927$ *and* $1,672,622$.

   Now that we are looking at whole numbers, we can see that $1,674,927 > 1,672,622$ (by (2b) above), which means that $1.674927 \times 10^{-27} > 1.672622 \times 10^{-27}$. Therefore, the mass of a neutron is heavier.

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Date: 5/23/14

engage^ny 138

3.   The average lifetime of the Z boson is approximately 3×10^{-25} seconds, and the average lifetime of a neutral rho meson is approximately 4.5×10^{-24} seconds.

a.   Without using the theorem from today's lesson, explain why the neutral rho meson has a longer average lifetime.

Since $3 \times 10^{-25} = 3 \times 10^{-1} \times 10^{-24}$, we can compare $3 \times 10^{-1} \times 10^{-24}$ and 4.5×10^{-24}. Based on Example 3 or by use of (1) above, we only need to compare 3×10^{-1} and 4.5, which is the same as 0.3 and 4.5. If we multiply each number by 10, we get whole numbers 3 and 45. Since $3 < 45$, then $3 \times 10^{-25} < 4.5 \times 10^{-24}$. Therefore, the neutral rho meson has a longer average lifetime.

b.   Approximately how much longer is the lifetime of a neutral rho meson than a Z boson?

$45:3$ or 15 times longer.

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Date:	5/23/14

engage^ny

139

Rapid White Board Exchange: Operations with Numbers Expressed in Scientific Notation

1. $(5 \times 10^4)^2 =$

   $\mathbf{2.5 \times 10^9}$

2. $(2 \times 10^9)^4 =$

   $\mathbf{1.6 \times 10^{37}}$

3. $\dfrac{(1.2 \times 10^4) + (2 \times 10^4) + (2.8 \times 10^4)}{3} =$

   $\mathbf{2 \times 10^4}$

4. $\dfrac{7 \times 10^{15}}{14 \times 10^9} =$

   $\mathbf{5 \times 10^5}$

5. $\dfrac{4 \times 10^2}{2 \times 10^8} =$

   $\mathbf{2 \times 10^{-6}}$

6. $\dfrac{(7 \times 10^9) + (6 \times 10^9)}{2} =$

   $\mathbf{6.5 \times 10^9}$

7. $(9 \times 10^{-4})^2 =$

   $\mathbf{8.1 \times 10^{-7}}$

8. $(9.3 \times 10^{10}) - (9 \times 10^{10}) =$

   $\mathbf{3 \times 10^9}$

Name _____   Date _____

1. You have been hired by a company to write a report on Internet companies' Wi-Fi ranges. They have requested that <u>all values be reported in feet using scientific notation</u>.

    Ivan's Internet Company boasts that its wireless access points have the greatest range. The company claims that you can access its signal up to 2,640 feet from its device. A competing company, Winnie's Wi-Fi, has devices that extend to up to $2\frac{1}{2}$ miles.

    a. Rewrite the range of each company's wireless access devices in feet using scientific notation and state which company actually has the greater range (5,280 feet = 1 mile).

    b. You can determine how many times greater the range of one internet company is than the other by writing their ranges as a ratio. Write and find the value of the ratio that compares the range of Winnie's wireless access devices to the range of Ivan's wireless access devices. Write a complete sentence describing how many times greater Winnie's Wi-Fi range is than Ivan's Wi-Fi range.

    c. UC Berkeley uses Wi-Fi over Long Distances (WiLD) to create long-distance, point-to-point links. UC Berkeley claims that connections can be made up to 10 miles away from its device. Write and find the value of the ratio that compares the range of Ivan's wireless access devices to the range of Berkeley's WiLD devices. Write your answer in a complete sentence.

2. There is still controversy about whether or not Pluto should be considered a planet. Though planets are mainly defined by their orbital path (the condition that prevented Pluto from remaining a planet), the issue of size is something to consider. The table below lists the planets, including Pluto, and their approximate diameters in meters.

Planet	Approximate Diameter (m)
Mercury	4.88×10^6
Venus	1.21×10^7
Earth	1.28×10^7
Mars	6.79×10^6
Jupiter	1.43×10^8
Saturn	1.2×10^8
Uranus	5.12×10^7
Neptune	4.96×10^7
Pluto	2.3×10^6

a. Name the planets (including Pluto) in order from smallest to largest.

b. Comparing only diameters, about how many times larger is Jupiter than Pluto?

c.   Again, comparing only diameters, find out about how many times larger Jupiter is compared to Mercury.

d.   Assume you were a voting member of the International Astronomical Union (IAU) and the classification of Pluto was based entirely on the length of the diameter. Would you vote to keep Pluto a planet or reclassify it? Why or why not?

e.   Just for fun, Scott wondered how big a planet would be if its diameter was the square of Pluto's diameter. If the diameter of Pluto in terms of meters were squared, what would be the diameter of the new planet (write answer in scientific notation)? Do you think it would meet any size requirement to remain a planet? Would it be larger or smaller than Jupiter?

3. Your friend Pat bought a fish tank that has a volume of 175 liters. The brochure for Pat's tank lists a "fun fact" that it would take 7.43×10^{18} tanks of that size to fill all the oceans in the world. Pat thinks the both of you can quickly calculate the volume of all the oceans in the world using the fun fact and the size of her tank.

   a. Given that 1 liter $= 1.0 \times 10^{-12}$ cubic kilometers, rewrite the size of the tank in cubic kilometers using scientific notation.

   b. Determine the volume of all the oceans in the world in cubic kilometers using the "fun fact".

   c. You liked Pat's fish so much you bought a fish tank of your own that holds an additional 75 liters. Pat asked you to figure out a different "fun fact" for your fish tank. Pat wants to know how many tanks of this new size would be needed to fill the Atlantic Ocean. The Atlantic Ocean has a volume of 323,600,000 cubic kilometers.

A Progression Toward Mastery					
Assessment Task Item		STEP 1 Missing or incorrect answer and little evidence of reasoning or application of mathematics to solve the problem.	STEP 2 Missing or incorrect answer but evidence of some reasoning or application of mathematics to solve the problem.	STEP 3 A correct answer with some evidence of reasoning or application of mathematics to solve the problem, or an incorrect answer with substantial evidence of solid reasoning or application of mathematics to solve the problem.	STEP 4 A correct answer supported by substantial evidence of solid reasoning or application of mathematics to solve the problem.
1	a–c 8.EE.A.3 8.EE.A.4	Student completed part (a) correctly by writing each company's Wi-Fi range in scientific notation and determined which was greater. Student was unable to write ratios in parts (b)–(c). OR Student was unable to perform operations with numbers written in scientific notation and did not complete parts (b)–(c). OR Student was able to write the ratios in parts (b)–(c) but was unable to find the value of the ratios.	Student completed part (a) correctly. Student was able to write ratios in parts (b)–(c). Student was able to perform operations with numbers written in scientific notation in parts (b)–(c) but made computational errors leading to incorrect answers. Student did not interpret calculations to answer questions.	Student answered at least two parts of (a)–(c) correctly. Student made a computational error that led to an incorrect answer. Student interpreted calculations correctly and justified the answers. Student used a complete sentence to answer part (b) or (c).	Student answered all parts of (a)–(c) correctly. Ratios written were correct and values were calculated accurately. Calculations were interpreted correctly and answers were justified. Student used a complete sentence to answer parts (b) and (c).
2	a–c 8.EE.A.3 8.EE.A.4	Student correctly ordered the planets in part (a). Student was unable to perform operations with numbers written in scientific notation.	Student completed two or three parts of (a)–(c) correctly. Calculations had minor errors. Student provided partial justifications for conclusions made.	Student completed two or three parts of (a)–(c) correctly. Calculations were precise. Student provided justifications for conclusions made.	Student completed all three parts of (a)–(c) correctly. Calculations were precise. Student responses demonstrated mathematical reasoning leading to strong justifications for conclusions made.

	d **8.EE.A.3** **8.EE.A.4**	Student stated a position but provided *no* explanation to defend it.	Student stated a position and provided *weak* arguments to defend it.	Student stated a position and provided a *reasonable* explanation to defend it.	Students stated a position and provided a *compelling* explanation to defend it.
	e **8.EE.A.3** **8.EE.A.4**	Student was unable to perform calculation or answer questions.	Student performed calculation but did not write answer in scientific notation. Student provided an explanation for why the new planet would remain a planet by stating it would be the largest.	Student correctly performed calculation. Student provided an explanation for why the new planet would remain a planet without reference to the calculation. Student correctly stated that the new planet would be the largest planet.	Student correctly performed calculation. Student provided an explanation for why the new planet would remain a planet, including reference to the calculation performed. Student correctly stated that the new planet would be the largest planet.
3	**a–c** **8.EE.A.3** **8.EE.A.4**	Student completed all parts of the problem incorrectly. Evidence that student has some understanding of scientific notation but cannot integrate use of properties of exponents to perform operations. Student made gross errors in computation.	Student completed one part of (a)–(c) correctly. Student made several minor errors in computation. Student performed operations on numbers written in scientific notation but did not rewrite answers in scientific notation.	Student completed two parts of (a)–(c) correctly. Student made a minor error in computation. Evidence that student understands scientific notation and can use properties of exponents with numbers in this form.	Student completed all parts of (a)–(c) correctly. Student had precise calculations. Evidence of mastery with respect to scientific notation usage and performing operations on numbers in this form using properties of exponents.

Name _____   Date _____

1. You have been hired by a company to write a report on Internet companies' Wi-Fi ranges. They have requested that <u>all values be reported in feet using scientific notation</u>.

    Ivan's Internet Company boasts that its wireless access points have the greatest range. The company claims that you can access its signal up to 2,640 feet from its device. A competing company, Winnie's Wi-Fi, has devices that extend to up to $2\frac{1}{2}$ miles.

    a. Rewrite the range of each company's wireless access devices in feet using scientific notation and state which company actually has the greater range (5,280 feet = 1 mile).

    IVAN'S RANGE: $2,640 = 2.64 \times 10^3$ ft

    WINNIE'S RANGE: $(2.5)5280 = 13200 = 1.32 \times 10^4$ ft.

    WINNIE'S WI-FI HAS THE GREATER RANGE.

    b. You can determine how many times greater the range of one internet company is than the other by writing their ranges as a ratio. Write and find the value of the ratio that compares the range of Winnie's wireless access devices to the range of Ivan's wireless access devices. Write a complete sentence describing how many times greater Winnie's Wi-Fi range is than Ivan's Wi-Fi range.

    WINNIE TO IVAN'S RATIO — $(1.32 \times 10^4) : (2.64 \times 10^3)$

    VALUE OF RATIO $\frac{1.32 \times 10^4}{2.64 \times 10^3} = \frac{1.32}{2.64} \times \frac{10^4}{10^3} = \frac{1}{2} \times 10 = 5$

    WINNIE'S WI-FI IS 5 TIMES GREATER IN RANGE THAN IVAN'S INTERNET COMPANY.

    c. UC Berkeley uses Wi-Fi over Long Distances (WiLD) to create long-distance, point-to-point links. UC Berkeley claims that connections can be made up to 10 miles away from its device. Write and find the value of the ratio that compares the range of Ivan's wireless access devices to the range of Berkeley's WiLD devices. Write your answer in a complete sentence.

    $(10)5280 = 52800 = 5.28 \times 10^4$

    IVAN'S TO BERKELEY RATIO : $(2.64 \times 10^3) : (5.28 \times 10^4)$

    VALUE OF RATIO — $\frac{2.64 \times 10^3}{5.28 \times 10^4} = \frac{2.64}{5.28} \times \frac{10^3}{10^4} = \frac{1}{2} \times \frac{1}{10} = \frac{1}{20}$

    IVAN'S INTERNET DEVICES HAVE A RANGE $\frac{1}{20}$ THE RANGE OF UC BERKELEY'S WILD DEVICES.

2. There is still controversy about whether or not Pluto should be considered a planet. Though planets are mainly defined by their orbital path (the condition that prevented Pluto from remaining a planet), the issue of size is something to consider. The table below lists the planets, including Pluto, and their approximate diameters in meters.

Planet	Approximate Diameter (m)
Mercury	4.88×10^6
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Saturn	1.2×10^8
Uranus	5.12×10^7
Neptune	4.96×10^7
Pluto	2.3×10^6

a. Name the planets (including Pluto) in order from smallest to largest.

PLUTO, MERCURY, MARS, VENUS, EARTH, NEPTUNE, URANUS, SATURN, JUPITER

b. Comparing only diameters, about how many times larger is Jupiter than Pluto?

$$\frac{1.43 \times 10^8}{2.3 \times 10^6} = \frac{1.43}{2.3} \times \frac{10^8}{10^6}$$

$$\approx 0.622 \times 10^2$$

$$\approx 62.2$$

THE DIAMETER OF JUPITER IS ABOUT 62 TIMES LARGER THAN PLUTO.

c. Again, comparing only diameters, find out about how many times larger Jupiter is compared to Mercury.

$$\frac{1.43 \times 10^8}{4.88 \times 10^6} = \frac{1.43}{4.88} \times \frac{10^8}{10^6}$$

$$\approx 0.293 \times 10^2$$

$$\approx 29.3$$

THE DIAMETER OF JUPITER IS ABOUT 29 TIMES LARGER THAN MERCURY.

d. Assume you were a voting member of the International Astronomical Union (IAU) and the classification of Pluto was based entirely on the length of the diameter. Would you vote to keep Pluto a planet or reclassify it? Why or why not?

I WOULD VOTE TO RECLASSIFY IT. KNOWING THAT JUPITER IS 29 TIMES LARGER THAN MERCURY MEANS MERCURY IS PRETTY SMALL. JUPITER IS 62 TIMES LARGER THAN PLUTO, WHICH MEANS PLUTO IS EVEN SMALLER THAN MERCURY. FOR THAT REASON I'D VOTE THAT THE LENGTH OF THE DIAMETER OF PLUTO IS TOO SMALL COMPARED TO OTHER PLANETS (EVEN THE SMALL ONE).

e. Just for fun, Scott wondered how big a planet would be if its diameter was the square of Pluto's diameter. If the diameter of Pluto in terms of meters were squared, what would be the diameter of the new planet (write answer in scientific notation)? Do you think it would meet any size requirement to remain a planet? Would it be larger or smaller than Jupiter?

$$\left(2.3 \times 10^6\right)^2 = 2.3^2 \times \left(10^6\right)^2$$

$$= 5.29 \times 10^{12}$$

YES, 5.29×10^{12} WOULD LIKELY MEET ANY SIZE REQUIREMENT FOR PLANETS. IT WOULD BE LARGER THAN JUPITER.

3. Your friend Pat bought a fish tank that has a volume of 175 liters. The brochure for Pat's tank lists a "fun fact" that it would take 7.43×10^{18} tanks of that size to fill all the oceans in the world. Pat thinks the both of you can quickly calculate the volume of all the oceans in the world using the fun fact and the size of her tank.

   a. Given that 1 liter $= 1.0 \times 10^{-12}$ cubic kilometers, rewrite the size of the tank in cubic kilometers using scientific notation.

   $$175 \text{ LITERS} = 175\left(1.0 \times 10^{-12}\right) \text{ CUBIC KILOMETERS}$$
   $$= 175 \times 10^{-12} \text{ KM}^3$$
   $$= 1.75 \times 10^{-10} \text{ KM}^3$$

   b. Determine the volume of all the oceans in the world in cubic kilometers using the "fun fact".

   $$\left(1.75 \times 10^{-10}\right)\left(7.43 \times 10^{18}\right) = (1.75 \times 7.43)\left(10^{-10} \times 10^{18}\right)$$
   $$= 13.0025 \times 10^{8}$$
   $$= 1.30025 \times 10^{9}$$

   THE VOLUME OF ALL THE OCEANS IN THE WORLD IS $\left(1.30025 \times 10^{9}\right) \text{ KM}^3$.

   c. You liked Pat's fish so much you bought a fish tank of your own that holds an additional 75 liters. Pat asked you to figure out a different "fun fact" for your fish tank. Pat wants to know how many tanks of this new size would be needed to fill the Atlantic Ocean. The Atlantic Ocean has a volume of 323,600,000 cubic kilometers.

   TANK : 175 + 75 = 250 LITERS
   $$250 \text{ LITERS} = 250\left(1.0 \times 10^{-12}\right) \text{ KM}^3$$
   $$= 250 \times 10^{-12}$$
   $$= 2.5 \times 10^{-10}$$

   ATLANTIC OCEAN : 323,600,000
   $$= 3.236 \times 10^{8} \text{ KM}^3$$

   $$\frac{3.236 \times 10^{8}}{2.5 \times 10^{-10}} = \frac{3.236}{2.5} \times \frac{10^{8}}{10^{-10}}$$
   $$= 1.2944 \times 10^{18}$$

   IT WOULD TAKE 1.2944×10^{18} TANKS (OF SIZE 250 LITERS) TO FILL THE ATLANTIC OCEAN.