Lesson 1: Construct an Equilateral Triangle

Classwork

Opening Exercise

Joe and Marty are in the park playing catch. Tony joins them, and the boys want to stand so that the distance between any two of them is the same. Where do they stand?

How do they figure this out precisely? What tool or tools could they use?

Fill in the blanks below as each term is discussed:

1. ____________________ The _______ between points A and B is the set consisting of A, B, and all points on the line AB between A and B.

2. ____________________ A segment from the center of a circle to a point on the circle.

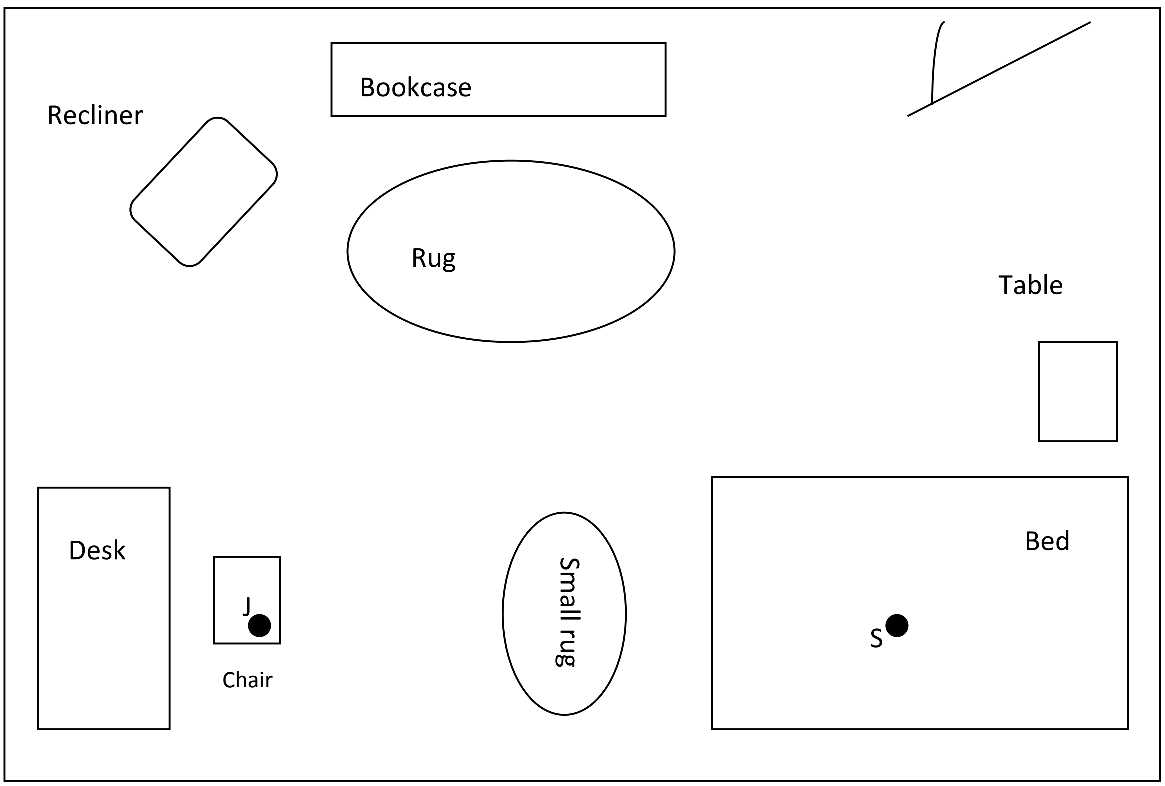
3. ____________________ Given a point C in the plane and a number $r > 0$, the _______ with center C and radius r is the set of all points in the plane that are distance r from point C.

Note that because a circle is defined in terms of a distance, r, we will often use a distance when naming the radius (e.g., "radius AB"). However, we may also refer to the specific segment, as in "radius $\overline{AB}$."

Example 1: Sitting Cats

You will need a compass and a straightedge.

Margie has three cats. She has heard that cats in a room position themselves at equal distances from one another and wants to test that theory. Margie notices that Simon, her tabby cat, is in the center of her bed (at S), while JoJo, her Siamese, is lying on her desk chair (at J). If the theory is true, where will she find Mack, her calico cat? Use the scale drawing of Margie's room shown below, together with (only) a compass and straightedge. Place an M where Mack will be if the theory is true.



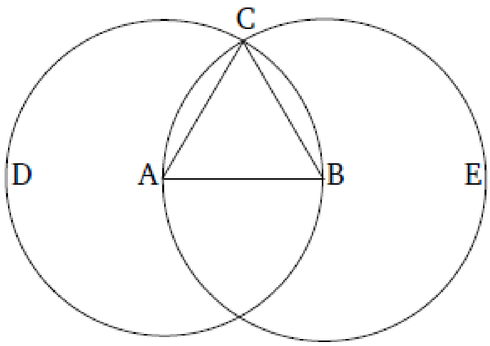
Mathematical Modeling Exercise: Euclid, Proposition 1

Let's see how Euclid approached this problem. Look at his first proposition, and compare his steps with yours.

In this margin, compare your steps with Euclid's.

Proposition 1

To construct an equilateral triangle on a given finite straight-line.



Let AB be the given finite straight-line.

So it is required to construct an equilateral triangle on the straight-line AB.

Let the circle BCD with center A and radius AB have been drawn [Post. 3], and again let the circle ACE with center B and radius BA have been drawn [Post. 3]. And let the straight-lines CA and CB have been joined from the point C, where the circles cut one another,† to the points A and B (respectively) [Post. 1].

And since the point A is the center of the circle CDB, AC is equal to AB [Def. 1.15]. Again, since the point B is the center of the circle CAE, BC is equal to BA [Def. 1.15]. But CA was also shown (to be) equal to AB. Thus, CA and CB are each equal to AB. But things equal to the same thing are also equal to one another [C.N. 1]. Thus, CA is also equal to CB. Thus, the three (straight-lines) CA, AB, and BC are equal to one another.

Thus, the triangle ABC is equilateral, and has been constructed on the given finite straight-line AB. (Which is) the very thing it was required to do.

Geometry Assumptions

In geometry, as in most fields, there are specific facts and definitions that we assume to be true. In any logical system, it helps to identify these assumptions as early as possible since the correctness of any proof hinges upon the truth of our assumptions. For example, in Proposition 1, when Euclid says, "Let AB be the given finite straight line," he assumed that, *given any two distinct points, there is exactly one line that contains them*. Of course, that assumes we have two points! It is best if we assume there are points in the plane as well: *Every plane contains at least three non-collinear points.*

Euclid continued on to show that the measures of each of the three sides of his triangle are equal. It makes sense to discuss the measure of a segment in terms of distance. *To every pair of points A and B, there corresponds a real number* $\text{dist}(A, B) \geq 0$*, called the distance from A to B.* Since the distance from A to B is equal to the distance from B to A, we can interchange A and B: $\text{dist}(A, B) = \text{dist}(B, A)$. Also, *$A$ and B coincide if and only if* $\text{dist}(A, B) = 0$.

Using distance, we can also assume that *every line has a coordinate system*, which just means that we can think of any line in the plane as a number line. Here's how: Given a line, l, pick a point A on l to be "0," and find the two points B and C such that $\text{dist}(A, B) = \text{dist}(A, C) = 1$. Label one of these points to be 1 (say point B), which means the other point C corresponds to -1. Every other point on the line then corresponds to a real number determined by the (positive or negative) distance between 0 and the point. In particular, if after placing a coordinate system on a line, if a point R corresponds to the number r, and a point S corresponds to the number s, then the distance from R to S is $\text{dist}(R, S) = |r - s|$.

History of Geometry: Examine the site http://geomhistory.com/home.html to see how geometry developed over time.

Relevant Vocabulary

Geometric Construction: A *geometric construction* is a set of instructions for drawing points, lines, circles, and figures in the plane.

The two most basic types of instructions are the following:

1. Given any two points A and B, a ruler can be used to draw the line AB or segment $\overline{AB}$.
2. Given any two points C and B, use a compass to draw the circle that has its center at C that passes through B. (Abbreviation: Draw circle C: center C, radius CB.)

Constructions also include steps in which the points where lines or circles intersect are selected and labeled. (Abbreviation: Mark the point of intersection of the lines AB and PQ by X, etc.)

Figure: A (two-dimensional) *figure* is a set of points in a plane.

Usually the term figure refers to certain common shapes such as triangle, square, rectangle, etc. However, the definition is broad enough to include any set of points, so a triangle with a line segment sticking out of it is also a figure.

Equilateral Triangle: An *equilateral triangle* is a triangle with all sides of equal length.

Collinear: Three or more points are collinear if there is a line containing all of the points; otherwise, the points are non-collinear.

Length of a Segment: The *length of the segment* $\overline{AB}$ is the distance from A to B and is denoted AB. Thus, $AB = \text{dist}(A, B)$.

In this course, you will have to write about distances between points and lengths of segments in many, if not most, Problem Sets. Instead of writing $\text{dist}(A, B)$ all of the time, which is a rather long and awkward notation, we will instead use the much simpler notation AB for both distance and length of segments. Even though the notation will always make the meaning of each statement clear, it is worthwhile to consider the context of the statement to ensure correct usage.

Here are some examples:

- $\overleftrightarrow{AB}$ intersects… — $\overleftrightarrow{AB}$ refers to a line.
- $AB + BC = AC$ — Only numbers can be added and AB is a length or distance.
- Find $\overline{AB}$ so that $\overline{AB} \parallel \overline{CD}$. — Only figures can be parallel and $\overline{AB}$ is a segment.
- $AB = 6$ — AB refers to the length of the segment AB or the distance from A to B.

Here are the standard notations for segments, lines, rays, distances, and lengths:

- A ray with vertex A that contains the point B: $\overrightarrow{AB}$ or "ray AB"
- A line that contains points A and B: $\overleftrightarrow{AB}$ or "line AB"
- A segment with endpoints A and B: $\overline{AB}$ or "segment AB"
- The length of segment $\overline{AB}$: AB
- The distance from A to B: $\text{dist}(A, B)$ or AB

Coordinate System on a Line: Given a line l, a *coordinate system on* l is a correspondence between the points on the line and the real numbers such that: (i) to every point on l, there corresponds exactly one real number; (ii) to every real number, there corresponds exactly one point of l; (iii) the distance between two distinct points on l is equal to the absolute value of the difference of the corresponding numbers.

Problem Set

1. Write a clear set of steps for the construction of an equilateral triangle. Use Euclid's Proposition 1 as a guide.

2. Suppose two circles are constructed using the following instructions:

   Draw circle: Center A, radius AB.

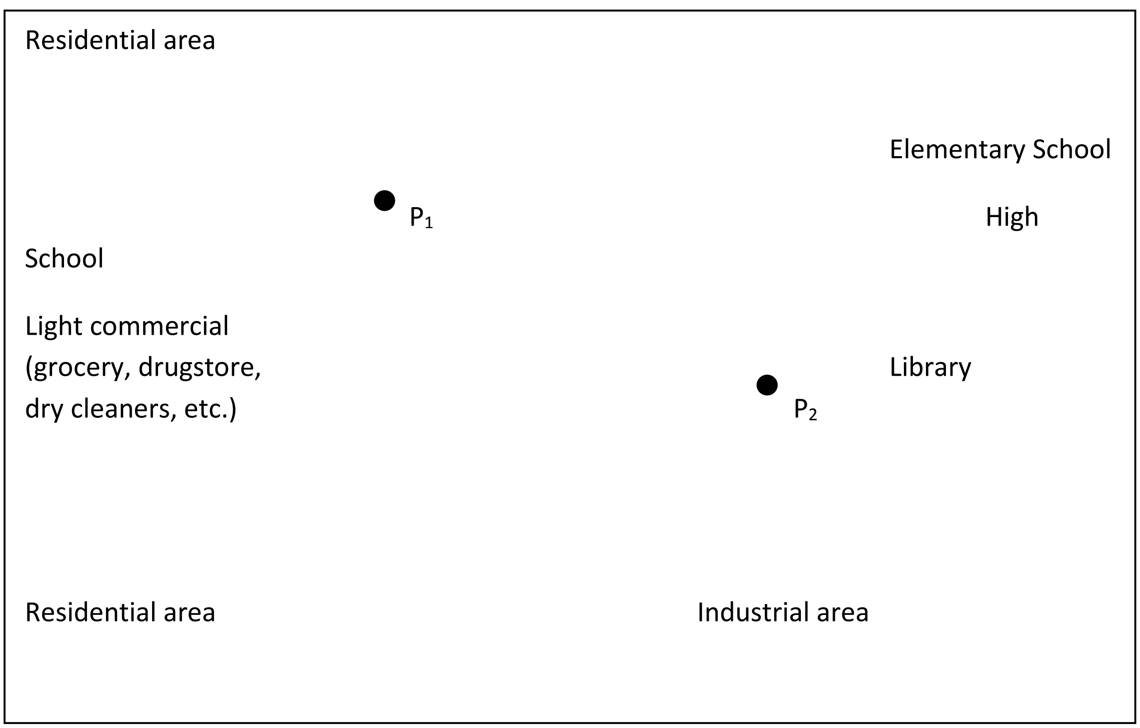
   Draw circle: Center C, radius CD.

   Under what conditions (in terms of distances AB, CD, AC) do the circles have

   a. One point in common?
   b. No points in common?
   c. Two points in common?
   d. More than two points in common? Why?

3. *You will need* a compass and straightedge.

   Cedar City boasts two city parks and is in the process of designing a third. The planning committee would like all three parks to be equidistant from one another to better serve the community. A sketch of the city appears below, with the centers of the existing parks labeled as P_1 and P_2. Identify two possible locations for the third park, and label them as P_{3a} and P_{3b} on the map. Clearly and precisely list the mathematical steps used to determine each of the two potential locations.



Lesson 2: Construct an Equilateral Triangle

Classwork

Opening Exercise

You will need a compass, a straightedge, and another student's Problem Set.

Directions:

Follow the directions of another student's Problem Set write-up to construct an equilateral triangle.

- What kinds of problems did you have as you followed your classmate's directions?
- Think about ways to avoid these problems. What criteria or expectations for writing steps in constructions should be included in a rubric for evaluating your writing? List at least three criteria.

Exploratory Challenge 1

You will need a compass and a straightedge.

Using the skills you have practiced, construct **three** equilateral triangles, where the first and second triangles share a common side and the second and third triangles share a common side. Clearly and precisely list the steps needed to accomplish this construction.

Switch your list of steps with a partner, and complete the construction according to your partner's steps. Revise your drawing and list of steps as needed.

Construct three equilateral triangles here:

Exploratory Challenge 2

On a separate piece of paper, use the skills you have developed in this lesson construct a **regular hexagon**. Clearly and precisely list the steps needed to accomplish this construction. Compare your results with a partner and revise your drawing and list of steps as needed.

Can you repeat the construction of a hexagon until the entire sheet is covered in hexagons (except the edges will be partial hexagons)?

Problem Set

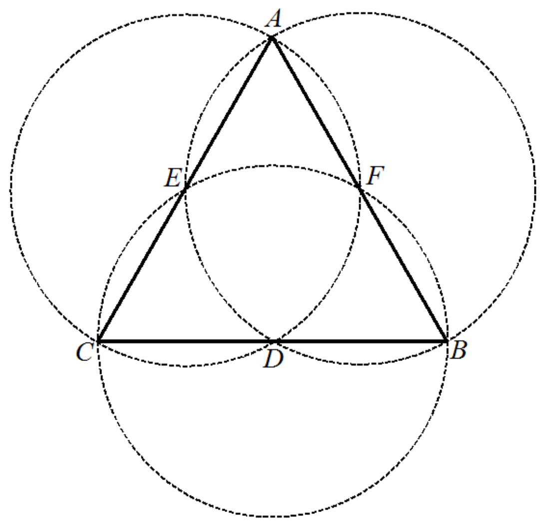
Why are *circles* so important to these constructions? Write out a concise explanation of the importance of circles in creating equilateral triangles. Why did Euclid use *circles* to create his equilateral triangles in Proposition 1? How does construction of a circle ensure that all relevant segments will be of equal length?

Lesson 3: Copy and Bisect an Angle

Classwork

Opening Exercise

In the following figure, circles have been constructed so that the endpoints of the diameter of each circle coincide with the endpoints of each segment of the equilateral triangle.



a. What is special about points D, E, and F? Explain how this can be confirmed with the use of a compass.

b. Draw DE, $\mathrm{E}F$, and FD. What kind of triangle must $\triangle DEF$ be?

c. What is special about the four triangles within $\triangle ABC$?

d. How many times greater is the area of $\triangle ABC$ than the area of $\triangle CDE$?

Discussion

Define the terms **angle**, **interior of an angle**, and **angle bisector**.

Angle: An *angle* is ____

____

____

____

Interior: The *interior of angle* $\angle BAC$ is the set of points in the intersection of the half-plane of $\overleftrightarrow{AC}$ that contains B and the half-plane of $\overleftrightarrow{AB}$ that contains C. The interior is easy to identify because it is always the "smaller" region of the two regions defined by the angle (the region that is convex). The other region is called the *exterior* of the angle.

Angle Bisector: If C is in the interior of $\angle AOB$, ____

____

____

____

When we say $\mathrm{m}\angle AOC = \mathrm{m}\angle COB$, we mean that the angle measures are equal.

Geometry Assumptions

In working with lines and angles, we again make specific assumptions that need to be identified. For example, in the definition of interior of an angle above, we assumed that an angle separated the plane into two disjoint sets. This follows from the assumption: *Given a line, the points of the plane that do not lie on the line form two sets called half-planes, such that (1) each of the sets is convex, and (2) if P is a point in one of the sets, and Q is a point in the other, then the segment PQ intersects the line.*

From this assumption, another obvious fact follows about a segment that intersects the sides of an angle: Given an angle $\angle AOB$, then for any point C in the interior of $\angle AOB$, the ray OC will always intersect the segment AB.

In this lesson, we move from working with line segments to working with angles, specifically with bisecting angles. Before we do this, we need to clarify our assumptions about measuring angles. These assumptions are based upon what we know about a protractor that measures up to $180°$ angles:

1. To every angle $\angle AOB$ there corresponds a quantity $\mathrm{m}\angle AOB$ called the degree or measure of the angle so that $0 < \mathrm{m}\angle AOB < 180$.

This number, of course, can be thought of as the angle measurement (in degrees) of the interior part of the angle, which is what we read off of a protractor when measuring an angle. In particular, we have also seen that we can use protractors to "add angles":

2. If C is a point in the interior of $\angle AOB$, then $\mathrm{m}\angle AOC + \mathrm{m}\angle COB = \mathrm{m}\angle AOB$.

Two angles $\angle BAC$ and $\angle CAD$ form a *linear pair* if $\overrightarrow{AB}$ and $\overrightarrow{AD}$ are opposite rays on a line, and $\overrightarrow{AC}$ is any other ray. In earlier grades, we abbreviated this situation and the fact that the angles on a line add up to $180°$ as, "$\angle s$ on a line." Now, we state it formally as one of our assumptions:

3. If two angles $\angle BAC$ and $\angle CAD$ form a linear pair, then they are supplementary, i.e., $\mathrm{m}\angle BAC + \mathrm{m}\angle CAD = 180$.

Protractors also help us to draw angles of a specified measure:

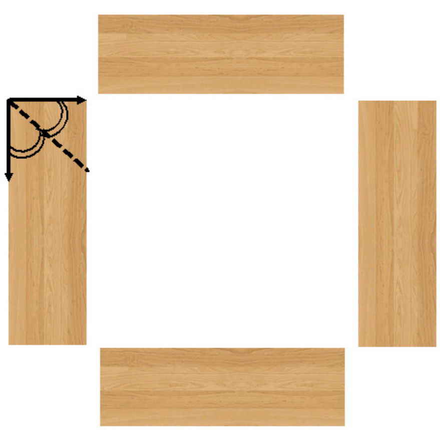
4. Let $\overrightarrow{OB}$ be a ray on the edge of the half-plane H. For every r such that $0° < r < 180°$, there is exactly one ray $\overrightarrow{OA}$ with A in H such that $\mathrm{m}\angle AOB = r$.

Mathematical Modeling Exercise 1: Investigate How to Bisect an Angle

You will need a compass and a straightedge.

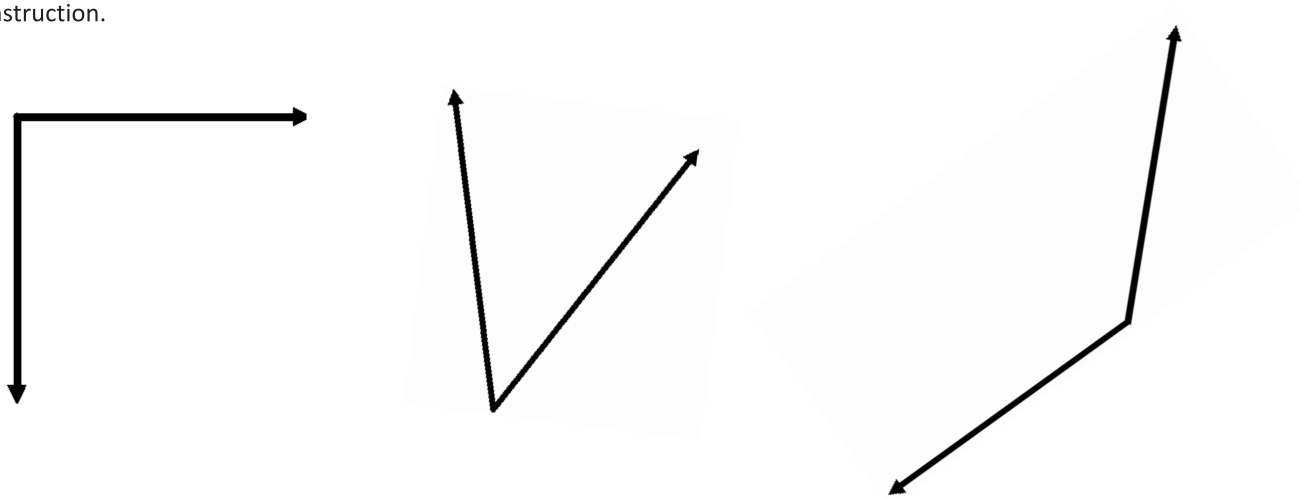
Joey and his brother, Jimmy, are working on making a picture frame as a birthday gift for their mother. Although they have the wooden pieces for the frame, they need to find the angle bisector to accurately fit the edges of the pieces together. Using your compass and straightedge, show how the boys bisected the corner angles of the wooden pieces below to create the finished frame on the right.

Before



After

Consider how the use of circles aids the construction of an angle bisector. Be sure to label the construction as it progresses and to include the labels in your steps. Experiment with the angles below to determine the correct steps for the construction.

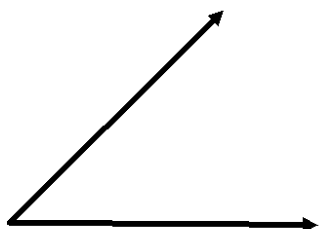


What steps did you take to bisect an angle? List the steps below:

Mathematical Modeling Exercise 2: Investigate How to Copy an Angle

You will need a compass and a straightedge.

You and your partner will be provided with a list of steps (in random order) needed to copy an angle using a compass and straightedge. Your task is to place the steps in the correct order, then follow the steps to copy the angle below.



Steps needed (in correct order):

1. ______________________________

2. ______________________________

3. ______________________________

4. ______________________________

5. ______________________________

6. ______________________________

7. ______________________________

8. ______________________________

9. ______________________________

Relevant Vocabulary

Midpoint: A point B is called a midpoint of $\overline{AC}$ if B is between A and C, and $AB = BC$.

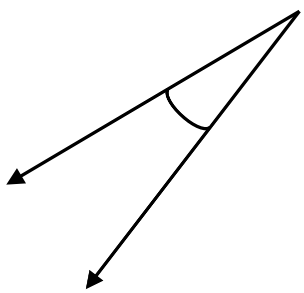
Degree: Subdivide the length around a circle into 360 arcs of equal length. A central angle for any of these arcs is called a one-degree angle and is said to have angle measure 1 degree. An angle that turns through n one-degree angles is said to have an angle measure of n degrees.

Zero and Straight Angle: A zero angle is just a ray and measures $0°$. A straight angle is a line and measures $180°$ (the ° is a symbol for *degree*).

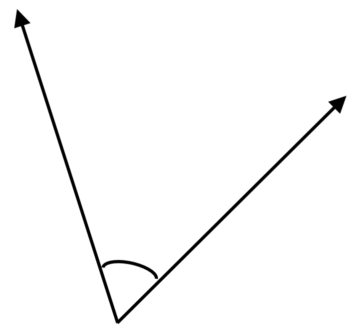
Problem Set

Bisect each angle below.

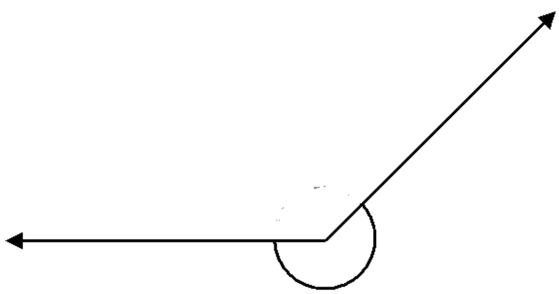
1.



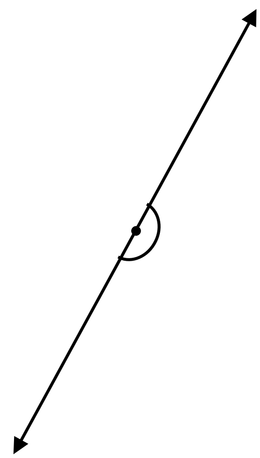
2.



3.

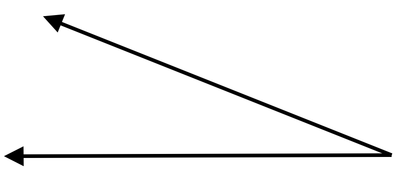


4.



Copy the angle below.

5.



Lesson 4: Construct a Perpendicular Bisector

Classwork

Opening Exercise

Choose **one** method below to check your Problem Set:

- Trace your copied angles and bisectors onto patty paper; then, fold the paper along the bisector you constructed. Did one ray exactly overlap the other?
- Work with your partner. Hold one partner's work over another's. Did your angles and bisectors coincide perfectly?

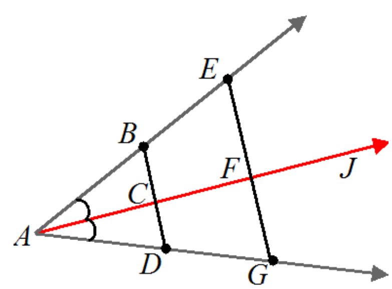
Use the following rubric to evaluate your Problem Set:

Needs Improvement	Satisfactory	Excellent
Few construction arcs visible	Some construction arcs visible	Construction arcs visible and appropriate
Few vertices or relevant intersections labeled	Most vertices and relevant intersections labeled	All vertices and relevant intersections labeled
Lines drawn without straightedge or not drawn correctly	Most lines neatly drawn with straightedge	Lines neatly drawn with straightedge
Fewer than 3 angle bisectors constructed correctly	3 of the 4 angle bisectors constructed correctly	Angle bisector constructed correctly

Discussion

In Lesson 3 we studied how to construct an angle bisector. We know we can verify the construction by folding an angle along the bisector. A correct construction means that one half of the original angle will coincide exactly with the other half so that each point of one ray of the angle maps onto a corresponding point on the other ray of the angle.

We now extend this observation. Imagine a segment that joins any pair of points that map onto each other when the original angle is folded along the bisector. The figure to the right illustrates two such segments.

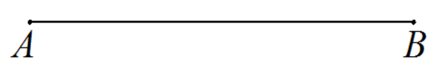


Let us examine one of the two segments, $\overline{EG}$. When the angle is folded along $\overrightarrow{AJ}$, E coincides with G. In fact, folding the angle demonstrates that E is the same distance from F as G is from F; $EF = FG$. The point that separates these equal halves of $\overline{EG}$ is F, which is, in fact, the midpoint of the segment and lies on the bisector $\overrightarrow{AJ}$. We can make this case for any segment that falls under the conditions above.

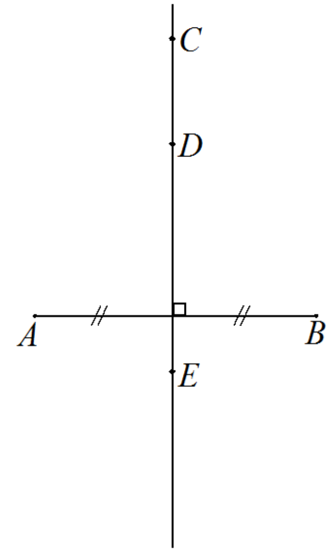
By using geometry facts we acquired in earlier school years, we can also show that the angles formed by the segment and the angle bisector are right angles. Again, by folding, we can show that $\angle EFJ$ and $\angle GFJ$ coincide and must have the same measure. The two angles also lie on a straight line, which means they sum to $180°$. Since they are equal in measure and sum to $180°$, they each have a measure of $90°$.

These arguments lead to a remark about symmetry with respect to a line and the definition of a perpendicular bisector. Two points are symmetric with respect to a line l if and only if l is the perpendicular bisector of the segment that joins the two points. A perpendicular bisector of a segment passes through the ____________________ of the segment and forms ______________________________ with the segment.

We now investigate how to construct a perpendicular bisector of a line segment using a compass and a straightedge. Using what you know about the construction of an angle bisector, experiment with your construction tools and the following line segment to establish the steps that determine this construction.



Precisely describe the steps you took to bisect the segment.



Now that you are familiar with the construction of a perpendicular bisector, we must make one last observation. Using your compass, string, or patty paper, examine the following pairs of segments:

i. $\overline{AC}, \overline{BC}$
ii. $\overline{AD}, \overline{BD}$
iii. $\overline{AE}, \overline{BE}$

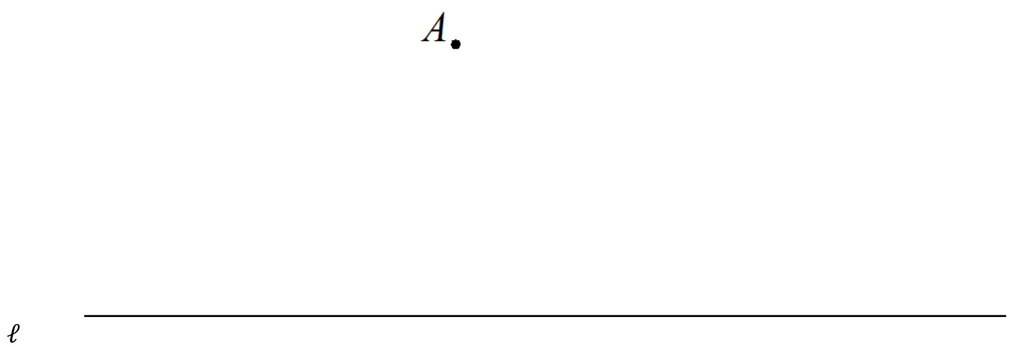
Based on your findings, fill in the observation below.

Observation:

Any point on the perpendicular bisector of a line segment is ______________________ from the endpoints of the line segment.

Mathematical Modeling Exercise

You know how to construct the perpendicular bisector of a segment. Now, you will investigate how to construct a perpendicular to a line ℓ from a point A not on ℓ. Think about how you have used circles in constructions so far and *why* the perpendicular bisector construction works the way it does. The first step of the instructions has been provided for you. Discover the construction and write the remaining steps.



Step 1. Draw circle A: center A, with radius so that circle A intersects line $\boldsymbol{\ell}$ in two points.

Relevant Vocabulary

Right Angle: An angle is called a *right angle* if its measure is $90°$.

Perpendicular: Two lines are *perpendicular* if they intersect in one point and if any of the angles formed by the intersection of the lines is a $90°$ (right) angle. Two segments or rays are perpendicular if the lines containing them are perpendicular lines.

Equidistant: A point A is said to be *equidistant* from two different points B and C if $AB = AC$. A point A is said to be *equidistant* from a point B and a line l if the distance between A and l is equal to AB.

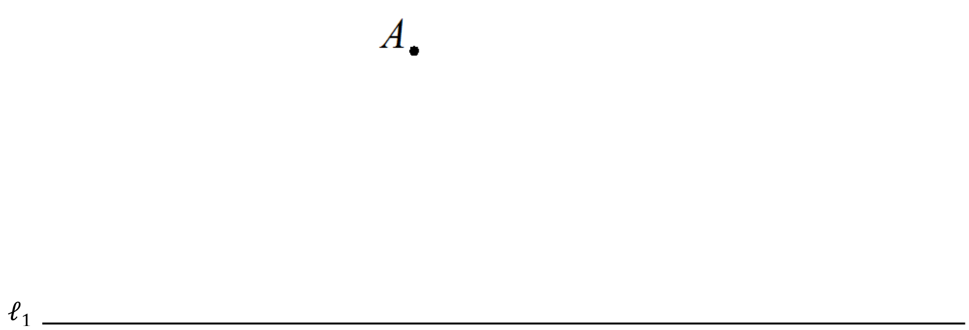
Problem Set

1. During this lesson, you constructed a perpendicular line to a line ℓ from a point A not on ℓ. We are going to use that construction to construct parallel lines:

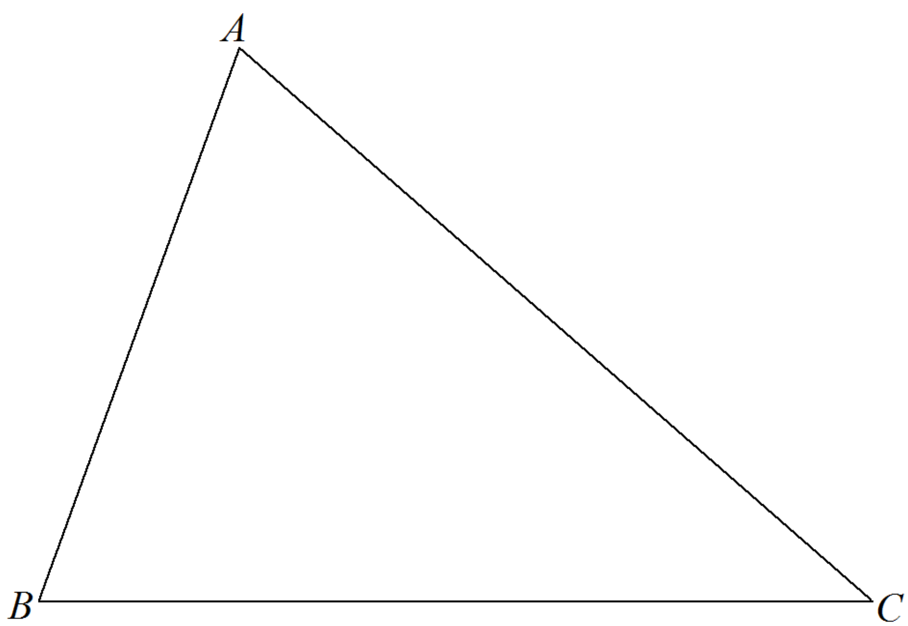
   To construct parallel lines ℓ_1 and ℓ_2:

   i. Construct a perpendicular line ℓ_3 to a line ℓ_1 from a point A not on ℓ_1.

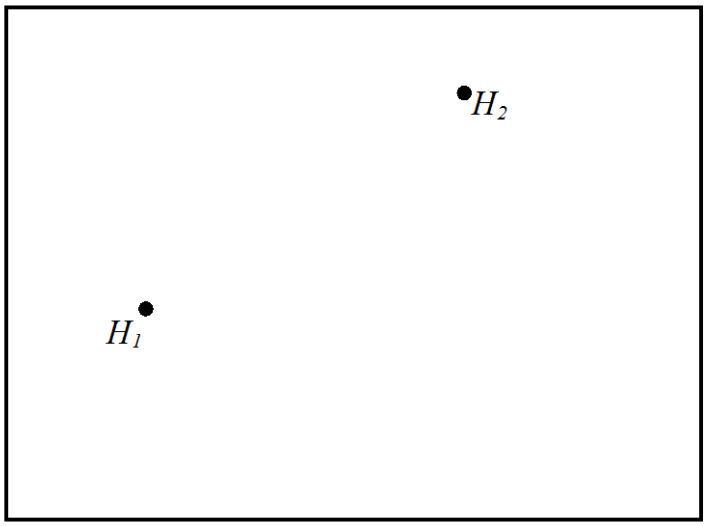
   ii. Construct a perpendicular line ℓ_2 to ℓ_3 through point A. *Hint:* Consider using the steps behind Problem 4 in the Lesson 3 Problem Set to accomplish this.

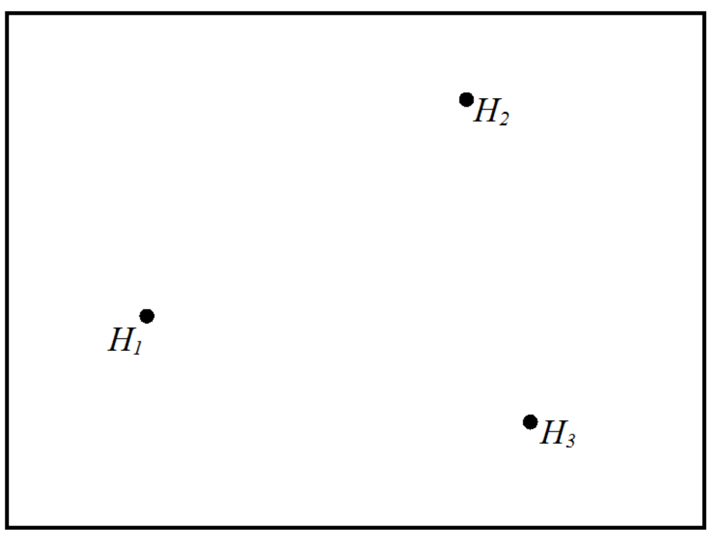


2. Construct the perpendicular bisector of $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ on the triangle below. What do you notice about the segments you have constructed?



3. Two homes are built on a plot of land. Both homeowners have dogs and are interested in putting up as much fencing as possible between their homes on the land but in a way that keeps the fence equidistant from each home. Use your construction tools to determine where the fence should go on the plot of land. How must the fencing be altered with the addition of a third home?





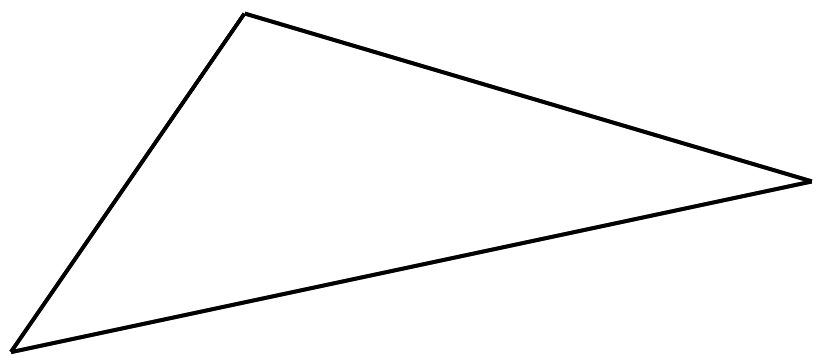
Lesson 5: Points of Concurrencies

Classwork

Opening Exercise

You will need a make-shift compass made from string and pencil.

Use these materials to construct the perpendicular bisectors of the three sides of the triangle below (like you did with Lesson 4, Problem Set 2).



How did using this tool differ from using a compass and straightedge? Compare your construction with that of your partner. Did you obtain the same results?

____

____

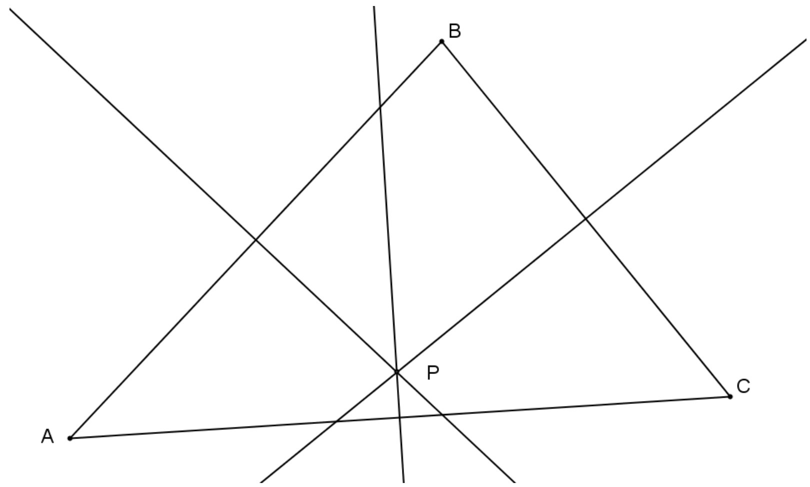
Exploratory Challenge

When three or more lines intersect in a single point, they are ____________________, and the point of intersection is the ____________________________.

You saw an example of a point of concurrency in yesterday's Problem Set (and in the Opening Exercise above) when all three perpendicular bisectors passed through a common point.

The point of concurrency of the three perpendicular bisectors is the ____.

The circumcenter of $\triangle ABC$ is shown below as point P.



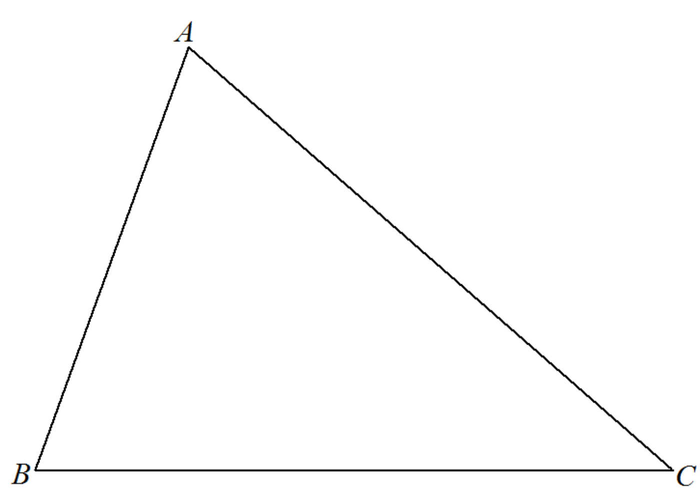
The questions that arise here are WHY are the three perpendicular bisectors concurrent? And WILL these bisectors be concurrent in all triangles? Recall that all points on the perpendicular bisector are equidistant from the endpoints of the segment, which means:

1. P is equidistant from A and B since it lies on the ___________________________ of $\overline{AB}$.
2. P is also _________________________ from B and C since it lies on the perpendicular bisector of $\overline{BC}$.
3. Therefore, P must also be equidistant from A and C.

Hence, $AP = BP = CP$, which suggests that P is the point of ________________________ of all three perpendicular bisectors.

You have also worked with angle bisectors. The construction of the three angle bisectors of a triangle also results in a point of concurrency, which we call the ________________________.

Use the triangle below to construct the angle bisectors of each angle in the triangle to locate the triangle's incenter.



1. State precisely the steps in your construction above.

___

___

___

2. Earlier in this lesson, we explained why the perpendicular bisectors of the sides of a triangle are always concurrent. Using similar reasoning, explain clearly why the angle bisectors are always concurrent at the incenter of a triangle.

___

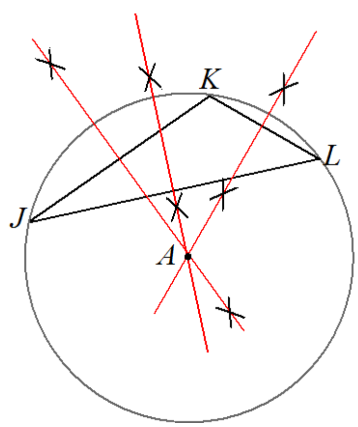
___

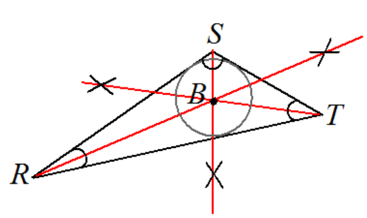
___

___

___

3. Observe the constructions below. Point A is the ___________________________ of $\triangle JKL$ (notice that it can fall outside of the triangle). Point B is the ___________________________ of triangle $\triangle RST$. The circumcenter of a triangle is the center of the circle that circumscribes that triangle. The incenter of the triangle is the center of the circle that is inscribed in that triangle.



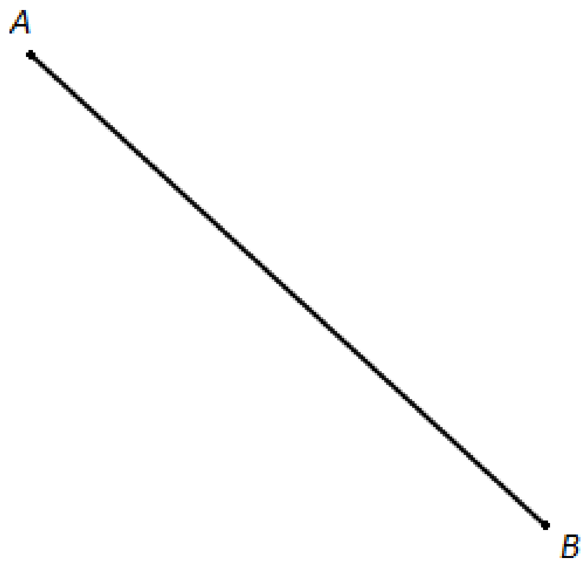


On a separate piece of paper, draw two triangles of your own below and demonstrate how the circumcenter and incenter have these special relationships.

4. How can you use what you have learned in Exercise 3 to find the center of a circle if the center is not shown?

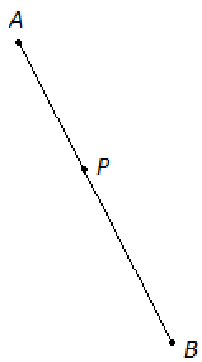
Problem Set

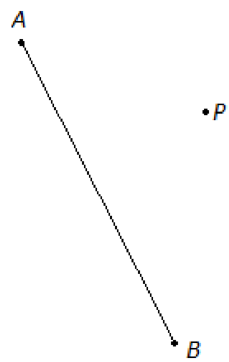
1. Given line segment AB, using a compass and straightedge, construct the set of points that are equidistant from A and B.



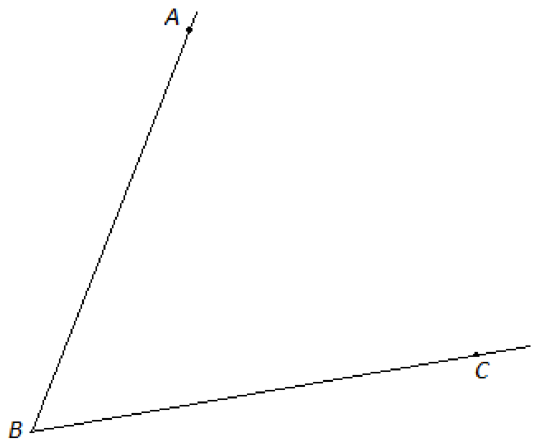
What figure did you end up constructing? Explain.

2. For each of the following, construct a line perpendicular to segment AB that goes through point P.





3. Using a compass and straightedge, construct the angle bisector of $\angle ABC$ shown below. What is true about every point that lies on the ray you created?



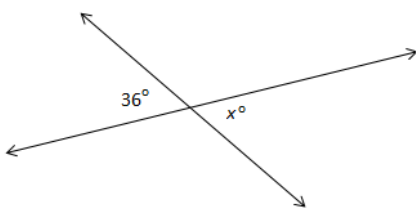
Lesson 6: Solve for Unknown Angles—Angles and Lines at a Point

Classwork

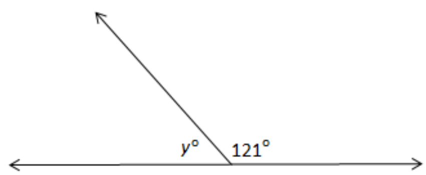
Opening Exercise

Determine the measure of the missing angle in each diagram.

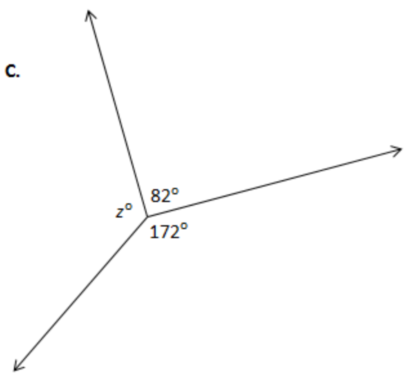
A.



B.



C.



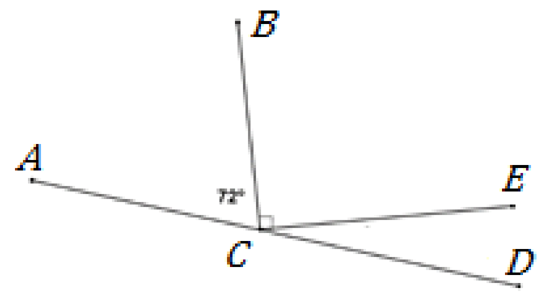
What facts about angles did you use?

Discussion

Two angles AOC and COB, with a common side $\overrightarrow{OC}$, are ________________ if C belongs to the interior of $\angle AOB$. The sum of angles on a straight line is $180°$ and two such angles are called a linear pair. Two angles are called supplementary if the sum of their measures is ________________, two angles are called complementary if the sum of their measures is ________________. Describing angles as supplementary or complementary refers only to the measures of their angles. The positions of the angles or whether the pair of angles is adjacent to each other is not part of the definition.

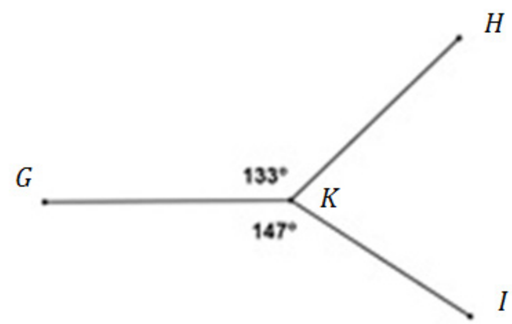
In the figure, line segment AD is drawn.

Find $\mathrm{m}\angle DCE$.



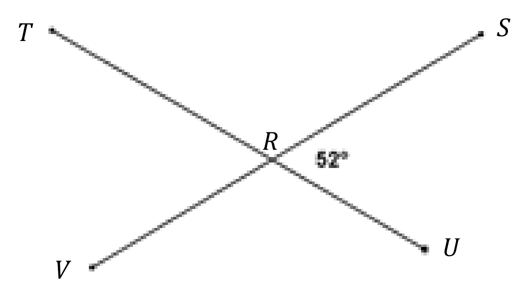
The total measure of adjacent angles around a point is ___________.

Find the measure of $\mathrm{m}\angle HKI$.



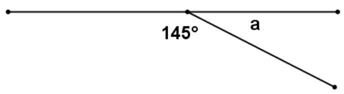
Vertical angles have __________ measure. Two angles are vertical if their sides form opposite rays.

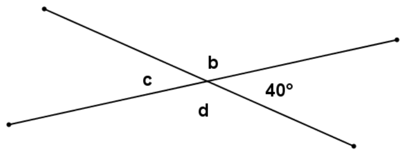
Find $\mathrm{m}\angle TRV$.

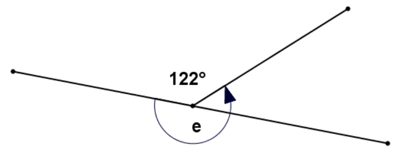


Example 1

Find the measures of each labeled angle. Give a reason for your solution.





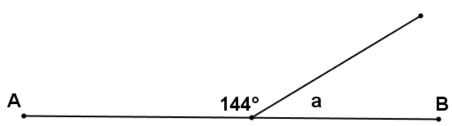


Angle	Angle measure	Reason
$\angle a$		
$\angle b$		
$\angle c$		
$\angle d$		
$\angle e$		

Exercises 1–12

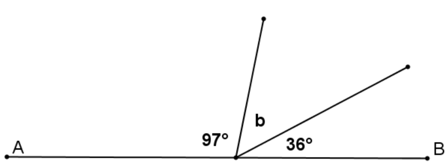
In the figures below, $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$ are straight-line segments. Find the measure of each marked angle or find the unknown numbers labeled by the variables in the diagrams. Give reasons for your calculations. Show all the steps to your solutions.

1.



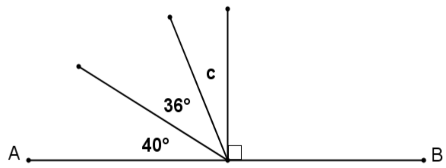
$\angle a =$ ____________________

2.



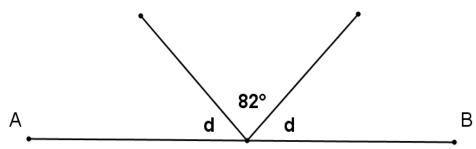
$\angle b =$ ____________________

3.



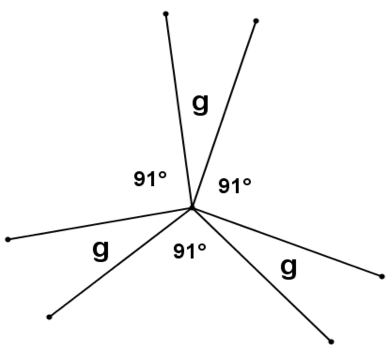
$\angle c =$ ______________________

4.



$\angle d =$ ______________________

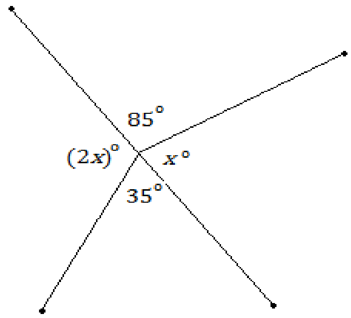
5.



$\angle g =$ ______________________

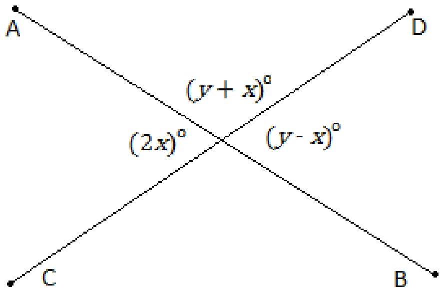
For Problems 6–12, find the values of x and y. Show all work.

6.



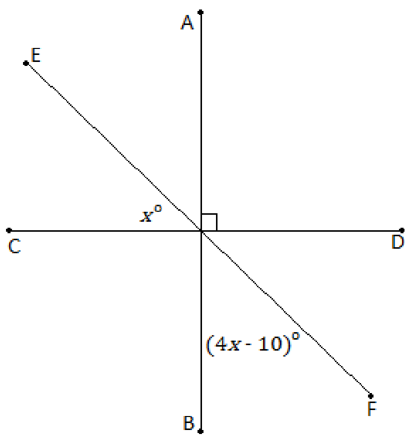
$x =$ ______________________

7.


A
D
$(y + x)^\circ$
$(2x)^\circ$
$(y - x)^\circ$
C
B

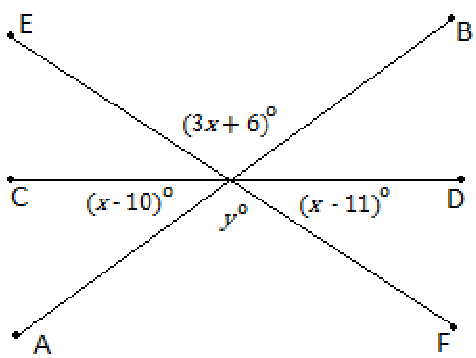
y = ________ x = ____________

8.


A
E
x°
C
D
$(4x - 10)^\circ$
B
F

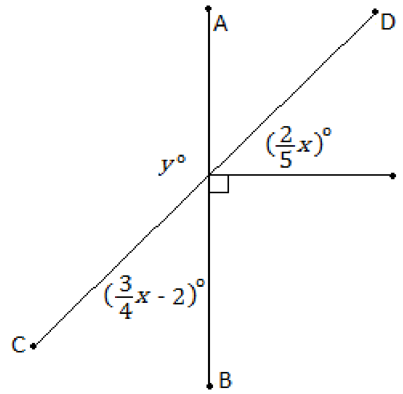
x = ____________________

9.


E
B
$(3x + 6)^\circ$
C
$(x - 10)^\circ$
y°
$(x - 11)^\circ$
D
A
F

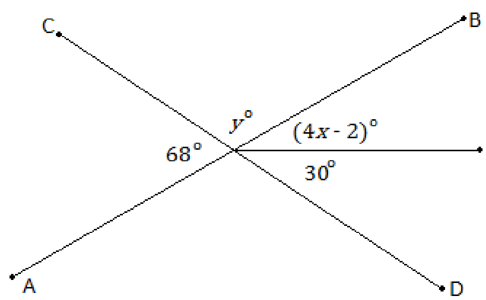
x = _______________________

10.



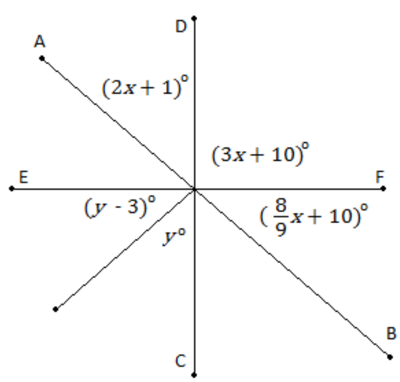
$x =$ ____________ $y =$ ____________

11.



$x =$ ____________________

12.



$x =$ ____________ $y =$ ____________

Relevant Vocabulary

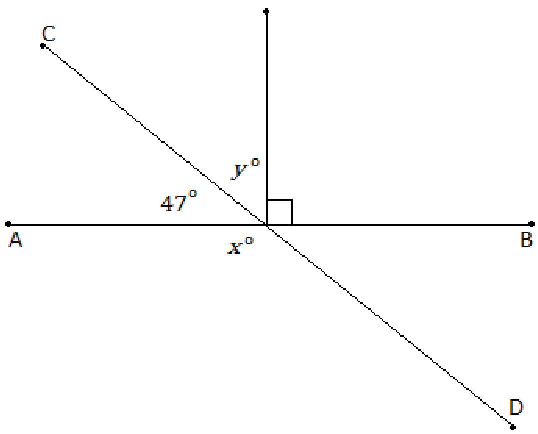
<u>Straight Angle</u>: If two rays with the same vertex are distinct and collinear, then the rays form a line called a *straight angle*.

<u>Vertical Angles</u>: Two angles are *vertical angles* (or vertically opposite angles) if their sides form two pairs of opposite rays.

Problem Set

In the figures below, $\overline{AB}$ and $\overline{CD}$ are straight line segments. Find the value of x and/or y in each diagram below. Show all the steps to your solution and give reasons for your calculations.

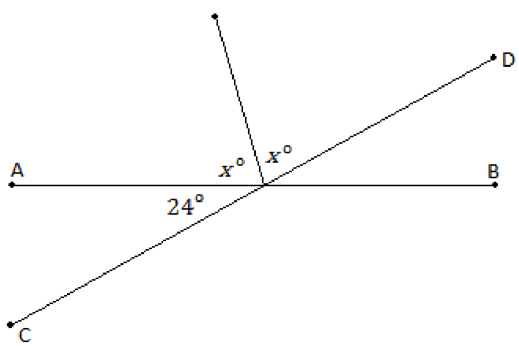
1.



x = ____________________

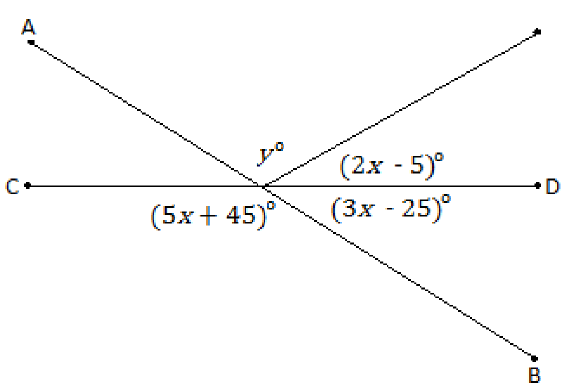
y = ____________________

2.



x = ____________________

3.



x = ____________________

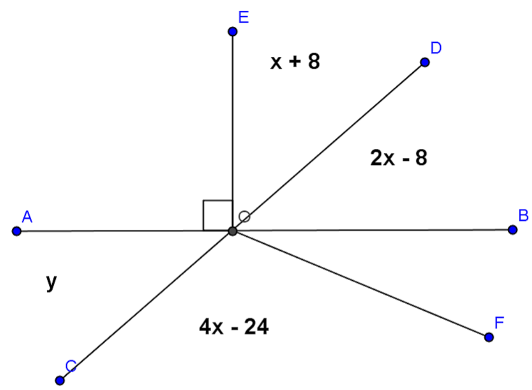
y = ____________________

Lesson 7: Solve for Unknown Angles—Transversals

Classwork

Opening Exercise

Use the diagram at the right to determine x and y.
$\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are straight lines.



$x =$ ________

$y =$ ________

Name a pair of vertical angles:

____

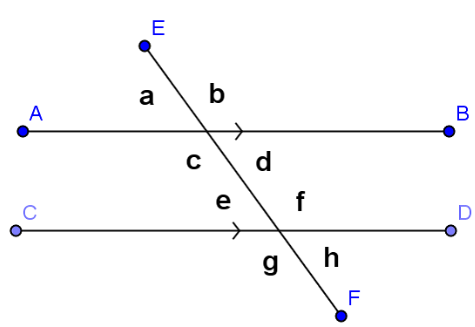
Find the measure of $\angle BOF$. Justify your calculation.

____

____

Discussion

Given line AB and line CD in a plane (see the diagram below), a third line EF is called a *transversal* if it intersects $\overleftrightarrow{AB}$ at a single point and intersects $\overleftrightarrow{CD}$ at a single but different point. Line AB and line CD are parallel if and only if the following types of angle pairs are congruent or supplementary.



- Corresponding angles are equal in measure

  ____

- Alternate interior angles are equal in measure

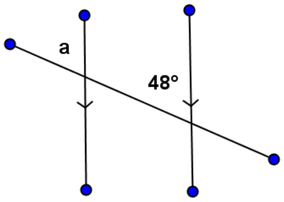
  ____

- Same side interior angles are supplementary

  ____

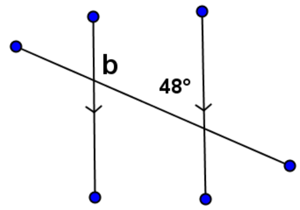
Examples

1.



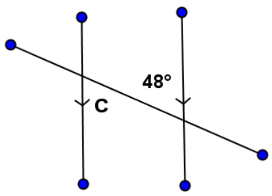
m∠a = ________

2.



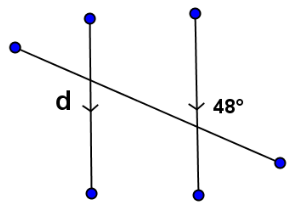
m∠b = ________

3.



m∠c =_________

4.

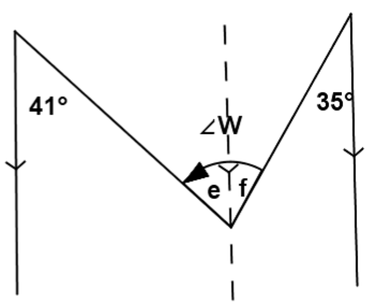


m∠d = ________

5. An ___________________________________is sometimes useful when solving for unknown angles.

In this figure, we can use the auxiliary line to find the measures of $\angle e$ and $\angle f$ (how?), then add the two measures together to find the measure of $\angle W$.

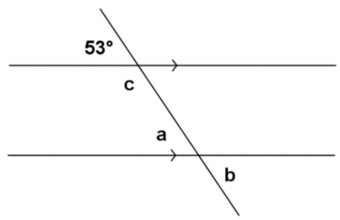
What is the measure of $\angle W$?



Exercises

In each exercise below, find the unknown (labeled) angles. Give reasons for your solutions.

1.

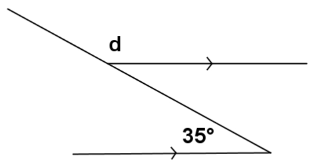


$m\angle a$ = ____________________

$m\angle b$ = ____________________

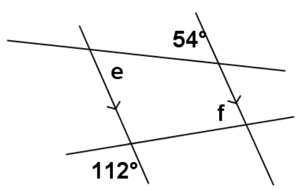
$m\angle c$ = ____________________

2.



$m\angle d$ = ____________________

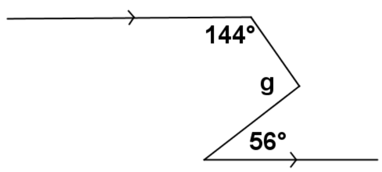
3.



$m\angle e$ = ____________________

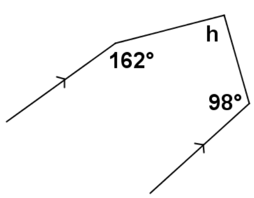
$m\angle f$ = ____________________

4.



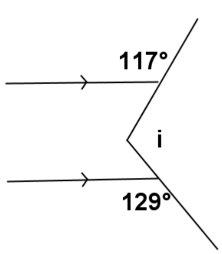
$m\angle g$ = ____________________

5.



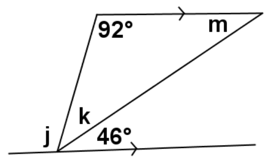
$m\angle h$ = ____________________

6.



$m\angle i$ = ____________________

7.

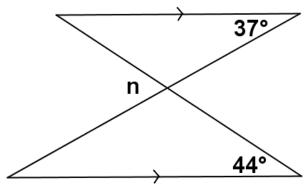


$m\angle j$ = ______________________

$m\angle k$ = ______________________

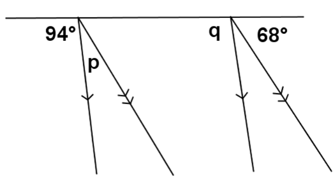
$m\angle m$= ______________________

8.



$m\angle n$ = ______________________

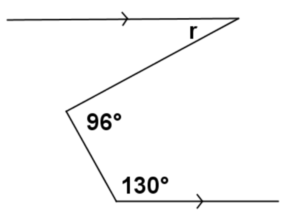
9.



$m\angle p$ = ______________________

$m\angle q$ = ______________________

10.



$m\angle r$ = ______________________

Relevant Vocabulary

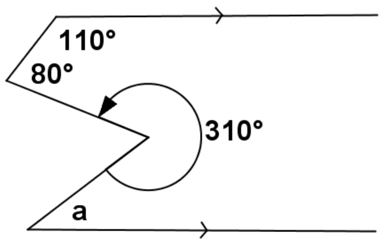
Alternate Interior Angles: Let line t be a transversal to lines l and m such that t intersects l at point P and intersects m at point Q. Let R be a point on line l and S be a point on line m such that the points R and S lie in opposite half-planes of t. Then $\angle RPQ$ and $\angle PQS$ are called *alternate interior angles* of the transversal t with respect to line m and line l.

Corresponding Angles*:* Let line t be a transversal to lines l and m. If $\angle x$ and $\angle y$ are alternate interior angles, and $\angle y$ and $\angle z$ are vertical angles, then $\angle x$ and $\angle z$ are *corresponding angles*.

Problem Set

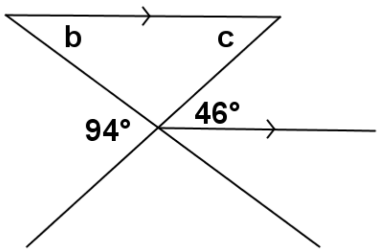
Find the unknown (labeled) angles. Give reasons for your solutions.

1.



m∠a = ____________________

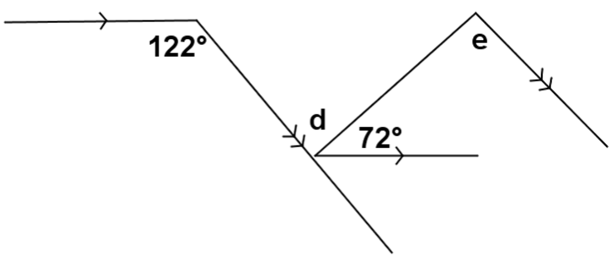
2.



m∠b = ____________________

m∠c = ____________________

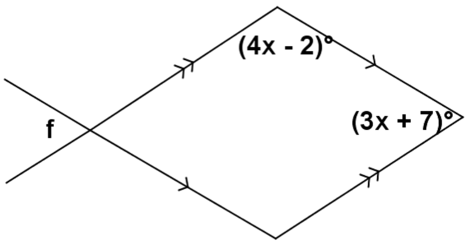
3.



m∠d = ____________________

m∠e = ____________________

4.



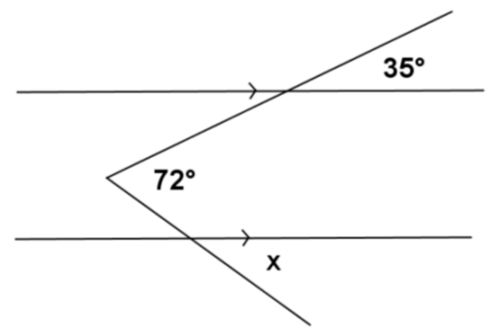
m∠f = ____________________

Lesson 8: Solve for Unknown Angles—Angles in a Triangle

Classwork

Opening Exercise

Find the measure of angle x in the figure to the right. Explain your calculations. (Hint: Draw an auxiliary line segment.)



Discussion

The sum of the 3 angle measures of any triangle is __________________.

Interior of a Triangle: A point lies in the *interior of a triangle* if it lies in the interior of each of the angles of the triangle.

In any triangle, the measure of the exterior angle is equal to the sum of the measures of the ________________ angles. These are sometimes also known as _________________ angles.

Base angles of an _______________ triangle are equal in measure.

Each angle of an _______________ triangle has a measure equal to $60°$.

Relevant Vocabulary

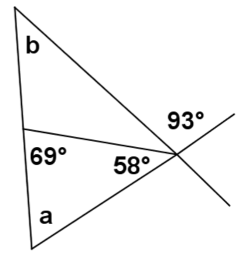
Isosceles Triangle: An *isosceles triangle* is a triangle with at least two sides of equal length.

Angles of a Triangle: Every triangle $\triangle ABC$ determines three angles, namely, $\angle BAC$, $\angle ABC$, and $\angle ACB$. These are called the *angles of* $\triangle ABC$.

Exterior Angle of a Triangle: Let $\angle ABC$ be an interior angle of a triangle $\triangle ABC$, and let D be a point on $\overleftrightarrow{AB}$ such that B is between A and D. Then $\angle CBD$ is an *exterior angle of the triangle* $\triangle ABC$.

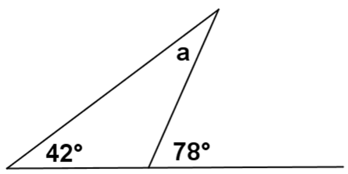
Exercises

1. Find the measures of a and b in the figure to the right. Justify your results.



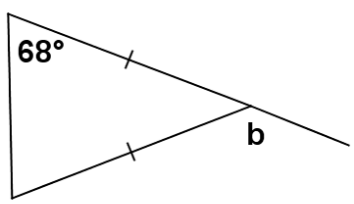
In each figure, determine the measures of the unknown (labeled) angles. Give reasons for your calculations.

2.



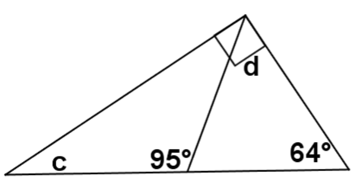
m∠a = ____________________

3.



m∠b = ____________________

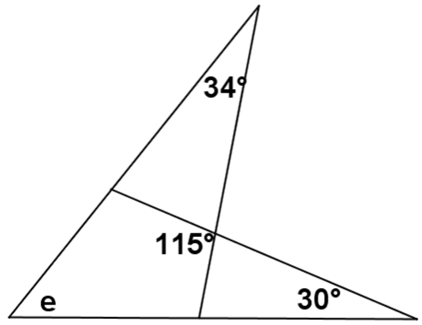
4.



m∠c = ____________________

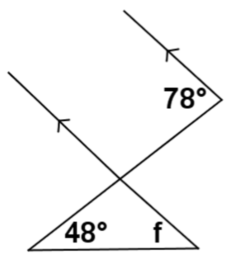
m∠d = ____________________

5.



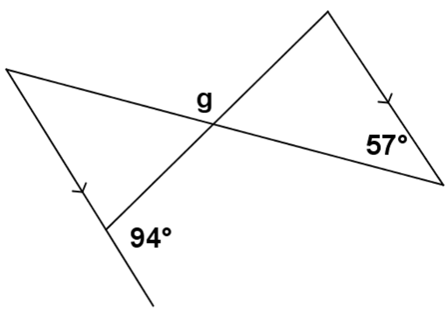
m∠e = ____________

6.



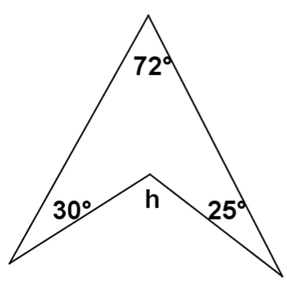
m∠f = ____________

7.



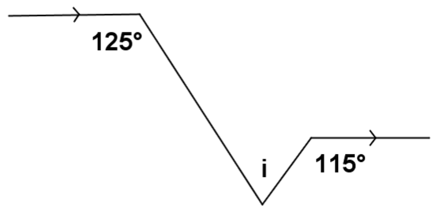
m∠g = ____________

8.



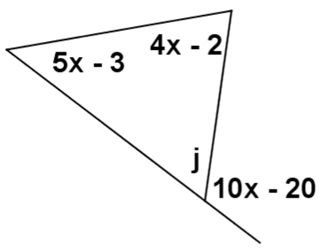
m∠h = ____________

9.



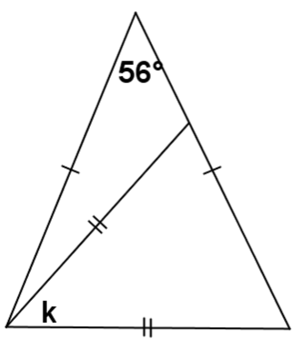
m∠i = ____________________

10.



m∠j = ____________________

11.

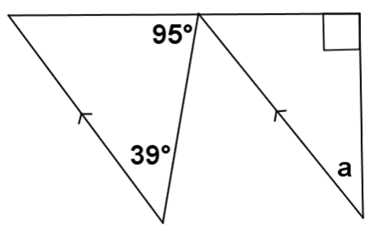


m∠k = ____________________

Problem Set

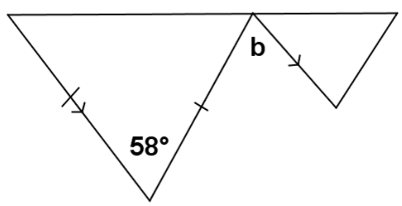
Find the unknown (labeled) angle in each figure. Justify your calculations.

1.



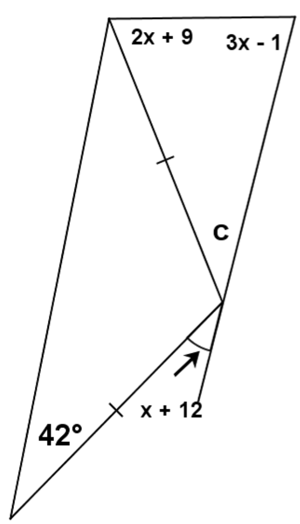
m∠k = ________________

2.



m∠k = ________________

3.



m∠k = ________________

Lesson 9: Unknown Angle Proofs—Writing Proofs

Classwork

Opening Exercise

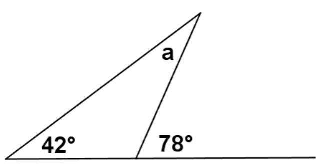
One of the main goals in studying geometry is to develop your ability to reason critically, to draw valid conclusions based upon observations and proven facts. Master detectives do this sort of thing all the time. Take a look as Sherlock Holmes uses seemingly insignificant observations to draw amazing conclusions.

Sherlock Holmes: Master of Deduction!

Could you follow Sherlock Holmes' reasoning as he described his thought process?

Discussion

In geometry, we follow a similar deductive thought process, much like Holmes' uses, to prove geometric claims. Let's revisit an old friend—solving for unknown angles. Remember this one?

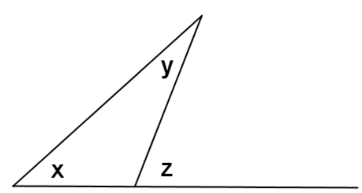


You needed to figure out the measure of a, and used the "fact" that an exterior angle of a triangle equals the sum of the measures of the opposite interior angles. The measure of $\angle a$ must, therefore, be $36°$.

Suppose that we rearrange the diagram just a little bit.

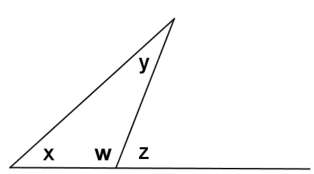
Instead of using numbers, we will use variables to represent angle measures.

Suppose further that we already have in our arsenal of facts the knowledge that the angles of a triangle sum to $180°$. Given the labeled diagram at the right, can we prove that $x + y = z$ (or, in other words, that the exterior angle of a triangle equals the sum of the remote interior angles)?



Proof:

Label $\angle w$, as shown in the diagram.



$\text{m}\angle x + \text{m}\angle y + \text{m}\angle w = 180°$	Sum of the angle measures in a triangle is $180°$
$\text{m}\angle w + \text{m}\angle z = 180°$	Linear pairs form supplementary angles.
$\text{m}\angle x + \text{m}\angle y + \text{m}\angle w = \text{m}\angle w + \text{m}\angle z$	Substitution property of equality
$\therefore \text{m}\angle x + \text{m}\angle y = \text{m}\angle z$	Subtraction property of equality

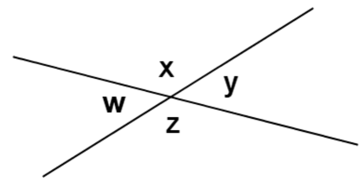
Notice that each step in the proof was justified by a previously known or demonstrated fact. We end up with a newly proven fact (that an exterior angle of *any* triangle is the sum of the measures of the opposite interior angles of the triangle). This ability to identify the steps used to reach a conclusion based on known facts is ***deductive reasoning*** (i.e., the same type of reasoning that Sherlock Holmes used to accurately describe the doctor's attacker in the video clip.)

Exercises

1. You know that angles on a line sum to $180°$.

   Prove that vertical angles are congruent.

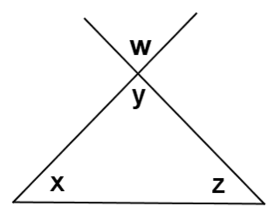
   Make a plan:



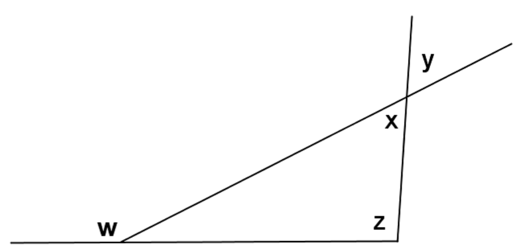
   - What do you know about $\angle w$ and $\angle x$? $\angle y$ and $\angle x$?

   - What conclusion can you draw based on both bits of knowledge?

   - Write out your proof:

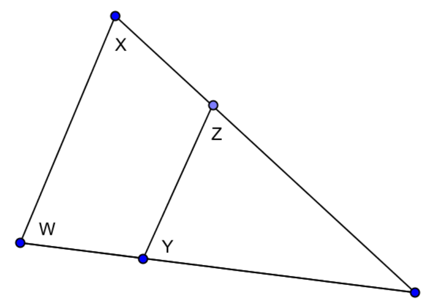


2. Given the diagram to the right, prove that $\mathrm{m}\angle w + \mathrm{m}\angle x + \mathrm{m}\angle z = 180°$. (Make a plan first. What do you know about $\angle x$, $\angle y$, and $\angle z$?)

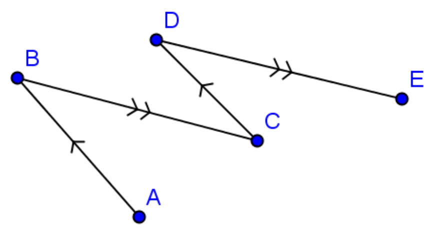


   Given the diagram to the right, prove that $\mathrm{m}\angle w = \mathrm{m}\angle y + \mathrm{m}\angle z$.

3. In the diagram to the right, prove that $m\angle y + m\angle z = m\angle w + m\angle x$. (You will need to write in a label in the diagram that is not labeled yet for this proof.)

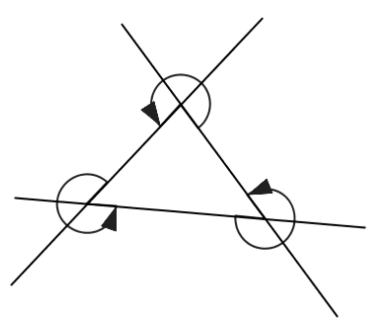


4. In the figure to the right, $\overline{AB} \,||\, \overline{CD}$ and $\overline{BC} \,||\, \overline{DE}$. Prove that $m\angle ABC = m\angle CDE$.

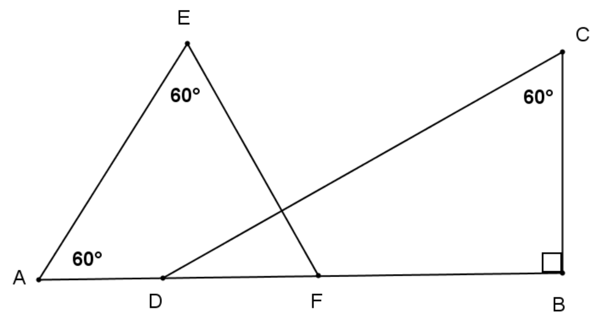


5. In the figure to the right, prove that the sum of the angles marked by arrows is 900

(You will need to write in several labels into the diagram for this proof.)

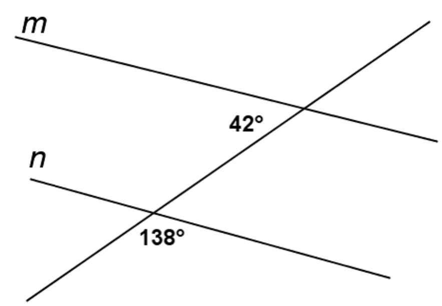


6. In the figure to the right, prove that $\overline{DC} \perp \overline{EF}$.

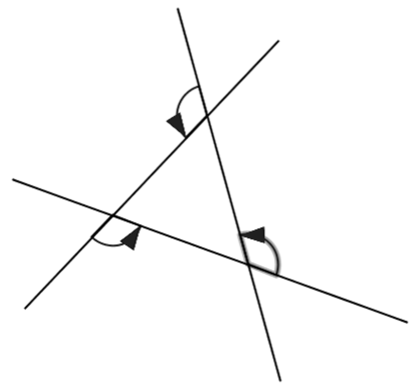


Problem Set

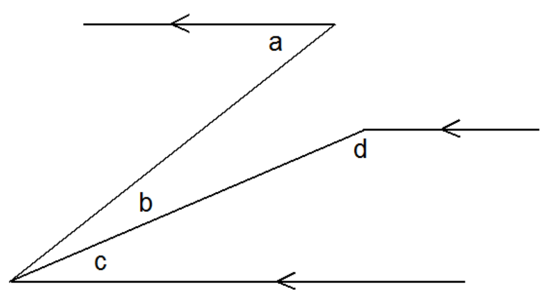
1. In the figure to the right, prove that $m \parallel n$.



2. In the diagram to the right, prove that the sum of the angles marked by arrows is $360°$.



3. In the diagram at the right, prove that $\mathrm{m}\angle a + \mathrm{m}\angle d - \mathrm{m}\angle b = 180$.

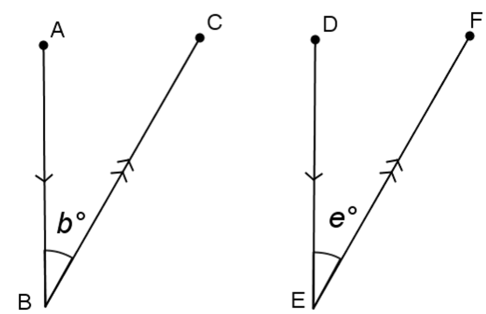


Lesson 10: Unknown Angle Proofs—Proofs with Constructions

Classwork

Opening Exercise

In the figure on the right, $\overline{AB} \parallel \overline{DE}$ and $\overline{BC} \parallel \overline{EF}$. Prove that $b = e$
(Hint: Extend $\overline{BC}$ and $\overline{ED}$.)



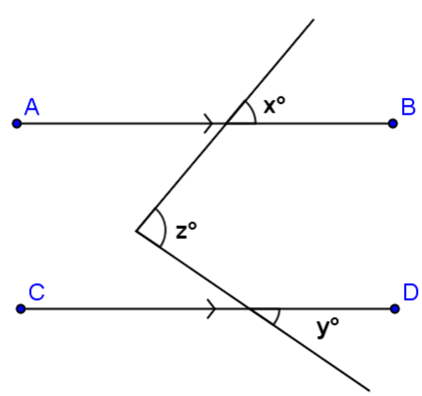
Proof:

In the previous lesson, you used deductive reasoning with labeled diagrams to prove specific conjectures. What is different about the proof above?

Adding or extending segments, lines, or rays (referred to as auxiliary lines) is frequently useful in demonstrating steps in the deductive reasoning process. Once $\overline{BC}$ and $\overline{ED}$ were extended, it was relatively simple to prove the two angles congruent based on our knowledge of alternate interior angles. Sometimes there are several possible extensions or additional lines that would work equally well.

For example, in this diagram, there are at least two possibilities for auxiliary lines. Can you spot them both?

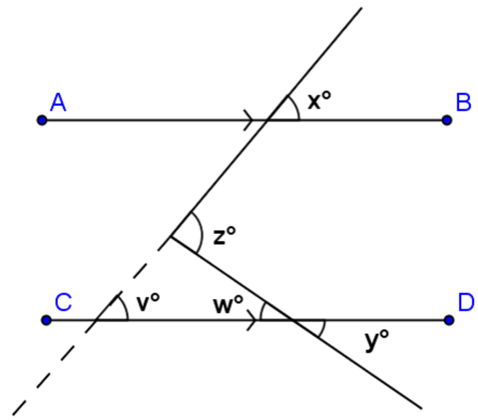
Given: $\overline{AB} \parallel \overline{CD}$.
Prove: $z = x + y$.



Discussion

Here is one possibility:

Given: $\overline{AB} \parallel \overline{CD}$.
Prove: $z = x + y$.



Extend the transversal as shown by the dotted line in the diagram.
Label angle measures v and w, as shown.

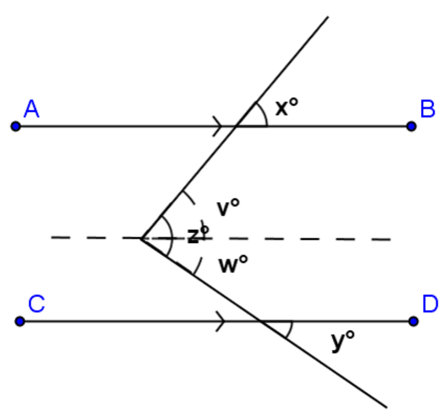
What do you know about v and x?

About angles w and y? How does this help you?

Write a proof using the auxiliary segment drawn in the diagram to the right.

Another possibility appears here:

Given: $\overline{AB} \,||\, \overline{CD}$.
Prove: $z = x + y$.



Draw a segment parallel to $\overline{AB}$ through the vertex of the angle measuring z degrees. This divides it into angles two parts as shown.

What do you know about angles v and x?

About w and y? How does this help you?

Write a proof using the auxiliary segment drawn in this diagram. Notice how this proof differs from the one above.

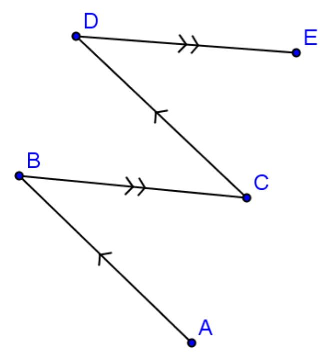
What do you know about v and x?

About w and y? How does this help you?

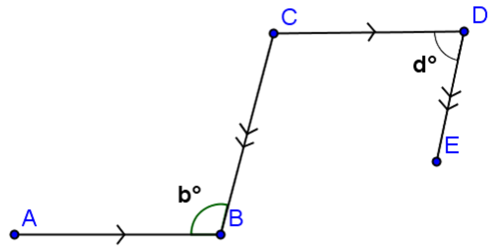
Write a proof using the auxiliary segment drawn in this diagram. Notice how this proof differs from the one above.

Examples

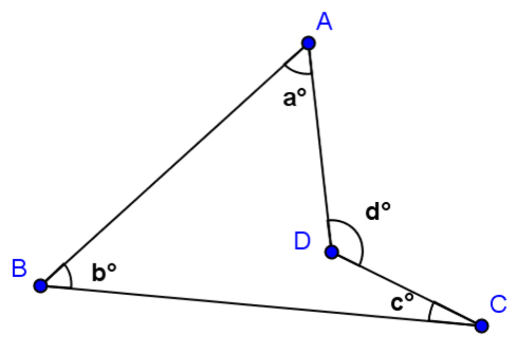
1. In the figure at the right, $\overline{AB} \parallel \overline{CD}$ and $\overline{BC} \parallel \overline{DE}$.
   Prove that $\mathrm{m}\angle ABC = \mathrm{m}\angle CDE$.
   (Is an auxiliary segment necessary?)



2. In the figure at the right, $\overline{AB} \parallel \overline{CD}$ and $\overline{BC} \parallel \overline{DE}$.
Prove that $b + d = 180$.

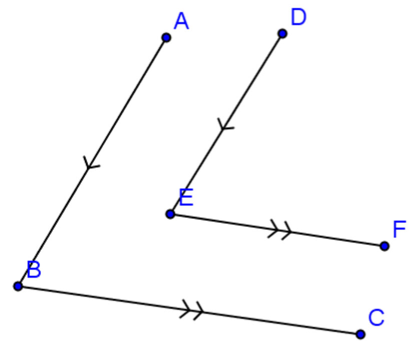


3. In the figure at the right, prove that $d = a + b + c$.

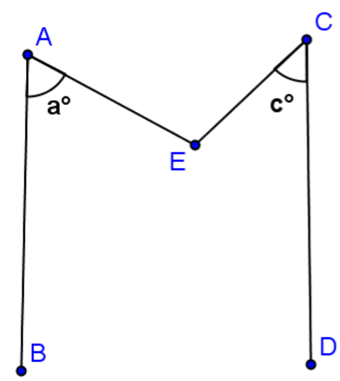


Problem Set

1. In the figure to the right, $\overline{AB} \parallel \overline{DE}$ and $\overline{BC} \parallel \overline{EF}$.
   Prove that $m\angle ABC = m\angle DEF$.



2. In the figure to the right, $\overline{AB} \parallel \overline{CD}$.
   Prove that $m\angle AEC = a° + c°$.



Lesson 11: Unknown Angle Proofs—Proofs of Known Facts

Classwork

Opening Exercise

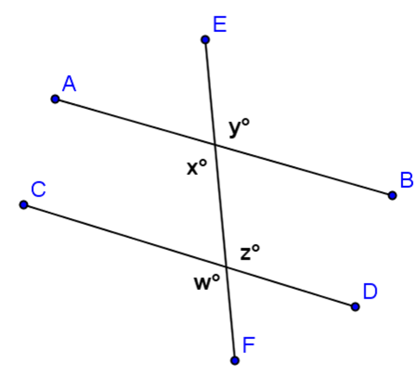
A **proof** of a mathematical statement is a detailed explanation of how that statement follows logically from other statements already accepted as true.

A **theorem** is a mathematical statement with a proof.

Discussion

Once a theorem has been proved, it can be added to our list of known facts and used in proofs of other theorems. For example, in Lesson 9 we proved that *vertical angles are of equal measure*, and we know (from earlier grades and by paper cutting and folding) that *if a transversal intersects two parallel lines, alternate interior angles are of equal measure*. How do these facts help us prove that corresponding angles are congruent?

In the diagram to the right, if you are given that $\overline{AB} \parallel \overline{CD}$ how can you use your knowledge of the congruence of vertical angles and alternate interior angles to prove that $x = w$?



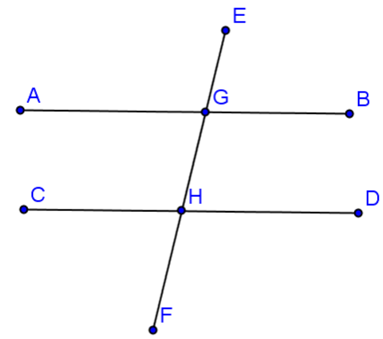
You now have available the following facts:

- Vertical angles are equal in measure.
- Alternate interior angles are equal in measure.
- Corresponding angles are equal in measure.

Use any or all of these facts to prove that *interior angles on the same side of the transversal are* ***supplementary***. Add any necessary labels to the diagram below, and then write out a proof including given facts and a statement of what needs to be proved.

Given: $\overline{AB} \parallel \overline{CD}$, transversal $\overline{EF}$

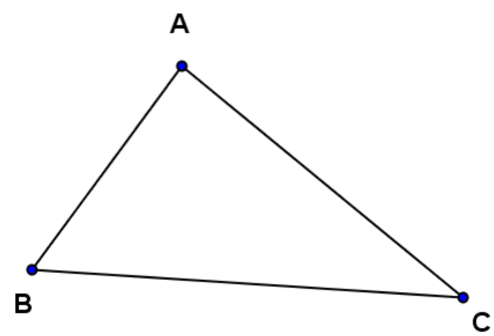
Prove: $\text{m}\angle BGH + \text{m}\angle DHG = 180°$



Now that you have proven this, you may add this theorem to your available facts.

- Interior angles on the same side of the transversal that intersects parallel lines sum to $180°$.

Use any of these four facts to prove that the three angles of a triangle sum to $180°$. For this proof, you will need to draw an auxiliary line, parallel to one of the triangle's sides and passing through the vertex opposite that side. Add any necessary labels and write out your proof.



Let's review the theorems we have now proven:

- Vertical angles are equal in measure.
- A transversal intersects a pair of lines. The pair of lines is parallel if and only if,
  - Alternate interior angles are equal in measure.
  - Corresponding angles are equal in measure.
- Interior angles on the same side of the transversal add to $180°$**.** The sum of the degree measures of the angles of a triangle is $180°$.

Side Trip: Take a moment to take a look at one of those really famous Greek guys we hear so much about in geometry, Eratosthenes. Over $2,000$ years ago, Eratosthenes used the geometry we have just been working with to find the diameter of Earth. He did not have cell towers, satellites, or any other advanced instruments available to scientists today. The only things Eratosthenes used were his eyes, his feet, and perhaps the ancient Greek equivalent to a protractor.

Watch this video to see how he did it, and try to spot the geometry we have been using throughout this lesson.

Eratosthenes solves a puzzle

Example 1

Construct a proof designed to demonstrate the following:

If two lines are perpendicular to the same line, they are parallel to each other.

(a) Draw and label a diagram, (b) state the given facts and the conjecture to be proved, and (c) write out a clear statement of your reasoning to justify each step.

Discussion

Each of the three parallel line theorems has a **converse** (or reversing) theorem as follows:

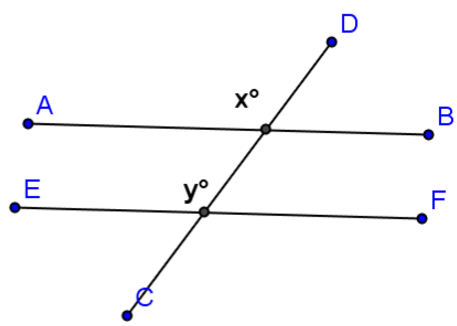
Original	Converse
If two parallel lines are cut by a transversal, then alternate interior angles are congruent.	If two lines are cut by a transversal such that alternate interior angles are congruent, then the lines are parallel.
If two parallel lines are cut by a transversal, then corresponding angles are congruent.	If two lines are cut by a transversal such that corresponding angles are congruent, then the lines are parallel.
If two parallel lines are cut by a transversal, then interior angles on the same side of the transversal add to $180°$.	If two lines are cut by a transversal such that interior angles on the same side of the transversal add to $180°$, then the lines are parallel.

Notice the similarities between the statements in the first column and those in the second. Think about when you would need to use the statements in the second column, i.e., the times when you are trying to prove two lines are parallel.

Example 2

In the figure at the right, $x = y$.

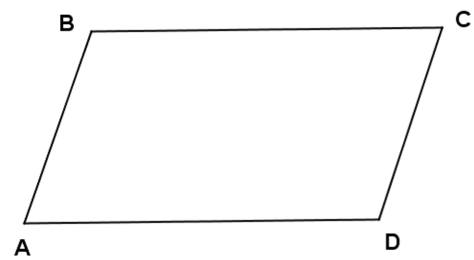
Prove that $\overline{AB} \parallel \overline{EF}$



Problem Set

1. Given: $\angle C$ and $\angle D$ are supplementary and $m\angle B = m\angle D$.

   Prove: $\overline{AB} \parallel \overline{CD}$



2. A theorem states that *in a plane, if a line is perpendicular to one of two parallel lines and intersects the other, then it is perpendicular to the other of the two parallel lines.*

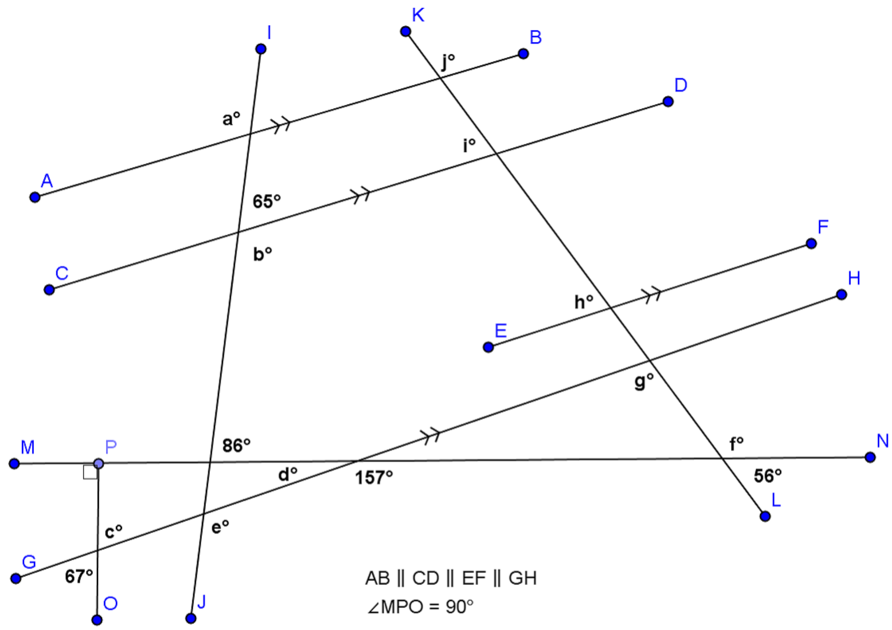
   Prove this theorem. (a) Construct and label an appropriate figure, (b) state the given information and the theorem to be proven, and (c) list the necessary steps to demonstrate the proof.

Lesson 12: Transformations—The Next Level

Classwork

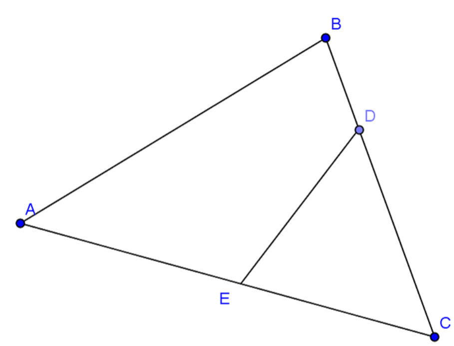
Opening Exercises 1–2

1. Find the measure of each lettered angle in the figure below.



a =	b =	c =	d =	e =
f =	g =	h =	i =	j =

2. Given: $\text{m}\angle CDE = \text{m}\angle BAC$
   Prove: $\text{m}\angle DEC = \text{m}\angle ABC$



Mathematical Modeling Exercise

You will work with a partner on this exercise and are allowed a protractor, compass, and straightedge.

- **<u>Partner A</u>**: Use the card your teacher gives you. Without showing the card to your partner, describe to your partner how to draw the transformation indicated on the card. When you have finished, compare your partner's drawing with the transformed image on your card. Did you describe the motion correctly?
- **<u>Partner B</u>**: Your partner is going to describe a transformation to be performed on the figure on your card. Follow your partner's instructions and then compare the image of your transformation to the image on your partner's card.

Discussion

Explaining how to transform figures without the benefit of a coordinate plane can be difficult without some important vocabulary. Let's review.

The word *transformation* has a specific meaning in geometry. A transformation F of the plane is a function that assigns to each point P of the plane a unique point $F(P)$ in the plane. Transformations that preserve lengths of segments and measures of angles are called ____________________. A *dilation* is an example of a transformation that preserves ____________________ measures but not the lengths of segments. In this lesson, we will work only with rigid transformations. We call a figure that is about to undergo a transformation the ____________________while the figure that has undergone the transformation is called the ____________________.

Rotation

Reflection

Translation

Using the figures above, identify specific information needed to perform the rigid motion shown.

For a rotation, we need to know:

For a reflection, we need to know:

For a translation, we need to know:

Geometry Assumptions

We have now done some work with all three basic types of rigid motions (rotations, reflections, and translations). At this point, we need to state our assumptions as to the properties of basic rigid motions:

a. *Any basic rigid motion preserves lines, rays, and segments. That is, for a basic rigid motion of the plane, the image of a line is a line, the image of a ray is a ray, and the image of a segment is a segment.*

b. *Any basic rigid motion preserves lengths of segments and angle measures of angles.*

Relevant Vocabulary

Basic Rigid Motion: A *basic rigid motion* is a rotation, reflection, or translation of the plane.

Basic rigid motions are examples of transformations. Given a transformation, the image of a point $\boldsymbol{A}$ is the point the transformation maps A to in the plane.

Distance-Preserving: A transformation is said to be *distance-preserving* if the distance between the images of two points is always equal to the distance between the pre-images of the two points.

Angle-Preserving: A transformation is said to be *angle-preserving* if (1) the image of any angle is again an angle and (2) for any given angle, the angle measure of the image of that angle is equal to the angle measure of the pre-image of that angle.

Problem Set

An example of a rotation applied to a figure and its image are provided. Use this representation to answer the questions that follow. For each question, a pair of figures (pre-image and image) are given as well as the center of rotation. For each question, identify and draw the following:

i. The circle that determines the rotation, using any point on the pre-image and its image.

ii. An angle, created with three points of your choice, which demonstrates the angle of rotation.

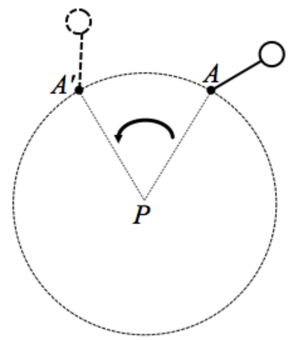
Example of a Rotation:

Pre-image: (solid line)

Image: (dotted line)

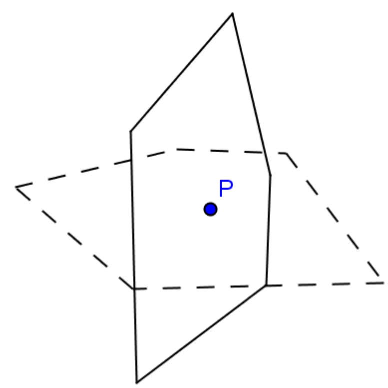
Center of rotation: P

Angle of rotation: $\angle APA'$



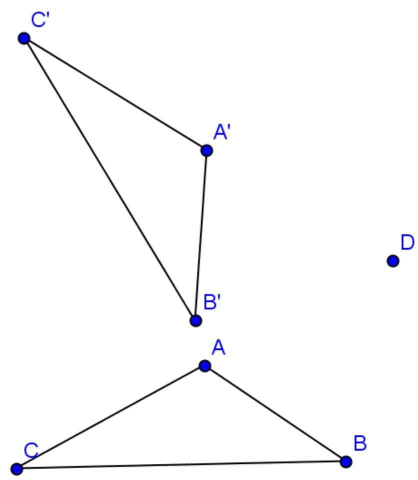
1. Pre-image: (solid line)
   Image: (dotted line)
   Center of rotation: P

   Angle of rotation: _______________



2. Pre-image: $\triangle ABC$
   Image: $\triangle A'B'C'$
   Center: D

   Angle of rotation: _______________



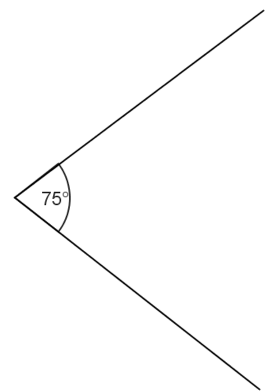
Lesson 13: Rotations

Classwork

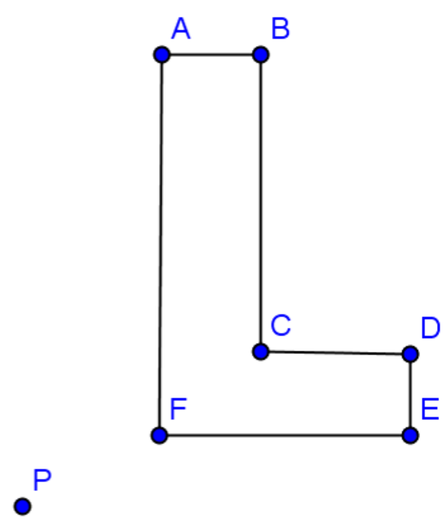
Opening Exercise

You will need a pair of scissors and a ruler.

Cut out the $75°$ angle at the right and use it as a guide to rotate the figure below $75°$ counterclockwise around the given center of rotation (Point P).



- Place the vertex of the $75°$ angle at point P.
- Line up one ray of the $75°$ angle with vertex A on the figure. Carefully measure the length from point P to vertex A.
- Measure that same distance along the other ray of the reference angle, and mark the location of your new point, A'.
- Repeat these steps for each vertex of the figure, labeling the new vertices as you find them.
- Connect the six segments that form the sides of your rotated image.



Discussion

First, we need to talk about the direction of the rotation. If you stand up and spin in place, you can either spin to your left or spin to your right. This spinning to your left or right can be rephrased using what we know about analog clocks: spinning to your left is spinning in a counterclockwise direction and spinning to your right is spinning in a clockwise direction. We need to have the same sort of notion for rotating figures in the plane. It turns out that there is a way to always choose a "counterclockwise half-plane" for any ray: The counterclockwise half-plane of a ray CP is the half-plane of $\overleftrightarrow{CP}$ that lies to the left as you move along $\overrightarrow{CP}$ in the direction from C to P. (The "clockwise half-plane" is then the half-plane that lies to the right as you move along $\overrightarrow{CP}$ in the direction from C to P.) We use this idea to state the definition of rotation.

For $0° < \theta < 180°$, the rotation of θ degrees around the center C is the transformation $R_{C,\theta}$ of the plane defined as follows:

1. For the center point C, $R_{C,\theta}(C) = C$, and
2. For any other point P, $R_{C,\theta}(P)$ is the point Q that lies in the counterclockwise half-plane of $\overrightarrow{CP}$, such that $CQ = CP$ and $\mathrm{m}\angle PCQ = \theta°$.

A rotation of 0 degrees around the center C is the identity transformation, i.e., for all points A in the plane, it is the rotation defined by the equation $R_{C,0}(A) = A$.

A rotation of $180°$ around the center C is the composition of two rotations of $90°$ around the center C. It is also the transformation that maps every point P (other than C) to the other endpoint of the diameter of circle with center C and radius CP.

Let's examine that definition more closely.

- A rotation leaves the center point C fixed. $R_{C,\theta}(C) = C$ states exactly that. The rotation function R with center point C that moves everything else in the plane $\theta°$, leaves only the center point itself unmoved.
- For every other point P, every point in the plane moves the exact same degree arc along the circle defined by the center of rotation and the angle $\theta°$.
- Found by turning in a counterclockwise direction along the circle from P to Q, such that $\mathrm{m}\angle QPC = \theta°$—all positive angle measures θ assume a counterclockwise motion; if citing a clockwise rotation, the answer should be labeled with "CW".
- $R_{C,\theta}(P)$ is the point Q that lies in the counterclockwise half-plane of ray $\overrightarrow{CP}$ such that $CQ = CP$. Visually, you can imagine rotating the point P in a counterclockwise arc around a circle with center C and radius CP to find the point Q.
- $\mathrm{m}\angle PCQ = \theta°$ — the point Q is the point on the circle with center C and radius CP such that the angle formed by the rays $\overrightarrow{CP}$ and $\overrightarrow{CQ}$ has an angle measure $\theta°$.

A composition of two rotations applied to a point is the image obtained by applying the second rotation to the image of the first rotation of the point. In mathematical notation, the image of a point A after "a composition of two rotations of $90°$ around the center C" can be described by the point $R_{C,90}\left(R_{C,90}(A)\right)$. The notation reads, "Apply $R_{C,90}$ to the point $R_{C,90}(A)$." So, we lose nothing by defining $R_{C,180}(A)$ to be that image. Then, $R_{C,180}(A) = R_{C,90}\left(R_{C,90}(A)\right)$ for all points A in the plane.

In fact, we can generalize this idea to define a rotation by any positive degree: For $\theta° > 180°$, a *rotation of* $\theta°$ *around the center* C is any composition of three or more rotations, such that each rotation is less than or equal to a $90°$ rotation and whose angle measures sum to $\theta°$. For example, a rotation of $240°$ is equal to the composition of three rotations by $80°$ about the same center, the composition of five rotations by $50°$, $50°$, $50°$, $50°$, and $40°$ about the same center, or the composition of 240 rotations by $1°$ about the same center.

Notice that we have been assuming that all rotations rotate in the counterclockwise direction. However, the inverse rotation (the rotation that "undoes" a given rotation) can be thought of as rotating in the clockwise direction. For example, rotate a point A by $30°$ around another point C to get the image $R_{C,30}(A)$. We can "undo" that rotation by rotating by $30°$ in the clockwise direction around the same center C. Fortunately, we have an easy way to describe a "rotation in the clockwise direction." If all positive degree rotations are in the counterclockwise direction, then we can define a negative degree rotation as a rotation in the clockwise direction (using the clockwise half-plane instead of the counterclockwise half-plane). Thus, $R_{C,-30}$ is a $30°$ rotation in the clockwise direction around the center C. Since a composition of two rotations around the same center is just the sum of the degrees of each rotation, we see that

$$R_{C,-30}\left(R_{C,30}(A)\right) = R_{C,0}(A) = A,$$

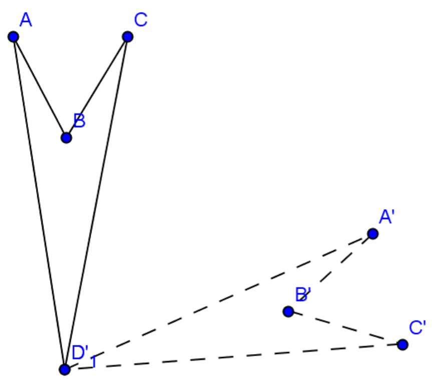
for all points A in the plane. Thus, we have defined how to perform a rotation for by any number of degrees—positive or negative.

As this is our first foray into close work with rigid motions, we emphasize an important fact about rotations. Rotations are one kind of rigid motion or transformation of the plane (a function that assigns to each point P of the plane a unique point $F(P)$) that preserves lengths of segments and measures of angles. Recall that Grade 8 investigations involved manipulatives that modeled rigid motions (e.g., transparencies) because you could actually *see* that a figure was not altered, as far as length or angle was concerned. It is important to hold onto this idea while studying all of the rigid motions.

Constructing rotations precisely can be challenging. Fortunately, computer software is readily available to help you create transformations easily. Geometry software (such as Geogebra) allows you to create plane figures and rotate them a given number of degrees around a specified center of rotation. The figures below were rotated using Geogebra. Determine the angle and direction of rotation that carries each pre-image onto its (dashed-line) image. Assume both angles of rotation are positive. The center of rotation for the Exercise 1 is point D and for Figure 2 is point E.

Exercises 1–3

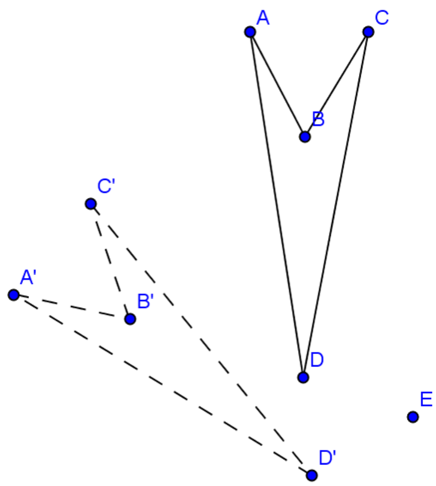
1.



To determine the angle of rotation, you measure the angle formed by connecting corresponding vertices to the center point of rotation. In Exercise 1, measure $\angle AD'A'$. What happened to $\angle D$? Can you see that D is the center of rotation, therefore, mapping D' onto itself? Before leaving Exercise 1, try drawing $\angle BD'B'$. Do you get the same angle measure? What about $\angle CD'C'$?

Try finding the angle and direction of rotation for Exercise 2 on your own.

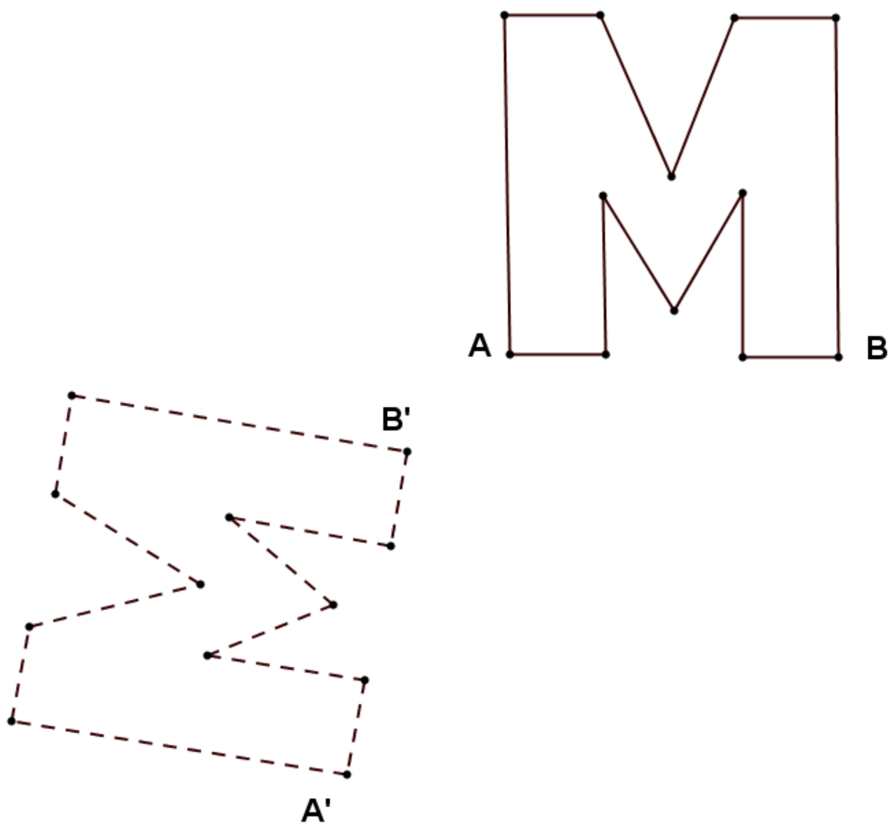
2.



Did you draw $\angle DED'$ or $\angle CEC'$?

Now that you can find the angle of rotation, let's move on to finding the center of rotation. Follow the directions below to locate the center of rotation taking the figure at the top right to its image at the bottom left.

3.



a. Draw a segment connecting points A and A'.

b. Using a compass and straightedge, find the perpendicular bisector of this segment.

c. Draw a segment connecting points B and B'.

d. Find the perpendicular bisector of this segment.

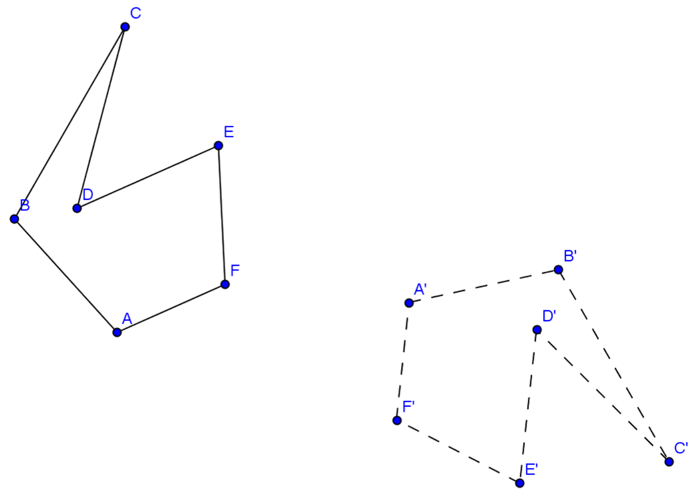
e. The point of intersection of the two perpendicular bisectors is the center of rotation. Label this point P.

Justify your construction by measuring angles $\angle APA'$ and $\angle BPB'$. Did you obtain the same measure?

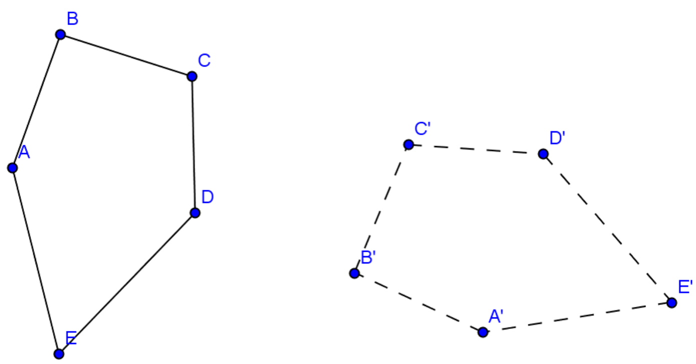
Exercises 4–5

Find the centers of rotation and angles of rotation for Exercises 4 and 5.

4.



5.



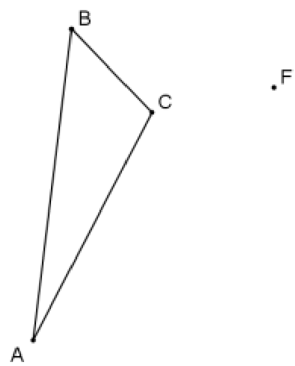
Lesson Summary

A rotation carries segments onto segments of equal length.

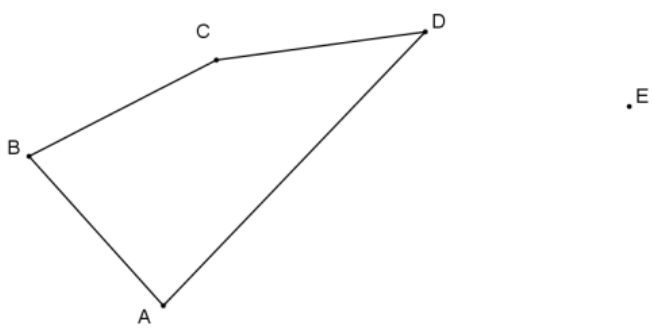
A rotation carries angles onto angles of equal measure.

Problem Set

1. Rotate the triangle ABC $60°$ around point F using a compass and straightedge only.



2. Rotate quadrilateral $ABCD$ $120°$ around point E using a straightedge and protractor.

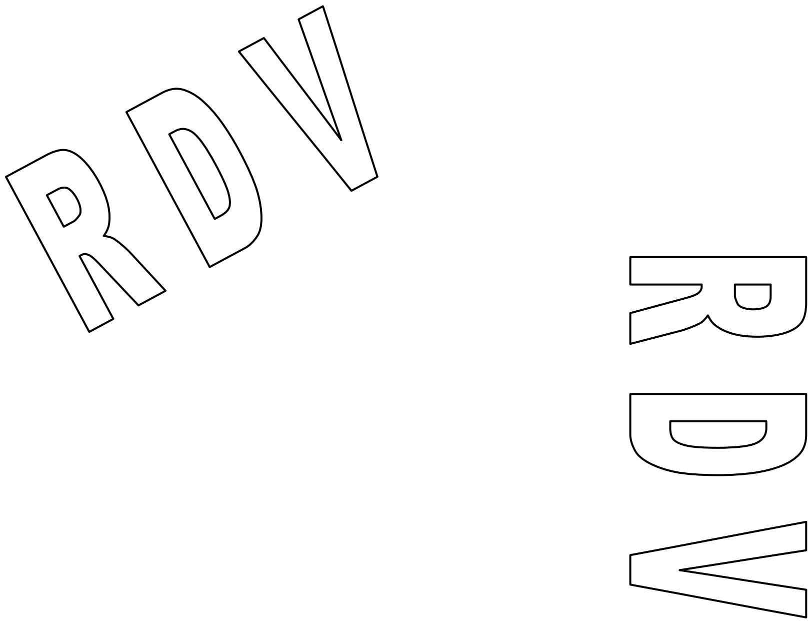


3. On your paper, construct a $45°$ angle using a compass and straightedge. Rotate the angle $180°$ around its vertex, again using only a compass and straightedge. What figure have you formed, and what are its angles called?

4. Draw a triangle with angles $90°$, $60°$, and $30°$ using only a compass and straightedge. Locate the midpoint of the longest side using your compass. Rotate the triangle $180°$ around the midpoint of the longest side. What figure have you formed?

5. On your paper, construct an equilateral triangle. Locate the midpoint of one side using your compass. Rotate the triangle $180°$ around this midpoint. What figure have you formed?

6. Use either your own initials (typed using WordArt on a word processor) or the initials provided below. If you create your own WordArt initials, copy, paste, and rotate to create a design similar to the one below. Find the center of rotation and the angle of rotation for your rotation design.



Lesson 14: Reflections

Classwork

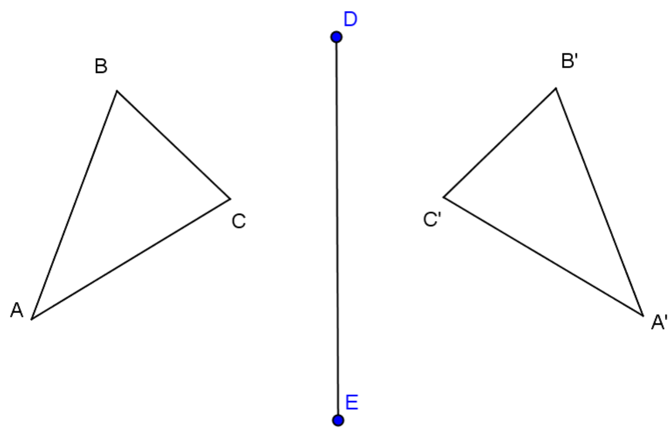
Exploratory Challenge

Think back to Lesson 12 where you were asked to describe to your partner how to reflect a figure across a line. The greatest challenge in providing the description was using the precise vocabulary necessary for accurate results. Let's explore the language that will yield the results we are looking for.

$\triangle ABC$ is reflected across $\overline{DE}$ and maps onto $\triangle A'B'C'$.

Use your compass and straightedge to construct the perpendicular bisector of each of the segments connecting A to A', B to B', and C to C'. What do you notice about these perpendicular bisectors?

Label the point at which $\overline{AA'}$ intersects $\overline{DE}$ as point O. What is true about AO and $A'O$? How do you know this is true?



Discussion

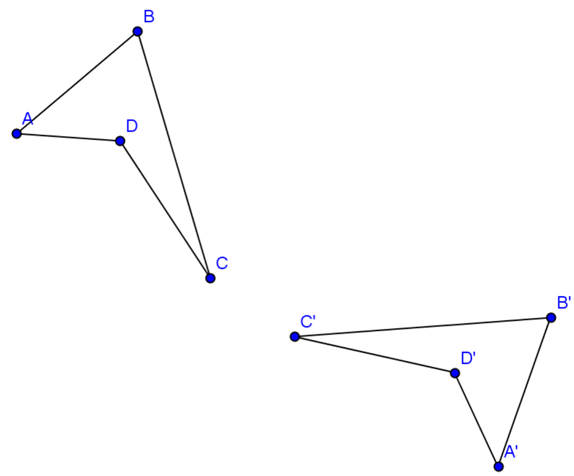
You just demonstrated that the line of reflection between a figure and its reflected image is also the perpendicular bisector of the segments connecting corresponding points on the figures.

In the Exploratory Challenge, you were given both the pre-image, image, and the line of reflection. For your next challenge, try finding the line of reflection provided a pre-image and image.

Example 1

Construct the segment that represents the line of reflection for quadrilateral $ABCD$ and its image $A'B'C'D'$.

What is true about each point on $ABCD$ and its corresponding point on $A'B'C'D'$?



Notice one very important fact about reflections. Every point in the original figure is carried to a corresponding point on the image by the same rule—a reflection across a specific line. This brings us to a critical definition:

Reflection: For a line l in the plane, a *reflection across* l is the transformation $r_l$of the plane defined as follows:

1. For any point P on the line l, $r_l(P) = P$, and
2. For any point P not on l, $r_l(P)$ is the point Q so that l is the perpendicular bisector of the segment PQ.

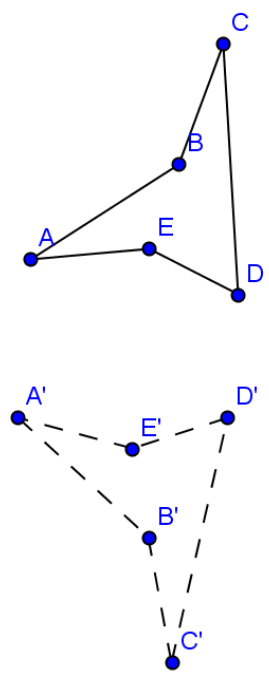
If the line is specified using two points, as in $\overleftrightarrow{AB}$, then the reflection is often denoted by $r_{\overline{AB}}$. Just as we did in the last lesson, let's examine this definition more closely:

- *A transformation of the plane*—the entire plane is transformed; what was once on one side of the line of reflection is now on the opposite side;
- $r_l(P) = P$ *means that the points on linel are left fixed*— the only part of the entire plane that is left fixed is the line of reflection itself;
- $r_l(P)$ is the point Q— the transformation r_l maps the point P to the point Q;
- So that l is the perpendicular bisector of the segment PQ— to find Q, first construct the perpendicular line m to the line l that passes through the point P. Label the intersection of l andm as N. Then locate the point Q on m on the other side of l such that $PN = NQ$.

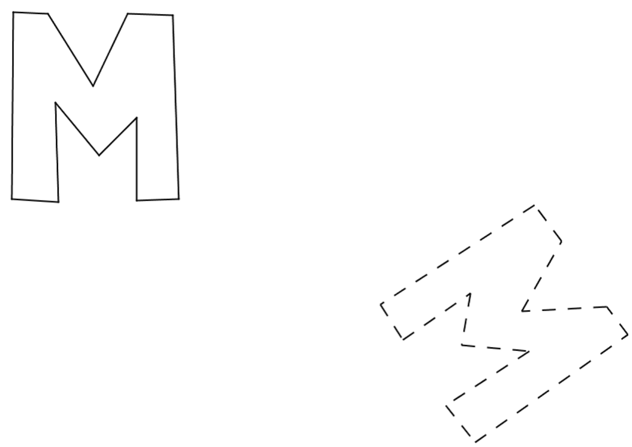
Examples 2–3

Construct the line of reflection across which each image below was reflected.

2.



3.

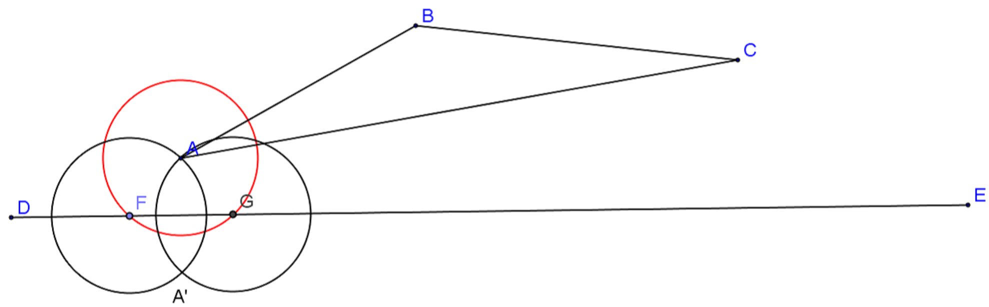


You have shown that a line of reflection is the perpendicular bisector of segments connecting corresponding points on a figure and its reflected image. You have also constructed a line of reflection between a figure and its reflected image. Now we need to explore methods for constructing the reflected image itself. The first few steps are provided for you in this next stage.

Example 4

The task at hand is to construct the reflection of $\triangle ABC$ over line DE. Follow the steps below to get started, then complete the construction on your own.

1. Construct circle A: center A, with radius such that the circle crosses $\overline{DE}$ at two points (labeled F and G).
2. Construct circle F: center F, radius FA and circle G: center G, radius GA. Label the [unlabeled] point of intersection between circles F and G as point A'. This is the reflection of vertex A across $\overline{DE}$.
3. Repeat steps 1 and 2 for vertices B and C to locate B' and C'.
4. Connect A', B', and C' to construct the reflected triangle.

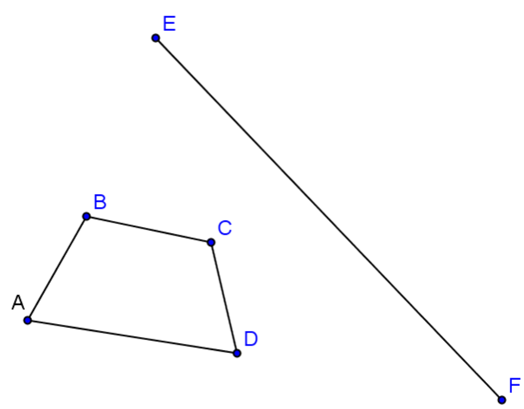


Things to consider:

When you found the line of reflection earlier, you did this by constructing perpendicular bisectors of segments joining two corresponding vertices. How does the reflection you constructed above relate to your earlier efforts at finding the line of reflection itself? Why did the construction above work?

Example 5

Now try a slightly more complex figure. Reflect $ABCD$ across line EF.



Lesson Summary

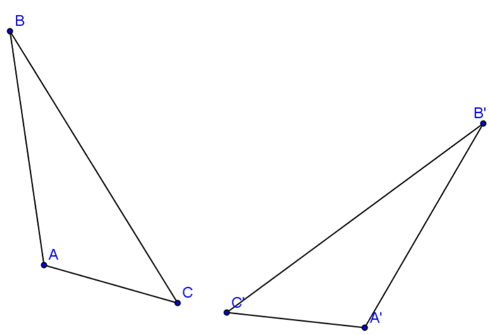
A reflection carries segments onto segments of equal length.

A reflection carries angles onto angles of equal measure.

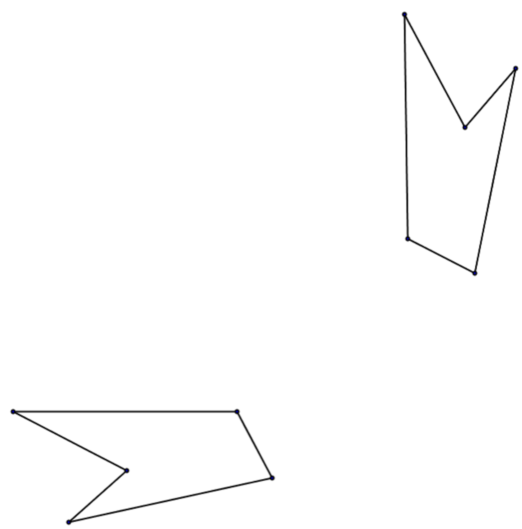
Problem Set

Construct the line of reflection for each pair of figures below.

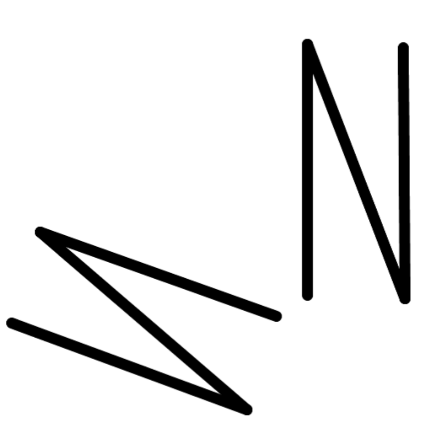
1.



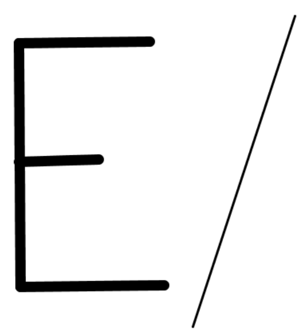
2.



3.



4. Reflect the given image across the line of reflection provided.



5. Draw a triangle ABC. Draw a line l through vertex C so that it intersects the triangle at more than just the vertex. Construct the reflection across l.

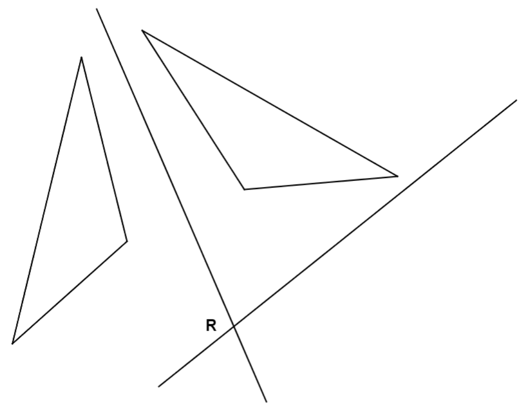
Lesson 15: Rotations, Reflections, and Symmetry

Classwork

Opening Exercise

The original triangle, labeled with "A," has been reflected across the first line, resulting in the image labeled with "B." Reflect the image across the second line.

Carlos looked at the image of the reflection across the second line and said, "That's not the image of triangle "A" after two reflections, that's the image of triangle "A" after a rotation!" Do you agree? Why or why not?

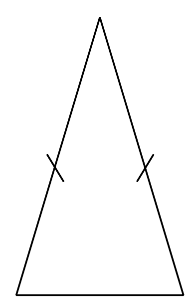


Discussion

When you reflect a figure across a line, the original figure and its image share a line of symmetry, which we have called the line of reflection. *When you reflect a figure across a line, then reflect the image across a line that intersects the first line, your final image is a rotation of the original figure.* The center of rotation is the point at which the two lines of reflection intersect. The angle of rotation is determined by connecting the center of rotation to a pair of corresponding vertices on the original figure and the final image. The figure above is a $210°$ rotation (or $150°$ clockwise rotation).

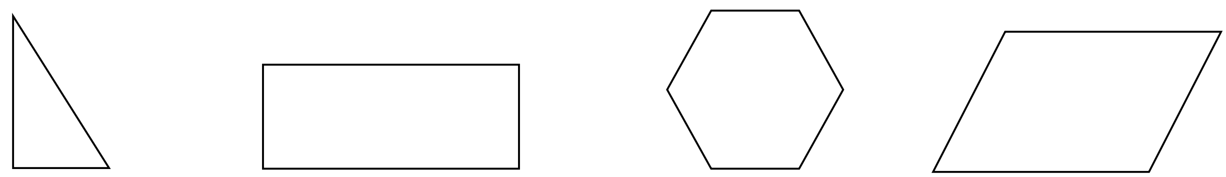
Exploratory Challenge

<u>**Line of Symmetry of a Figure**</u>: This is an isosceles triangle. By definition, an isosceles triangle has at least two congruent sides. A line of symmetry of the triangle can be drawn from the top vertex to the midpoint of the base, decomposing the original triangle into two congruent right triangles. This line of symmetry can be thought of as a reflection across itself that takes the isosceles triangle to itself. Every point of the triangle on one side of the line of symmetry has a corresponding point on the triangle on the other side of the line of symmetry, given by reflecting the point across the line. In particular, the line of symmetry is equidistant from all corresponding pairs of points. Another way of thinking about line symmetry is that a figure has line symmetry if there exists a line (or lines) such that the image of the figure when reflected over the line is itself.



Does every figure have a line of symmetry?

Which of the following have multiple lines of symmetry?



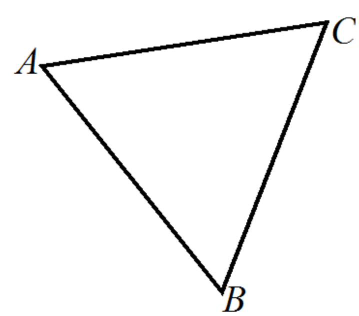
Use your compass and straightedge to draw one line of symmetry on each figure above that has at least one line of symmetry. Then, sketch any remaining lines of symmetry that exist. What did you do to justify that the lines you constructed were, in fact, lines of symmetry? How can you be certain that you have found all lines of symmetry?

Rotational Symmetry of a Figure: A nontrivial rotational symmetry of a figure is a rotation of the plane that maps the figure back to itself such that the rotation is greater than $0°$ but less than $360°$. Three of the four polygons above have a nontrivial rotational symmetry. Can you identify the polygon that does not have such symmetry?

When we studied rotations two lessons ago, we located both a center of rotation and an angle of rotation.

Identify the center of rotation in the equilateral triangle $\triangle ABC$ below and label it D. Follow the directions in the paragraph below to locate the center precisely.

To identify the center of rotation in the equilateral triangle, the simplest method is finding the perpendicular bisector of at least two of the sides. The intersection of these two bisectors gives us the center of rotation. Hence, the center of rotation of an equilateral triangle is also the circumcenter of the triangle. In Lesson 5 of this module, you also located another special point of concurrency in triangles—the incenter. What do you notice about the incenter and circumcenter in the equilateral triangle?



In any regular polygon, how do you determine the angle of rotation? Use the equilateral triangle above to determine the method for calculating the angle of rotation, and try it out on the rectangle, hexagon, and parallelogram above.

Identity Symmetry: A symmetry of a figure is a basic rigid motion that maps the figure back onto itself. There is a special transformation that trivially maps any figure in the plane back to itself called the *identity transformation*. This transformation, like the function f defined on the real number line by the equation $f(x) = x$, maps each point in the plane back to the same point (in the same way that f maps 3 to 3, π to π, and so forth). It may seem strange to discuss the "do nothing" identity symmetry (the symmetry of a figure under the identity transformation), but it is actually quite useful when listing all of the symmetries of a figure.

Let us look at an example to see why. The equilateral triangle $\triangle ABC$ above has two nontrivial rotations about its circumcenter D, a rotation by $120°$ and a rotation by $240°$. Notice that performing two $120°$ rotations back-to-back is the same as performing one $240°$ rotation. We can write these two back-to-back rotations explicitly, as follows:

- First, rotate the triangle by $120°$ about D: $R_{D,120°}(\triangle ABC)$.
- Next, rotate the image of the first rotation by $120°$: $R_{D,120°}\left(R_{D,120°}(\triangle ABC)\right)$.

Rotating $\triangle ABC$ by $120°$ twice in a row is the same as rotating $\triangle ABC$ once by $120° + 120° = 240°$. Hence, rotating by $120°$ twice is equivalent to one rotation by $240°$:

$$R_{D,120°}\left(R_{D,120°}(\triangle ABC)\right) = R_{D,240°}(\triangle ABC).$$

In later lessons, we will see that this can be written compactly as $R_{D,120°} \cdot R_{D,120°} = R_{D,240°}$. What if we rotated by $120°$ one more time? That is, what if we rotated $\triangle ABC$ by $120°$ three times in a row? That would be equivalent to rotating $\triangle ABC$ once by $120° + 120° + 120°$ or $360°$. But a rotation by $360°$ is equivalent to doing nothing, i.e., the identity transformation! If we use I to denote the identity transformation ($I(P) = P$ for every point P in the plane), we can write this equivalency as follows:

$$R_{D,120°}\left(R_{D,120°}\left(R_{D,120°}(\triangle ABC)\right)\right) = I(\triangle ABC).$$

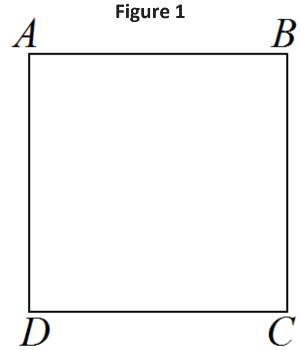
Continuing in this way, we see that rotating $\triangle ABC$ by $120°$ four times in a row is the same as rotating once by $120°$, rotating five times in a row is the same as $R_{D,240°}$, and so on. In fact, for a whole number n, rotating $\triangle ABC$ by $120°$ n times in a row is equivalent to performing one of the following three transformations:

$$\{R_{D,120°}, \quad R_{D,240°}, \quad I\}.$$

Hence, by including identity transformation I in our list of rotational symmetries, we can write any number of rotations of $\triangle ABC$ by $120°$ using only three transformations. For this reason, we include the identity transformation as a type of symmetry as well.

Exercises 1–3

Figure 1



Use Figure 1 to answer the questions below.

1. Draw all lines of symmetry. Locate the center of rotational symmetry.

2. Describe all symmetries explicitly.

   a. What kinds are there?

   b. How many are rotations? (Include a "$360°$ rotational symmetry," i.e., the identity symmetry.)

   c. How many are reflections?

3. Prove that you have found all possible symmetries.

   a. How many places can vertex A be moved to by some symmetry of the square that you have identified? (Note that the vertex to which you move A by some specific symmetry is known as the image of A under that symmetry. Did you remember the identity symmetry?)

   b. For a given symmetry, if you know the image of A, how many possibilities exist for the image of B?

   c. Verify that there is symmetry for all possible images of A and B.

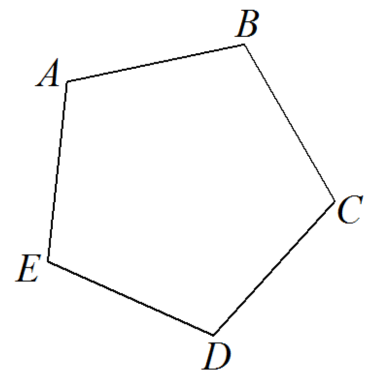
   d. Using part (b), count the number of possible images of A and B. This is the total number of symmetries of the square. Does your answer match up with the sum of the numbers from Exercise 2 parts (b) and (c)?

Relevant Vocabulary

Regular Polygon: A polygon is *regular* if all sides have equal length and all interior angles have equal measure.

Problem Set

Figure 1



Use Figure 1 to answer Problems 1–3.

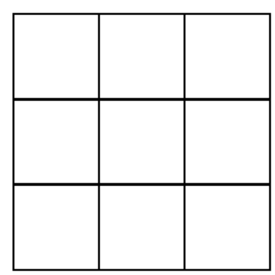
1. Draw all lines of symmetry. Locate the center of rotational symmetry.

2. Describe all symmetries explicitly.
   a. What kinds are there?
   b. How many are rotations (including the identity symmetry)?
   c. How many are reflections?

3. Now that you have found the symmetries of the pentagon, consider these questions:
   a. How many places can vertex A be moved to by some symmetry of the pentagon? (Note that the vertex to which you move A by some specific symmetry is known as the image of A under that symmetry. Did you remember the identity symmetry?)
   b. For a given symmetry, if you know the image of A, how many possibilities exist for the image of B?
   c. Verify that there is symmetry for all possible images of A and B.
   d. Using part (b), count the number of possible images of A and B. This is the total number of symmetries of the figure. Does your answer match up with the sum of the numbers from Problem parts (b) and (c)?

Use Figure 2 to answer Problem 4.

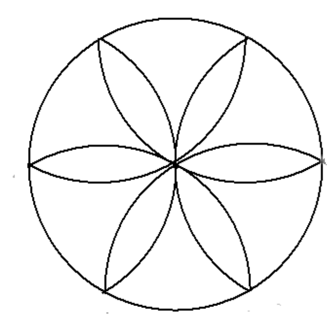
Figure 2



4. Shade exactly two of the nine smaller squares so that the resulting figure has
   a. Only one vertical and one horizontal line of symmetry.
   b. Only two lines of symmetry about the diagonals.
   c. Only one horizontal line of symmetry.
   d. Only one line of symmetry about a diagonal.
   e. No line of symmetry.

Use Figure 3 to answer Problem 5.

Figure 3



5. Describe all the symmetries explicitly.
   a. How many are rotations (including the identity symmetry)?
   b. How many are reflections?
   c. How could you shade the figure so that the resulting figure only has three possible rotational symmetries (including the identity symmetry)?

6. Decide whether each of the statements is true or false. Provide a counterexample if the answer is false.

   a. If a figure has exactly two lines of symmetry, it has exactly two rotational symmetries (including the identity symmetry).

   b. If a figure has at least three lines of symmetry, it has at least three rotational symmetries (including the identity symmetry).

   c. If a figure has exactly two rotational symmetries (including the identity symmetry), it has exactly two lines of symmetry.

   d. If a figure has at least three rotational symmetries (including the identity symmetry), it has at least three lines of symmetry.

Lesson 16: Translations

Classwork

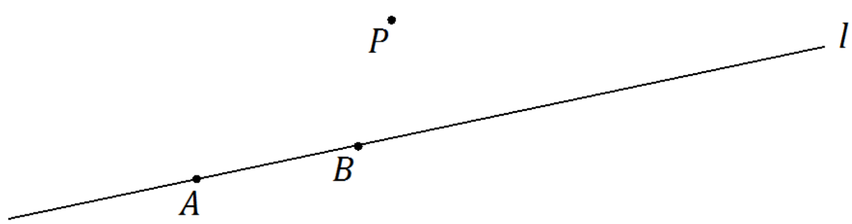
Exploratory Challenge

In Lesson 4, you completed a construction exercise that resulted in a pair of parallel lines (Problem 1 from the Problem Set). Now we examine an alternate construction.

Construct the line parallel to a given line AB through a given point P.

1. Draw circle P: Center P, radius AB.
2. Draw circle B: Center B, radius AP.
3. Label the intersection of circle P and circle B as Q.
4. Draw $\overleftrightarrow{PQ}$.

Note: Circles P and B intersect in two locations. Pick the intersection Q so that points A and Q are in opposite half-planes of line PB.



Discussion

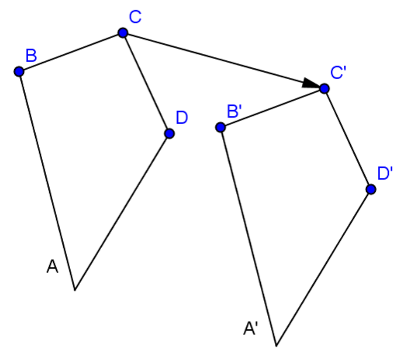
To perform a translation, we need to use the above construction. Let us investigate the definition of translation.

For vector $\overrightarrow{AB}$, the *translation along* $\overrightarrow{AB}$ is the transformation $T_{\overrightarrow{AB}}$ of the plane defined as follows:

1. For any point P on the line AB, $T_{\overrightarrow{AB}}(P)$ is the point Q on $\overleftrightarrow{AB}$ so that $\overrightarrow{PQ}$ has the same length and the same direction as $\overrightarrow{AB}$, and
2. For any point P not on $\overleftrightarrow{AB}$, $T_{\overrightarrow{AB}}(P)$ is the point Q obtained as follows. Let l be the line passing through P and parallel to $\overleftrightarrow{AB}$. Let m be the line passing through B and parallel to line AP. The point Q is the intersection of l and m.

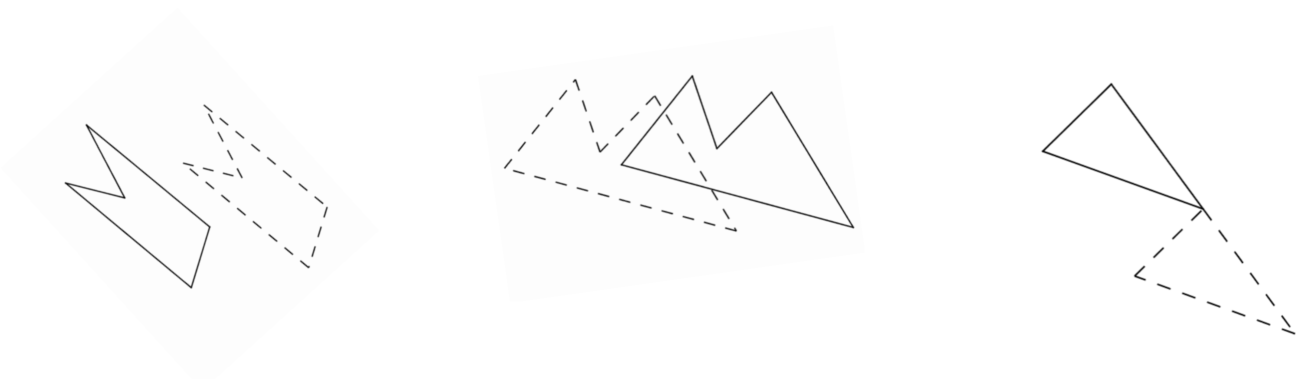
Note: The parallel line construction above shows a quick way to find the point Q in part 2 of the definition of translation!

In the figure to the right, quadrilateral $ABCD$ has been translated the length and direction of vector $\overrightarrow{CC'}$. Notice that the distance and direction from each vertex to its corresponding vertex on the image are identical to that of $\overrightarrow{CC'}$.



Example 1

Draw the vector that defines each translation below.

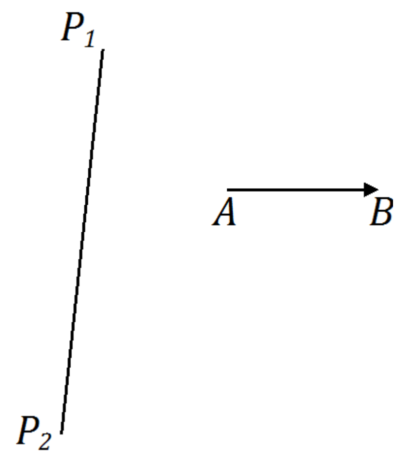


Finding the vector is relatively straightforward. Applying a vector to translate a figure is more challenging. To translate a figure, we must construct parallel lines to the vector through the vertices of the original figure and then find the points on those parallel lines that are the same direction and distance away as given by the vector.

Example 2

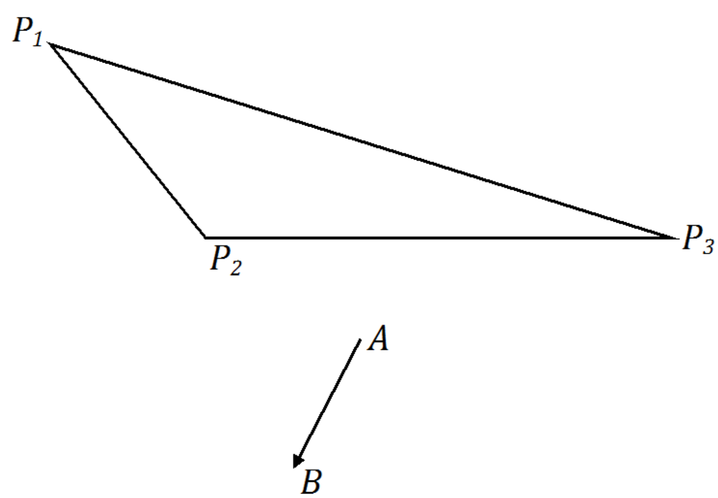
Use your compass and straightedge to apply $T_{\overrightarrow{AB}}$ to segment P_1P_2.

Note: Use the steps from the Exploratory Challenge *twice* for this question, creating two lines parallel to $\overrightarrow{AB}$: one through P_1 and one through P_2.



Example 3

Use your compass and straightedge to apply $T_{\overrightarrow{AB}}$ to $\triangle P_1P_2P_3$.



Relevant Vocabulary

Parallel: Two lines are *parallel* if they lie in the same plane and do not intersect. Two segments or rays are parallel if the lines containing them are parallel lines.

Lesson Summary

A translation carries segments onto segments of equal length.

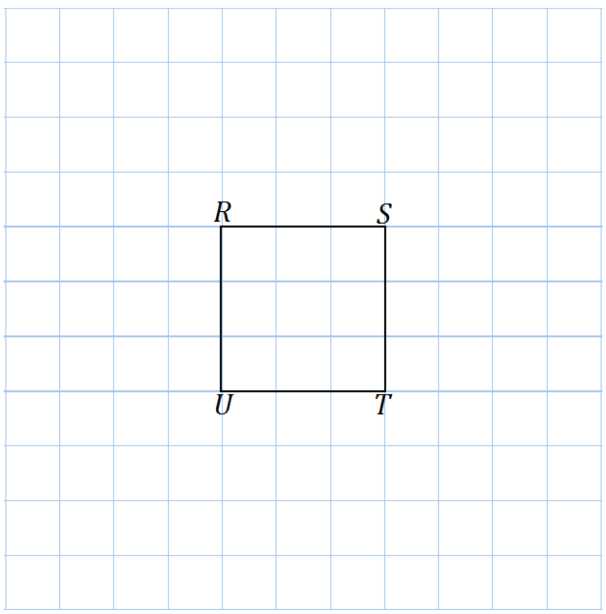
A translation carries angles onto angles of equal measure.

Problem Set

Translate each figure according to the instructions provided.

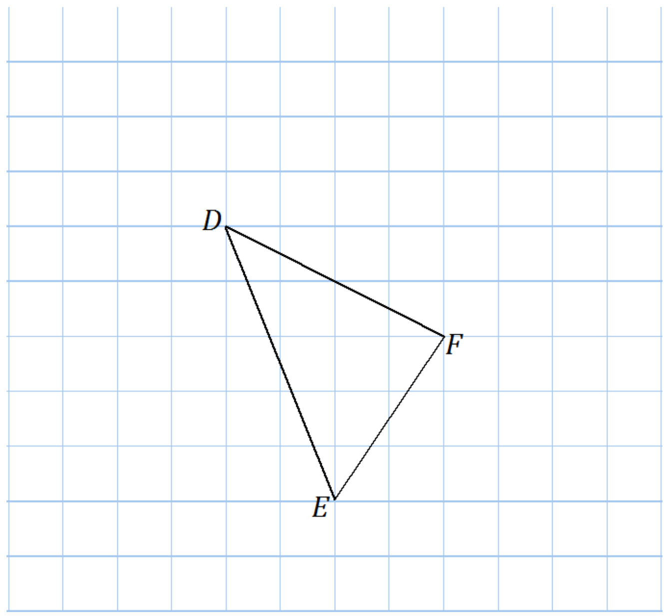
1. 2 units down and 3 units left.

   Draw the vector that defines the translation.

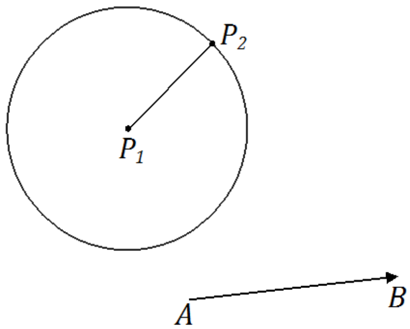


2. 1 unit up and 2 units right.

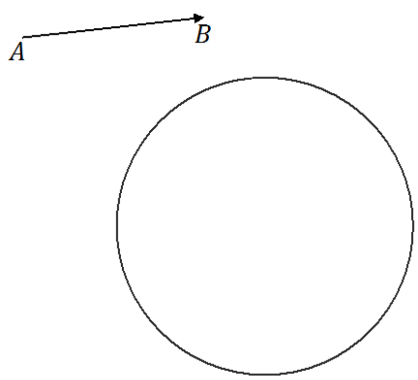
   Draw the vector that defines the translation.



3. Use your compass and straightedge to apply $T_{\overrightarrow{AB}}$ to the circle below (center P_1, radius $\overline{P_1P_2}$).

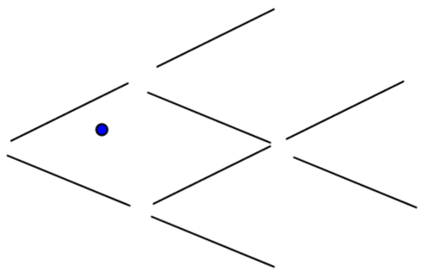


4. Use your compass and straightedge to apply $T_{\overrightarrow{AB}}$ to the circle below.
Hint: You will need to first find the center of the circle. You can use what you learned in Lesson 4 to do this.

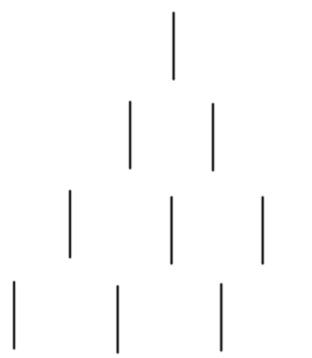


Two classic toothpick puzzles appear below. Solve each puzzle.

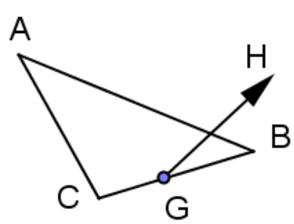
5. Each segment on the fish represents a toothpick. Move (translate) exactly three toothpicks and the eye to make the fish swim in the opposite direction. Show the translation vectors needed to move each of the three toothpicks and the eye.



6. Again, each segment represents a single toothpick. Move (translate) exactly three toothpicks to make the "triangle" point downward. Show the translation vectors needed to move each of the three toothpicks.



7. Apply $T_{\overrightarrow{GH}}$ to translate $\triangle ABC$.

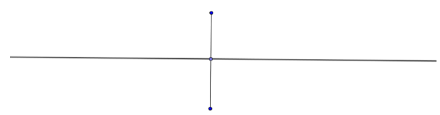


Lesson 17: Characterize Points on a Perpendicular Bisector

Classwork

Opening Exercise

In Lesson 3, you bisected angles, including straight angles. You related the bisection of straight angles in Lesson 3 to the construction of perpendicular bisectors in Lesson 4. Review the process of constructing a perpendicular bisector with the segment below. Then complete the definition of perpendicular lines below your construction.



Use the compass and straightedge construction from Lesson 4.

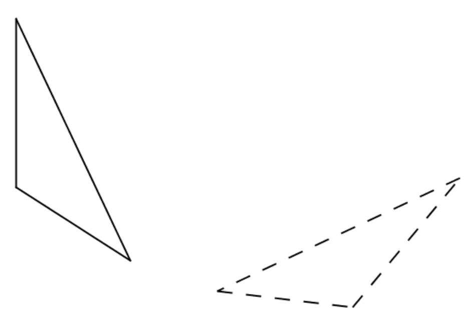
Two lines are perpendicular if they ______________ , and if any of the angles formed by the intersection of the lines is a ______________ angle. Two segments are perpendicular if the lines containing them are ______________.

Discussion

The line you constructed in the opening exercise is called the perpendicular bisector of the segment. As you learned in Lesson 14, the perpendicular bisector is also known as the line of reflection of the segment. With a line of reflection, any point on one side of the line (pre-image) is the same distance from the line as its image on the opposite side of the line

Example 1

Is it possible to find or construct a line of reflection that is NOT a perpendicular bisector of a segment connecting a point on the pre-image to its image? Try to locate a line of reflection between the two figures at the right without constructing any perpendicular bisectors.



Discussion

Why were your attempts impossible? Look back at the definition of *reflection* from Lesson 14.

For a line l in the plane, a *reflection across* l is the transformation r_l of the plane defined as follows:

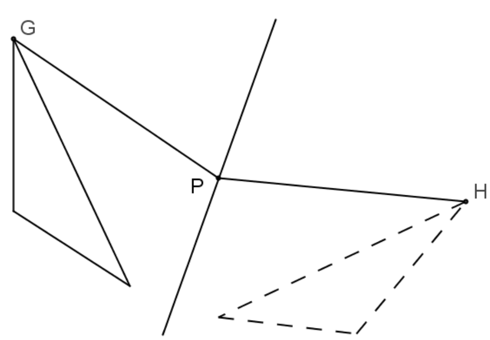
1. For any point P on the line l, $r_l(P) = P$, and
2. For any point P not on l, $r_l(P)$ is the point Q so that l is the perpendicular bisector of the segment PQ.

The key lies in the use of the term *perpendicular bisector*. For a point P not on L, explain how to construct the point Q so that L is the perpendicular bisector of the segment PQ.

Now, let's think about the problem from another perspective. We have determined that any point on the pre-image figure is the same distance from the line of reflection as its image. Therefore, the two points are equidistant from the point at which the line of reflection (*perpendicular bisector*) intersects the segment connecting the pre-image point to its image. What about other points on the perpendicular bisector? Are they also equidistant from the pre-image and image points? Let's investigate.

Example 2

Using the same figure from the previous investigation, but with the line of reflection, is it possible to conclude that any point on the perpendicular bisector is equidistant from any pair of pre-image and image points? For example, is $GP = HP$ in the figure? The point P is clearly NOT on the segment connecting the pre-image point G to its image H. How can you be certain that $GP = HP$? If r is the reflection, then $r(G) = H$ and $r(P) = P$. Since r preserves distances, $GP = HP$.

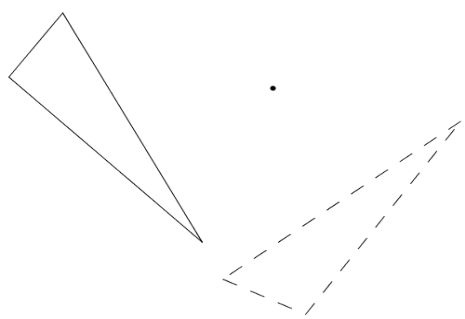


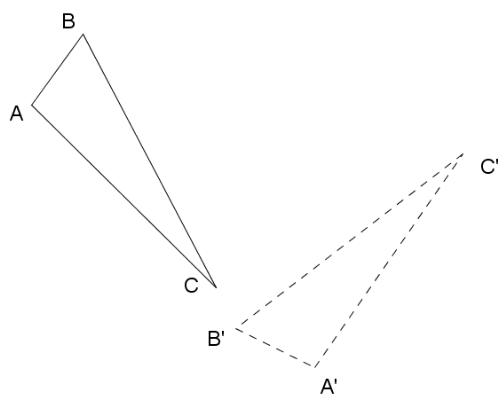
Discussion

We have explored perpendicular bisectors as they relate to reflections and have determined that they are essential to reflections. Are perpendicular lines, specifically, *perpendicular bisectors*, essential to the other two types of rigid motions: rotations and translations? Translations involve constructing parallel lines (which can certainly be done by constructing perpendiculars but are not essential to constructing parallels). However, perpendicular bisectors play an important role in rotations. In Lesson 13, we found that the intersection of the perpendicular bisectors of two segments connecting pairs of pre-image to image points determined the center of rotation.

Example 3

Find the center of rotation for the transformation below. How are perpendicular bisectors a major part of finding the center of rotation? Why are they essential?





As you explore this figure, also note another feature of rotations. As with all rigid motions, rotations preserve distance. A transformation is said to be *distance-preserving* (or *length-preserving*) if the distance between the images of two points is always equal to the distance between the original two points. Which of the statements below is true of the distances in the figure? Justify your response.

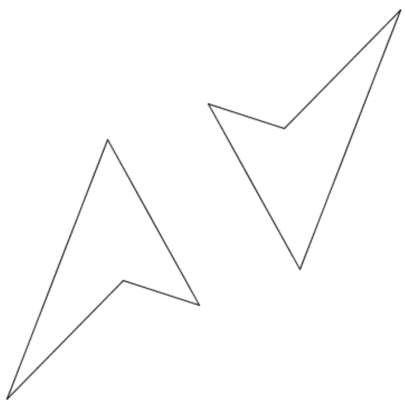
a. $AB = A'B'$

b. $AA' = BB'$

Exercises 1–5

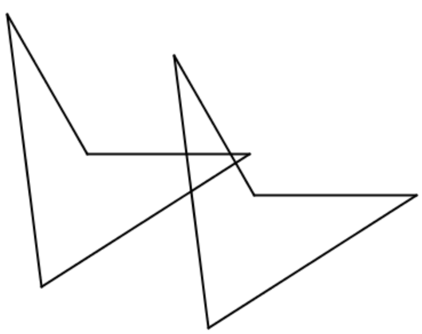
In each pre-image/image combination below (a) identify the type of transformation; (b) state whether perpendicular bisectors play a role in constructing the transformation and, if so, what role; and (c) cite an illustration of the distance-preserving characteristic of the transformation (e.g., identify two congruent segments from the pre-image to the image). For the last requirement, you will have to label vertices on the pre-image and image.

1.



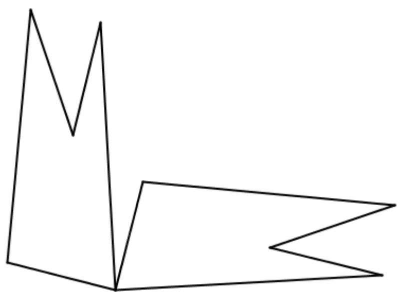
Transformation	Perpendicular bisectors?	Examples of distance preservation

2.



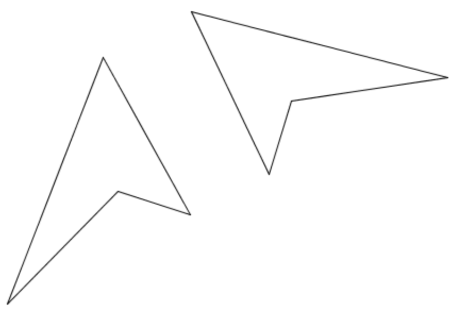
Transformation	Perpendicular bisectors?	Examples of distance preservation

3.



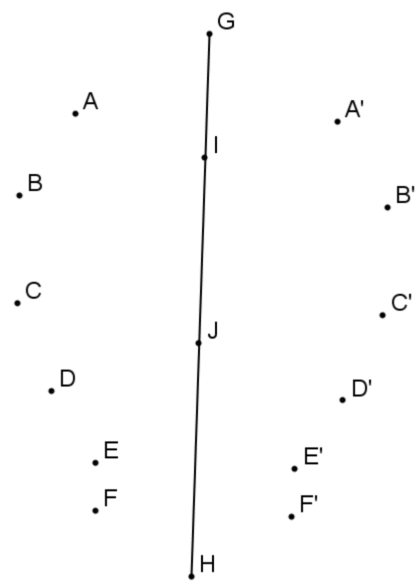
Transformation	Perpendicular bisectors?	Examples of distance preservation

4.



Transformation	Perpendicular bisectors?	Examples of distance preservation

5. In the figure to the right, GH is a line of reflection. State and justify two conclusions about distances in this figure. At least one of your statements should refer to perpendicular bisectors.



Problem Set

Create/construct two problems involving transformations—one reflection and one rotation—that require the use of perpendicular bisectors. Your reflection problem may require locating the line of reflection or using the line of reflection to construct the image. Your rotation problem should require location of the point of rotation. (Why should your rotation problem NOT require construction of the rotated image?) Create the problems on one page, and construct the solutions on another. Another student will be solving your problems in the next class period.

Lesson 18: Looking More Carefully at Parallel Lines

Classwork

Opening Exercise

Exchange Problem Sets with a classmate. Solve the problems posed by your classmate while he or she solves yours. Compare your solutions, and then discuss and resolve any discrepancies. Why were you asked only to locate the point of rotation, rather than to rotate a pre-image to obtain the image? How did you use perpendicular bisectors in constructing your solutions?

Discussion

We say that two lines are *parallel* if they lie in the same plane and do not intersect. Two segments or rays are parallel if the lines containing them are parallel.

Example 1

Why is the phrase *in the plane* critical to the definition of parallel lines? Explain and illustrate your reasoning.

In Lesson 7, we recalled some basic facts learned in earlier grades about pairs of lines and angles created by a transversal to those lines. One of those basic facts is the following:

> *Suppose a transversal intersects a pair of lines. The lines are parallel if and only if a pair of alternate interior angles are equal in measure.*

Our goal in this lesson is to prove this theorem using basic rigid motions, geometry assumptions, and a geometry assumption we will introduce in this lesson called the *parallel postulate.* Of all of the geometry assumptions we have given so far, the parallel postulate gets a special name because of the special role it played in the history of mathematics. (Euclid included a version of the parallel postulate in his books, and for 2,000 years people tried to show that it was not a necessary assumption. Not only did it turn out that the assumption was necessary for Euclidean geometry, but study of the parallel postulate lead to the creation of non-Euclidean geometries.)

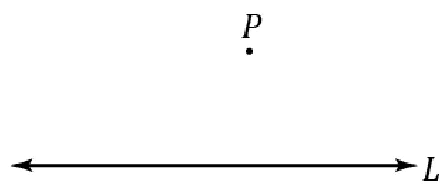
The basic fact above really has two parts, which we prove separately:

1. Suppose a transversal intersects a pair of lines. If two alternate interior angles are equal in measure, then the pair of lines are parallel.
2. Suppose a transversal intersects a pair of lines. If the lines are parallel, then the pair of alternate interior angles are equal in measure.

The second part turns out to be an equivalent form of the parallel postulate. To build up to the theorem, first we need to do a construction.

Example 2

Given a line l and a point P not on the line, follow the steps below to rotate l by 180° to a line l' that passes through P:



a. Label any point A on l.

b. Find the midpoint of segment AP using a ruler. (Measure the length of segment AP, and locate the point that is distance $\frac{AP}{2}$ from A between A and P.) Label the midpoint C.

c. Perform a 180° rotation around center C. To quickly find the image of $\boldsymbol{l}$ under this rotation by hand:
   i. Pick another point B on l.
   ii. Draw $\overleftrightarrow{CB}$.
   iii. Draw circle: center C, radius CB
   iv. Label the other point where the circle intersects $\overleftrightarrow{CB}$ by Q.
   v. Draw $\overleftrightarrow{PQ}$.

d. Label the image of the rotation by 180° of l by $l' = R_{C,180}(l)$.

How does your construction relate to the geometry assumption stated above to rotations? Complete the statement below to clarify your observations:

$R_{C,180}$ is a $180°$ ___________ around C. Rotations preserve __________ therefore $R_{C,180}$ maps the line l to the line ___________. What is $R_{C,180}(A)$? ___________.

Example 3

The lines l and l' in the picture above certainly look parallel, but we do not have to rely on "looks."

Claim: In the construction above, l is parallel to l'.

Proof: We will show that assuming they are not parallel leads to a contradiction. If they are not parallel, then they must intersect somewhere, call that point X. Since X is on l', it must be the image of some point S on l under the $R_{C,180}$ rotation, i.e., $R_{C,180}(S) = X$. Since $R_{C,180}$ is a 180˚ rotation, S and X must be the endpoints of a diameter of a circle that has center C. In particular, $\overleftrightarrow{SX}$ must contain C. Since S is a point on l, and X is a different point on l (it was the intersection of both lines), we have that $l = \overleftrightarrow{SX}$ because there is only one line through two points. But $\overleftrightarrow{SX}$ also contains C, which means that l contains C. However, C was constructed so that it was not on l. This is absurd.

There are only two possibilities for any two distinct lines l and l' in a plane: either the lines are parallel or they are not parallel. Since assuming the lines were not parallel lead to a false conclusion, the only possibility left is that l and l' were parallel to begin with.

Example 4

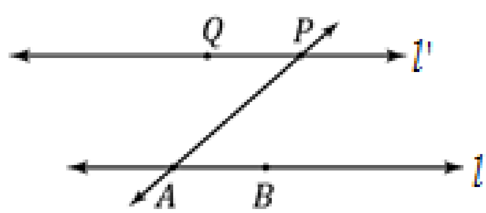
The construction and claim together implies the following theorem.

Theorem: Given a line l and a point P not on the line, then there exists line l' that contains P and is parallel to l.

This is a theorem we have justified before using compass and straightedge constructions, but now we see it follows directly from basic rigid motions and our geometry assumptions.

Example 5

We are now ready to prove the first part of the basic fact above. We have two lines, l and l', and all we know is that a transversal $\overleftrightarrow{AP}$ intersects l and l' such that a pair of alternate interior angles are equal in measure. (In the picture below we are assuming $\mathrm{m}\angle QPA = \mathrm{m}\angle BAP$.)



Let C be the midpoint of $\overline{AP}$. What happens if you rotate 180° around the center C? Is there enough information to show that $R_{C,180}(l) = l'$?

a. What is the image of the segment AP?

b. In particular, what is the image of the point A?

c. Why are the points Q and $R_{C,180}(B)$ on the same side of $\overleftrightarrow{AP}$?

d. What is the image of $R_{C,180}(\angle BAP)$? $\angle QPA$ Why?

e. Why is $R_{C,180}(l) = l'$?

We have just proved that a rotation by 180° takes l to l'. By the claim in Example 3, lines l and l' must be parallel, which is summarized below.

Theorem: Suppose a transversal intersects a pair of lines. If a pair of alternate interior angles are equal in measure, then the pair of lines are parallel.

Discussion

In Example 5, suppose we had used a different rotation to construct a line parallel to l that contains P. Such constructions are certainly plentiful. For example, for every other point D on l, we can find the midpoint of segment PD, and use the construction in Example 2 to construct a different 180˚ rotation around a different center such that the image of the line l is a parallel line through the point P. Are any of these parallel lines through P different? In other words,

Can we draw a line other than the line l' through P that never meets l?

The answer may surprise you; it stumped mathematicians and physicists for centuries. In nature, the answer is that it is sometimes possible and sometimes not. This is because there are places in the universe (near massive stars, for example) where the model geometry of space is not "plane-like" or flat, but is actually quite curved. To rule out these other types of "strange but beautiful" geometries, we must assume that the answer to the question above is only one line. That choice becomes one of our geometry assumptions:

(Parallel Postulate) Through a given external point there is at most one line parallel to a given line.

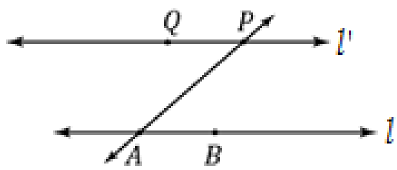
In other words, we assume that for any point P in the plane not lying on a line ℓ, every line in the plane that contains P intersects ℓ except at most one line—the one we call *parallel* to ℓ.

Example 6

We can use the parallel postulate to prove the second part of the basic fact.

Theorem: Suppose a transversal intersects a pair of lines. If the pair of lines are parallel, then the pair of alternate interior angles are equal in measure.

Proof: Suppose that a transversal $\overleftrightarrow{AP}$ intersects line l at A and l' at P; pick and label another point B on l and choose a point Q on l' on the opposite side of $\overleftrightarrow{AP}$ as B. The picture might look like the figure below:



Let C be the midpoint of segment $\overline{AP}$, and apply a rotation by 180˚ around the center C. As in previous discussions, the image of l is the line $R_{C,180}(l)$ which is parallel to l and contains point P. Since l' and $R_{C,180}(l)$ are both parallel to l and contain P, by the parallel postulate, they must be the same line: $R_{C,180}(l) = l'$. In particular, $R_{C,180}(\angle BAP) = \angle QPA$. Since rotations preserve angle measures, $\mathrm{m}\angle BAP = \mathrm{m}\angle QPA$, which was what we needed to show.

Discussion

It is important to point out that, although we only proved the alternate interior angles theorem, the same sort of proofs can be done in the exact same way to prove the corresponding angles theorem and the interior angles theorem. Thus, all of the proofs we have done so far (in class and in the Problem Sets) that use these facts are really based, in part, on our assumptions about rigid motions!

Example 7

We end this lesson with a theorem that we will just state, but can be easily proved using the parallel postulate.

Theorem: If three distinct lines l_1, l_2, and l_3 in the plane have the property that $l_1 \parallel l_2$ and $l_2 \parallel l_3$, then $l_1 \parallel l_3$. (In proofs, this can be written as, “If two lines are parallel to the same line, then they are parallel to each other.”)

Relevant Vocabulary

Parallel: Two lines are *parallel* if they lie in the same plane and do not intersect. Two segments or rays are parallel if the lines containing them are parallel lines.

Transversal: Given a pair of lines l and m in a plane, a third line t is a *transversal* if it intersects l at a single point and intersects m at a single but different point.

The definition of transversal rules out the possibility that any two of the lines l, m, and t are the same line.

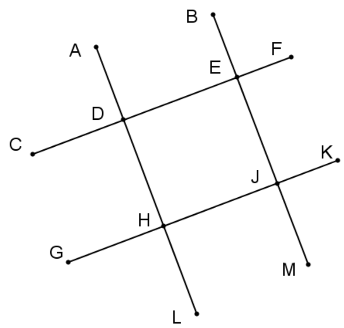
Alternate Interior Angles: Let line t be a transversal to lines l and m such that t intersects l at point P and intersects m at point Q. Let R be a point on l and S be a point on m such that the points R and S lie in opposite half-planes of t. Then the angle $\angle RPQ$ and the angle $\angle PQS$ are called *alternate interior angles* of the transversal t with respect to m and l.

Corresponding Angles: Let line t be a transversal to lines l and m. If $\angle x$ and $\angle y$ are alternate interior angles, and $\angle y$ and $\angle z$ are vertical angles, then $\angle x$ and $\angle z$ are *corresponding angles*.

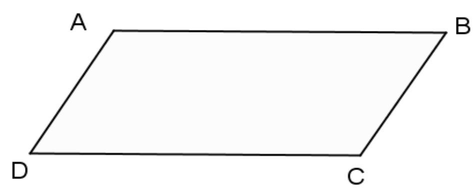
Problem Set

Notice that we are frequently asked two types of questions about parallel lines. If we are told that two lines are parallel, then we may be required to use this information to prove the congruence of two angles (corresponding, alternate interior, etc.). On the other hand, if we are given the fact that two angles are congruent (or perhaps supplementary), we may have to prove that two lines are parallel.

1. In the figure, $AL \parallel BM$, $AL \perp CF$, and $GK \perp BM$. Prove that $CF \parallel GK$.

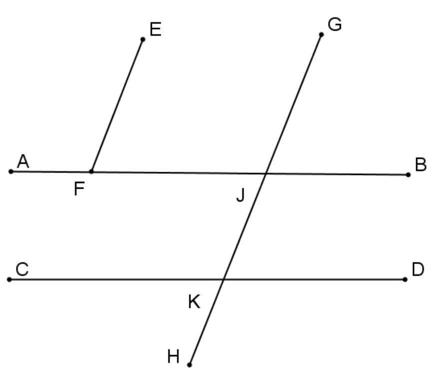


2. Given that $\angle B$ and $\angle C$ are supplementary and $\mathrm{m}\angle A = \mathrm{m}\angle \mathrm{C}$, prove that $AD \parallel BC$.

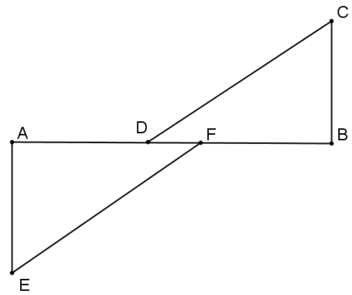


3. Mathematicians state that *if a transversal to two parallel lines is perpendicular to one of the lines, then it is perpendicular to the other*. Prove this statement. (Include a labeled drawing with your proof.)

4. In the figure, $AB \parallel CD$ and $EF \parallel GH$. Prove that $\angle AFE = \angle DKH$.



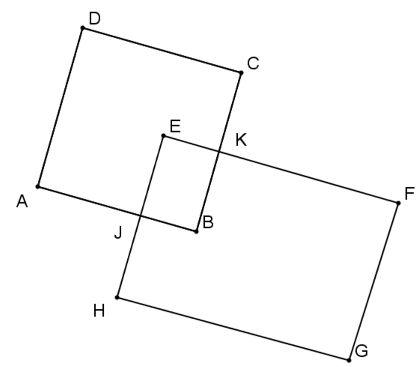
5. In the figure, $\angle E$ and $\angle AFE$ are complementary and $\angle C$ and $\angle BDC$ are complementary. Prove that $AE \parallel CB$.



6. Given a line l and a point P not on the line, the following directions can be used to draw a line m perpendicular to the line l through the point P based upon a rotation by 180°:
   a. Pick and label a point A on the line l so that the circle (center P, radius AP) intersects l twice.
   b. Use a protractor to draw a perpendicular line n through the point A (by constructing a 90° angle).
   c. Use the directions in Example 2 to construct a parallel line m through the point P.

   Do the construction. Why is the line m perpendicular to the line l in the figure you drew? Why is the line m the only perpendicular line to l through P?

Problems 7–10 all refer to the figure to the right. The exercises are otherwise unrelated to each other.



7. $\overline{AD} \parallel \overline{BC}$ and $\angle EJB$ is supplementary to $\angle JBK$. Prove that $\overline{AD} \parallel \overline{JE}$.

8. $\overline{AD} \parallel \overline{FG}$ and $\overline{EJ} \parallel \overline{FG}$. Prove that $\angle DAJ$ and $\angle EJA$ are supplementary.

9. $\mathrm{m}\angle C = \mathrm{m}\angle G$ and $\angle B$ is supplementary to $\angle G$. Prove that $\overline{DC} \parallel \overline{AB}$.

10. $\overline{AB} \parallel \overline{EF}$, $\overline{EF} \perp \overline{CB}$, and $\angle EKC$ is supplementary to $\angle KCD$. Prove that $\overline{AB} \parallel \overline{DC}$.

Lesson 19: Construct and Apply a Sequence of Rigid Motions

Classwork

Opening

We have been using the idea of congruence already (but in a casual and unsystematic way). In Grade 8, we introduced and experimented with concepts around congruence through *physical models, transparencies or geometry software.* Specifically, we had to.

> *(1) Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations. And (2) describe a sequence that exhibits the congruence between two congruent figures.* (**8.G.A.2**)

As with so many other concepts in high school Geometry, congruence is familiar, but we now study it with greater precision and focus on the language with which we discuss it.

Let us recall some facts related to congruence that appeared previously in this unit.

1. We observed that rotations, translations, and reflections—and thus all rigid motions—preserve the lengths of segments and the measures of angles. We think of two segments (respectively, angles) as the *same* in an important respect if they have the same length (respectively, degree measure), and thus, sameness of these objects relating to measure is well characterized by the existence of a rigid motion mapping one thing to another. Defining *congruence* by means of rigid motions extends this notion of sameness to arbitrary figures, while clarifying the meaning in an articulate way.
2. We noted that a symmetry is a rigid motion that carries a figure to itself.

So how do these facts about rigid motions and symmetry relate to congruence? We define two figures in the plane as congruent if there exists a finite composition of basic rigid motions that maps one figure onto the other.

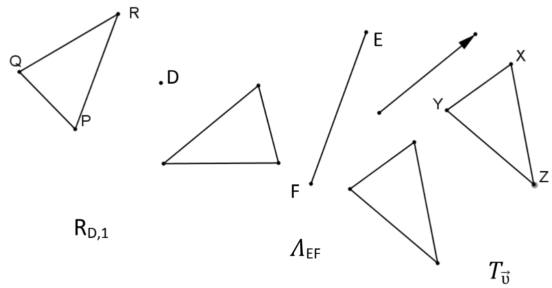
It might seem easy to equate two figures being congruent to having same size same shape. The phrase "same size and same shape" has intuitive meaning and helps to paint a mental picture, but is not a definition. As in a court of law, to establish guilt it is not enough to point out that the defendant looks like a sneaky, unsavory type. We need to point to exact pieces of evidence concerning the specific charges. It is also not enough that the defendant did something bad. It must be a violation of a specific law. Same size, same shape is on the level of, "He looks like a sneaky, bad guy who deserves to be in jail."

It is also not enough to say that they are alike in all respects except position in the plane. We are saying that there is some particular rigid motion that carries one to another. Almost always, when we use congruence in an explanation or proof, we need to refer to the rigid motion. To show that two figures are congruent, we only need to show that there is a transformation that maps one directly onto the other. However, once we know that there is a transformation, then we know that there are actually many such transformations and it can be useful to consider more than one. We see this when discussing the symmetries of a figure. A symmetry is nothing other than a congruence of an object with itself. A figure may have many different rigid motions that map it onto itself. For example, there are six different rigid motions that take one equilateral triangle with side length 1 to another such triangle. Whenever this occurs, it is because of a symmetry in the objects being compared.

Lastly, we discuss the relationship between *congruence* and *correspondence*. A correspondence between two figures is a function from the parts of one figure to the parts of the other, with no requirements concerning same measure or existence of rigid motions. If we have rigid motion T that takes one figure to another, then we have a correspondence between the parts. For example, if the first figure contains segment AB, then the second includes a corresponding segment $T(A)T(B)$. But we do not need to have a congruence to have a correspondence. We might list the parts of one figure and pair them with the parts of another. With two triangles, we might match vertex to vertex. Then the sides and angles in the first have corresponding parts in the second. But being able to set up a correspondence like this does not mean that there is a rigid motion that produces it. The sides of the first might be paired with sides of different length in the second. Correspondence in this sense is important in triangle similarity.

Discussion

We now examine a figure being mapped onto another through a composition of rigid motions.



To map $\triangle PQR$ to $\triangle XYZ$ here, we first rotate $\triangle PQR$ $120°$ ($R_{D,120°}$), around the point, D. Then reflect the image ($r_{\overline{EF}}$) across $\overleftrightarrow{EF}$. Finally, translate the second image ($T_{\vec{v}}$) along the given vector to obtain $\triangle XYZ$. Since each transformation is a rigid motion, $\triangle PQR \cong \triangle XYZ$. We use function notation to describe the composition of the rotation, reflection, and translation:

$$T_{\vec{v}}\left(r_{\overline{EF}}\left(RD, 120°\left(\triangle PQR\right)\right)\right) = \triangle XYZ$$

Notice that (as with all composite functions) the innermost function/transformation (the rotation) is performed first, and the outermost (the translation) last.

Example 1

i. Draw and label a triangle $\triangle PQR$ in the space below.

ii. Use your construction tools to apply one of each of the rigid motions we have studied to it in a sequence of your choice.

iii. Use function notation to describe your chosen composition here. Label the resulting image as $\triangle XYZ$:
____

iv. Complete the following sentences: (Some blanks are single words, others are phrases.)

Triangle $\triangle PQR$ is _______________ to $\triangle XYZ$ because ___________________________ map point P to point X, point Q to point Y, and point R to point Z. Rigid motions map segments onto ____and angles onto ________________________________.

Example 2

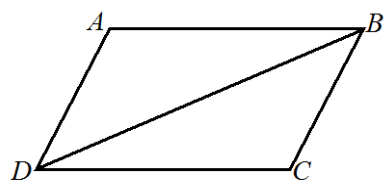
On a separate piece of paper, trace the series of figures in your composition but do NOT include the center of rotation, the line of reflection, or the vector of the applied translation.

Swap papers with a partner and determine the composition of transformations your partner used. Use function notation to show the composition of transformations that renders $\triangle PQR \cong \triangle XYZ$.

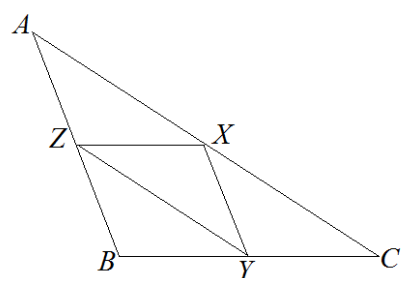
Problem Set

1. Use your understanding of congruence to explain why a triangle cannot be congruent to a quadrilateral.
   a. Why can't a triangle be congruent to a quadrilateral?
   b. Why can't an isosceles triangle be congruent to a triangle that is not isosceles?

2. Use the figures below to answer each question:
   a. $\triangle ABD \cong \triangle CDB$. What rigid motion(s) maps $\overline{CD}$ onto $\overline{AB}$? Find two possible solutions.



   b. All of the smaller sized triangles are congruent to each other. What rigid motion(s) map $\overline{ZB}$ onto $\overline{AZ}$? Find two possible solutions.



Lesson 20: Applications of Congruence in Terms of Rigid Motions

Classwork

Opening

Every congruence gives rise to a correspondence.

Under our definition of congruence, when we say that one figure is congruent to another, we mean that there is a rigid motion that maps the first onto the second. That rigid motion is called a congruence.

Recall the Grade 7 definition: *A correspondence between two triangles is a pairing of each vertex of one triangle with one and only one vertex of the other triangle.* When reasoning about figures, it is useful to be able to refer to corresponding parts (e.g., sides and angles) of the two figures. We look at one part of the first figure and compare it to the corresponding part of the other. Where does a correspondence come from? We might be told by someone how to make the vertices correspond. Conversely, we might make our own correspondence by matching the parts of one triangle with the parts of another triangle based on appearance. Finally, if we have a congruence between two figures, the congruence gives rise to a correspondence.

A rigid motion F always produces a one-to-one correspondence between the points in a figure (the *pre-image*) and points in its image. If P is a point in the figure, then the corresponding point in the image is $F(P)$. A rigid motion also maps each part of the figure to a corresponding part of the image. As a result, *corresponding parts of congruent figures are congruent* since the very same rigid motion that makes a congruence between the figures also makes a congruence between each part of the figure and the corresponding part of the image.

In proofs, we frequently refer to the fact that corresponding angles, sides, or parts of congruent triangles are congruent. This is simply a repetition of the definition of congruence. If $\triangle ABC$ is congruent to $\triangle DEG$ because there is a rigid motion F such that $F(A) = D$, $F(B) = E$, and $F(C\ ) = G$, then $\overline{AB}$ is congruent to $\overline{DE}$, $\triangle ABC$ is congruent to $\triangle DEG$, and so forth because the rigid motion F takes $\overline{AB}$ to $\overline{DE}$ and $\angle BAC$ to $\angle EDF$.

There are correspondences that do not come from congruences.

The sides (and angles) of two figures might be compared even when the figures are not congruent. For example, a carpenter might want to know if two windows in an old house are the same, so the screen for one could be interchanged with the screen for the other. He might list the parts of the first window and the analogous parts of the second, thus making a correspondence between the parts of the two windows. Checking part by part, he might find that the angles in the frame of one window are slightly different from the angles in the frame of the other, possibly because the house has tilted slightly as it aged. He has used a correspondence to help describe the differences between the windows, not to describe a congruence.

In general, given any two triangles, one could make a table with two columns and three rows, and then list the vertices of the first triangle in the first column and the vertices of the second triangle in the second column in a random way. This would create a correspondence between the triangles, though generally not a very useful one. No one would expect a random correspondence to be very useful, but it is a correspondence nonetheless.

Later, when we study similarity, we will find that it is very useful to be able to set up correspondences between triangles despite the fact that the triangles are not congruent. Correspondences help us to keep track of which part of one figure we are comparing to that of another. It makes the rules for associating part to part explicit and systematic so that other people can plainly see what parts go together.

Discussion

Let's review function notation for rigid motions.

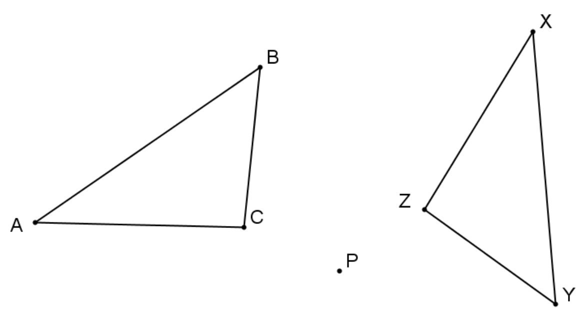
a. To name a translation, we use the symbol $T_{\overrightarrow{AB}}$. We use the letter T to signify that we are referring to a translation and the letters A and B to indicate the translation that moves each point in the direction from A to B along a line parallel to line AB by distance AB. The image of a point P is denoted $T_{\overrightarrow{AB}}(P)$. Specifically, $T_{\overrightarrow{AB}}(A) = B$.

b. To name a reflection, we use the symbol r_l, where l is the line of reflection. The image of a point P is denoted $r_l(P)$. In particular, if A is a point on l, $r_l(A) = A$. For any point P, line l is the perpendicular bisector of segment $Pr_l(P)$.

c. To name a rotation, we use the symbol R_{C,x° to remind us of the word rotation. C is the center point of the rotation, and x represents the degree of the rotation counterclockwise around the center point. Note that a positive degree measure refers to a counterclockwise rotation, while a negative degree measure refers to a clockwise rotation.

Example 1

In each figure below, the triangle on the left has been mapped to the one on the right by a $240°$ rotation about P. Identify all six pairs of corresponding parts (vertices and sides).



Corresponding vertices	Corresponding sides

What rigid motion mapped $\triangle ABC$ onto $\triangle XYZ$? Write the transformation in function notation.

Example 2

Given a triangle with vertices A, B, and C, list all the possible correspondences of the triangle with itself.

Example 3

Give an example of two quadrilaterals and a correspondence between their vertices such that (a) corresponding sides are congruent, but (b) corresponding angles are not congruent.

Problem Set

1. Given two triangles, one with vertices A, B, and C, and the other with vertices X, Y, and Z, there are six different correspondences of the first with the second.
   a. One such correspondence is the following:

      $A \rightarrow Z$

      $B \rightarrow X$

      $C \rightarrow Y$

      Write the other five correspondences.
   b. If all six of these correspondences come from congruences, then what can you say about $\triangle ABC$?
   c. If two of the correspondences come from congruences, but the others do not, then what can you say about $\triangle ABC$?
   d. Why can there be no two triangles where three of the correspondences come from congruences but the others do not?

2. Give an example of two triangles and a correspondence between them such that (a) all three corresponding angles are congruent, but (b) corresponding sides are not congruent.

3. Give an example of two triangles and a correspondence between their vertices such that (a) one angle in the first is congruent to the corresponding angle in the second and (b) two sides of the first are congruent to the corresponding sides of the second, but (c) the triangles themselves are not congruent.

4. Give an example of two quadrilaterals and a correspondence between their vertices such that (a) all four corresponding angles are congruent and (b) two sides of the first are congruent to two sides of the second, but (c) the two quadrilaterals are not congruent.

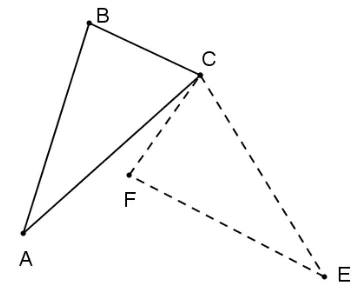
5. A particular rigid motion, M, takes point P as input and gives point P' as output. That is, $M(P) = P'$. The same rigid motion maps point Q to point Q'. Since rigid motions preserve distance, is it reasonable to state that $P' = QQ'$? Does it matter which type of rigid motion M is? Justify your response for each of the three types of rigid motion. Be specific. If it is indeed the case, for some class of transformations, that $PP' = QQ'$ is true for all P and Q, explain why. If not, offer a counter-example.

Lesson 21: Correspondence and Transformations

Classwork

Opening Exercise

The figure to the right represents a rotation of $\triangle ABC$ $80°$ around vertex C. Name the triangle formed by the image of $\triangle ABC$. Write the rotation in function notation, and name all corresponding angles and sides.

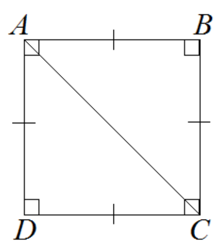


Discussion

In the Opening Exercise, we explicitly showed a single rigid motion, which mapped every side and every angle of $\triangle ABC$ onto $\triangle EFC$. Each corresponding pair of sides and each corresponding pair of angles was congruent. When each side and each angle on the pre-image maps onto its corresponding side or angle on the image, the two triangles are congruent. Conversely, if two triangles are congruent, then each side and angle on the pre-image is congruent to its corresponding side or angle on the image.

Example 1

$ABCD$ is a square, and AC is one diagonal of the square. $\triangle ABC$ is a reflection of $\triangle ADC$ across segment AC. Complete the table below identifying the missing corresponding angles and sides.



Corresponding angles	Corresponding sides
$\angle BAC \rightarrow$	$AB \rightarrow$
$\angle ABC \rightarrow$	$BC \rightarrow$
$\angle BCA \rightarrow$	$AC \rightarrow$

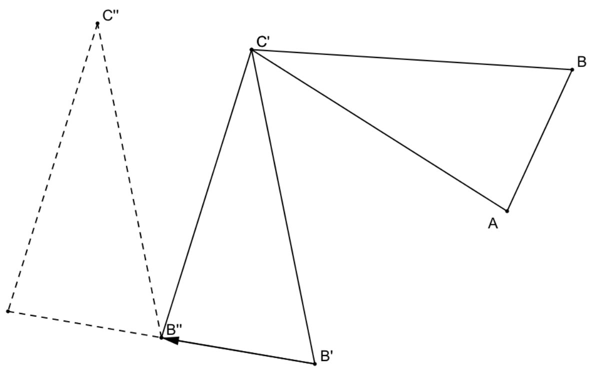
a. Are the corresponding sides and angles congruent? Justify your response.

b. Is $\triangle ABC \cong \triangle ADC$? Justify your response.

Exercises 1–3

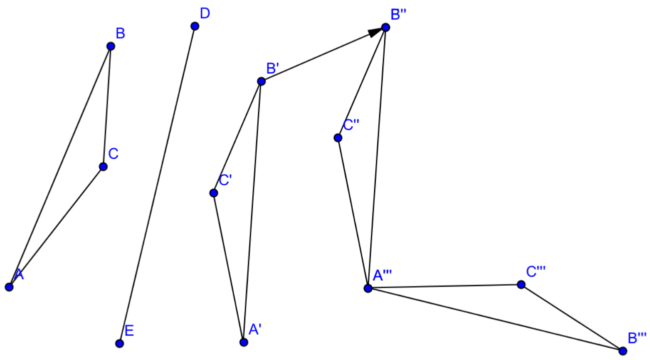
Each exercise below shows a sequence of rigid motions that map a pre-image onto a final image. Identify each rigid motion in the sequence, writing the composition using function notation. Trace the congruence of each set of corresponding sides and angles through all steps in the sequence, proving that the pre-image is congruent to the final image by showing that every side and every angle in the pre-image maps onto its corresponding side and angle in the image. Finally, make a statement about the congruence of the pre-image and final image.

1.



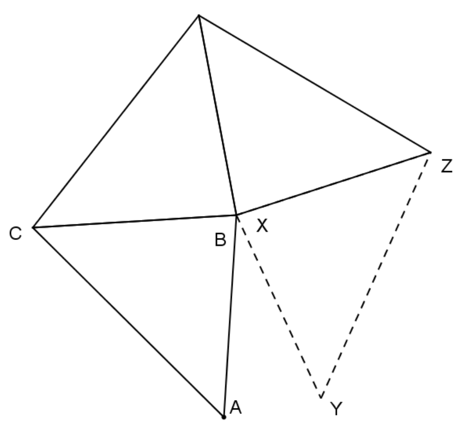
Sequence of rigid motions (2)	
Composition in function notation	
Sequence of corresponding sides	
Sequence of corresponding angles	
Triangle congruence statement	

2.



Sequence of rigid motions (3)	
Composition in function notation	
Sequence of corresponding sides	
Sequence of corresponding angles	
Triangle congruence statement	

3.



Sequence of rigid motions (3)	
Composition in function notation	
(Sequence of corresponding sides	
Sequence of corresponding angles	
Triangle congruence statement	

Problem Set

1. Exercise 3 above mapped $\triangle ABC$ onto $\triangle YXZ$ in three "steps." Construct a fourth step that would map $\triangle YXZ$ back onto $\triangle ABC$.

2. Explain triangle congruence in terms of rigid motions. Use the terms corresponding sides and corresponding angles in your explanation.

Lesson 22: Congruence Criteria for Triangles—SAS

Classwork

Opening Exercise

Answer the following question. Then discuss your answer with a partner.

Do you think it is possible to know that there is a rigid motion that takes one triangle to another without actually showing the particular rigid motion? Why or why not?

____

____

____

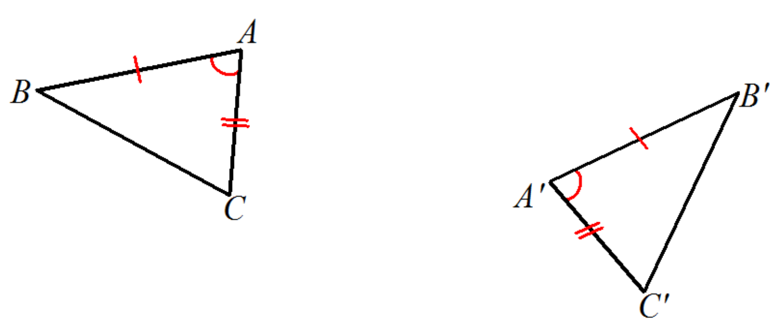
Discussion

It is true that we will not need to show the rigid motion to be able to know that there is one. We are going to show that there are criteria that refer to a few parts of the two triangles and a correspondence between them that guarantee congruency (i.e., existence of rigid motion). We start with the Side-Angle-Side (SAS) criteria.

Side-Angle-Side Triangle Congruence Criteria (SAS): Given two triangles $\triangle ABC$ and $\triangle A'B'C'$ so that $AB = A'B'$ (Side), $\text{m}\angle A = \text{m}\angle A'$ (Angle), $AC = A'C'$ (Side). Then the triangles are congruent.

The steps below show the most general case for determining a congruence between two triangles that satisfy the SAS criteria. Note that not all steps are needed for every pair of triangles. For example, sometimes the triangles will already share a vertex. Sometimes a reflection will be needed, sometimes not. It is important to understand that we can always use the steps below—some or all of them—to determine a congruence between the two triangles that satisfies the SAS criteria

Proof: Provided the two distinct triangles below, assume $AB = A'B'$ (Side), $\text{m}\angle A = \text{m}\angle A'$ (Angle), $AC = A'C'$ (Side).



By our definition of congruence, we will have to find a composition of rigid motions will map $\triangle A'B'C'$ to $\triangle ABC$. We must find a congruence F so that $F(\triangle A'B'C') = \triangle ABC$. First, use a translation T to map a common vertex.

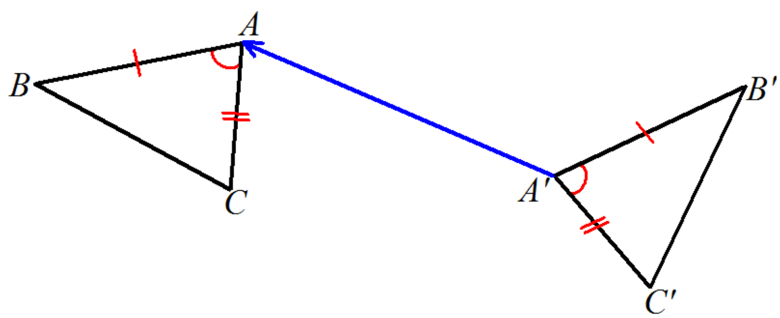
Which two points determine the appropriate vector?

Can any other pair of points be used? ________ Why or why not?

____

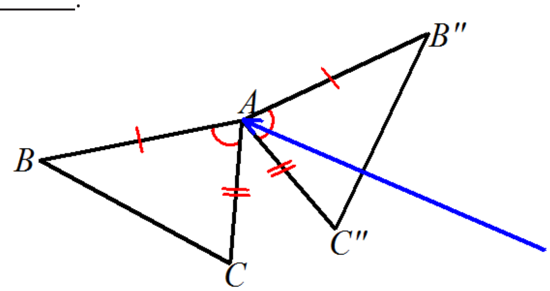
State the vector in the picture below that can be used to translate $\triangle A'B'C'$: _____________

Using a dotted line, draw an intermediate position of $\triangle A'B'C'$ as it moves along the vector:

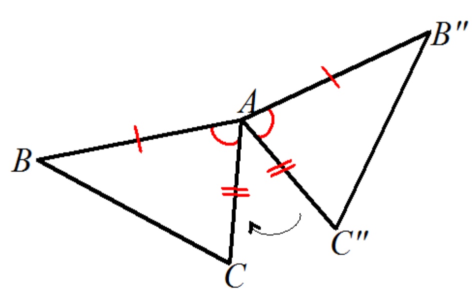


After the translation (below), $T_{\overrightarrow{A'A}}(\triangle A'B'C')$ shares one vertex with $\triangle ABC$, A. In fact, we can say

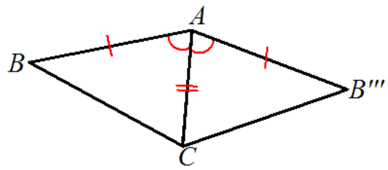
T______($\triangle$ __________) $= \triangle$ __________.



Next, use a clockwise rotation $R_{\angle CAC''}$ to bring the sides $\overline{AC''}$ to $\overline{AC}$ (or counterclockwise rotation to bring $\overline{AB''}$ to $\overline{AB}$).

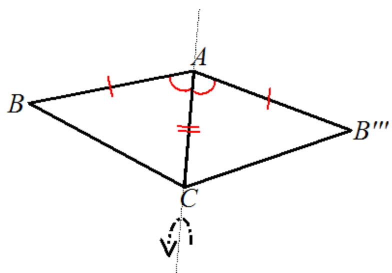


A rotation of appropriate measure will map $\overrightarrow{AC''}$ to $\overrightarrow{AC}$, but how can we be sure that vertex C'' maps to C? Recall that part of our assumption is that the lengths of sides in question are equal, ensuring that the rotation maps C'' to C. ($AC = AC''$; the translation performed is a rigid motion, and thereby did not alter the length when $\overline{AC'}$ became $\overline{AC''}$.)



After the rotation $R_{\angle CAC''}(\triangle AB''C'')$, a total of two vertices are shared with $\triangle ABC$, A and C. Therefore,

Finally, if B''' and B are on opposite sides of the line that joins AC, a reflection $r_{\overline{AC}}$ brings B''' to the same side as B.



Since a reflection is a rigid motion and it preserves angle measures, we know that $m\angle B'''AC = m\angle BAC$ and so $\overrightarrow{AB'''}$ maps to $\overrightarrow{AB}$. If, however, $\overrightarrow{AB'''}$ coincides with $\overrightarrow{AB}$, can we be certain that B''' actually maps to B? We can, because not only are we certain that the rays coincide but also by our assumption that $AB = AB'''$. (Our assumption began as $AB = A'B'$, but the translation and rotation have preserved this length now as AB'''.) Taken together, these two pieces of information ensure that the reflection over $\overline{AC}$ brings B''' to B.

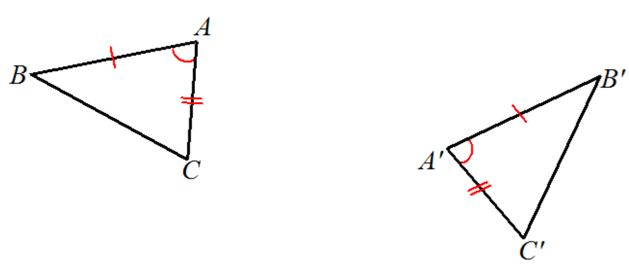
Another way to visually confirm this is to draw the marks of the ______________________construction for $\overline{AC}$.

Write the transformations used to correctly notate the congruence (the composition of transformations) that take $\triangle A'B'C' \cong \triangle ABC$:

F __________

G __________

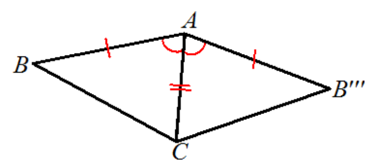
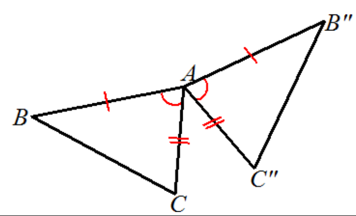
H __________



We have now shown a sequence of rigid motions that takes $\triangle A'B'C'$ to $\triangle ABC$ with the use of just three criteria from each triangle: two sides and an included angle. Given any two distinct triangles, we could perform a similar proof. There is another situation when the triangles are not distinct, where a modified proof will be needed to show that the triangles map onto each other. Examine these below. Note that when using the Side-Angle-Side triangle congruence criteria as a reason in a proof, you need only state the congruence and "SAS."

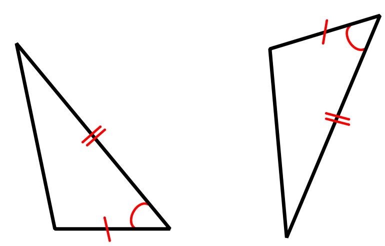
Example 1

What if we had the SAS criteria for two triangles that were not distinct? Consider the following two cases. How would the transformations needed to demonstrate congruence change?

Case	Diagram	Transformations Needed
Shared Side		
Shared Vertex		

Exercises 1–4

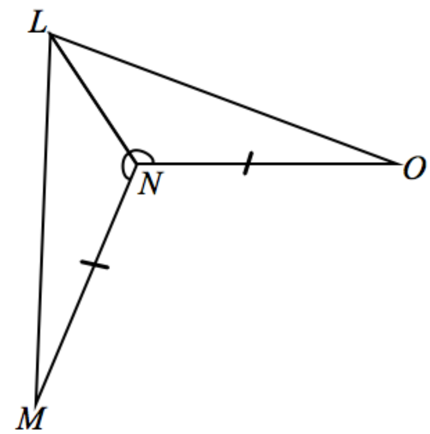
1. Given: Triangles with a pair of corresponding sides of equal length and a pair of included angles of equal measure. Sketch and label three phases of the sequence of rigid motions that prove the two triangles to be congruent.



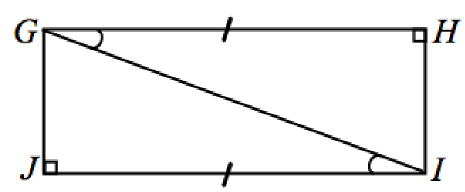
Justify whether the triangles meet the SAS congruence criteria; explicitly state which pairs of sides or angles are congruent and why. If the triangles do meet the SAS congruence criteria, describe the rigid motion(s) that would map one triangle onto the other.

Given: Triangles with a pair of corresponding sides of equal length and a pair of included angles of equal measure. Sketch and label three phases of the sequence of rigid motions that prove the two triangles to be congruent.

2. Given: $\angle LMN = \angle LNO$, $MN = OM$.
   Do $\triangle LMN$ and $\triangle LOM$ meet the SAS criteria?



3. Given: $\angle HGI = \angle JIG$, $HG = JI$.
   Do $\triangle HGI$ and $\triangle JIG$ meet the SAS criteria?

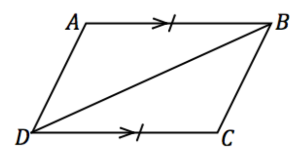


4. Is it true that we could also have proved $\triangle HGI$ and $\cong \triangle JIG$ meet the SAS criteria if we had been given that $\angle HGI \cong \angle JIG$ and $\overline{HG} \cong \overline{JI}$? Explain why or why not.

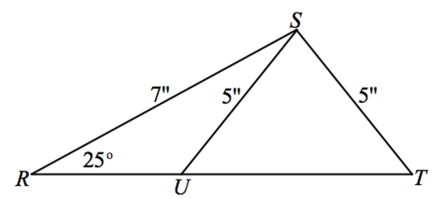
Problem Set

Justify whether the triangles meet the SAS congruence criteria; explicitly state which pairs of sides or angles are congruent and why. If the triangles do meet the SAS congruence criteria, describe the rigid motion(s) that would map one triangle onto the other.

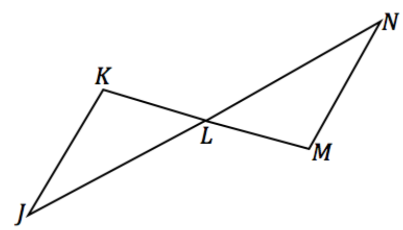
1. Given: $\overline{AB} \parallel \overline{CD}$, $AB = CD$
   Do $\triangle ABD$ and $\triangle CDB$ meet the SAS criteria?



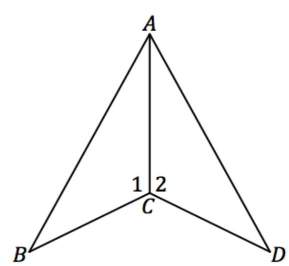
2. Given: $m\angle R = 25°$, $RT = 7"$, $SU = 5"$, $ST = 5"$
   Do $\triangle RSU$ and $\triangle RST$ meet the SAS criteria?



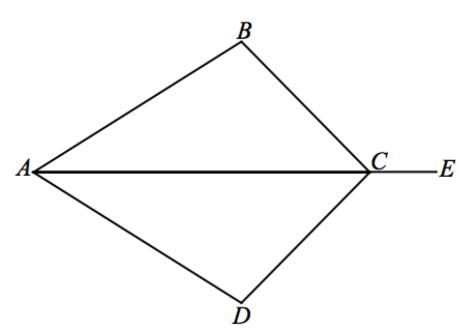
3. Given: $\overline{KM}$ and $\overline{JN}$ bisect each other.
   Do $\triangle JKL$ and $\triangle NML$ meet the SAS criteria?



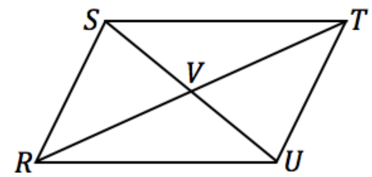
4. Given: $m\angle 1 = m\angle 2$, $BC = DC$
   Do $\triangle ABC$ and $\triangle ADC$ meet the SAS criteria?



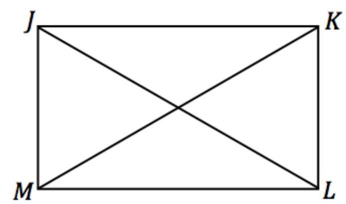
5. Given: $\overline{AE}$ bisects angle $\angle BCD$, $BC = DC$
   Do $\triangle CAB$ and $\triangle CAD$ meet the SAS criteria?



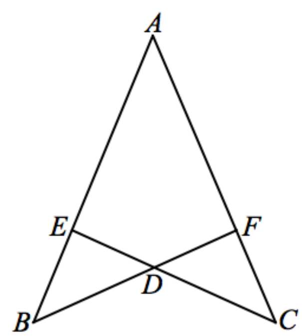
6. Given: $\overline{SU}$ and $\overline{RT}$ bisect each other
   Do $\triangle SVR$ and $\triangle UVT$ meet the SAS criteria?



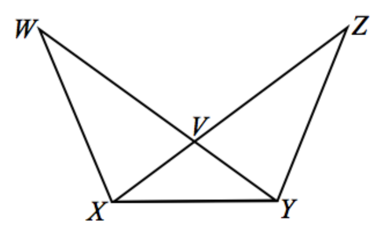
7. Given: $JM = KL$, $\overline{JM} \perp \overline{ML}$, $\overline{KL} \perp \overline{ML}$
   Do $\triangle JML$ and $\triangle KLM$ meet the SAS criteria?



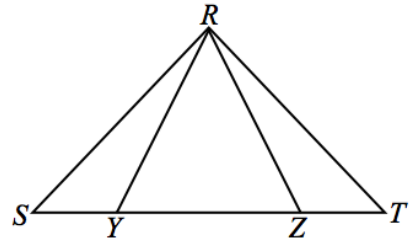
8. Given: $\overline{BF} \perp \overline{AC}$, $\overline{CE} \perp \overline{AB}$
   Do $\triangle BED$ and $\triangle CFD$ meet the SAS criteria?



9. Given: $\text{m}\angle VXY = \text{m}\angle VYX$
   Do $\triangle VXW$ and $\triangle VYZ$ meet the SAS criteria?



10. Given: $\triangle RST$ is isosceles, $SY = TZ$.
    Do $\triangle RSY$ and $\triangle RTZ$ meet the SAS criteria?



Lesson 23: Base Angles of Isosceles Triangles

Classwork

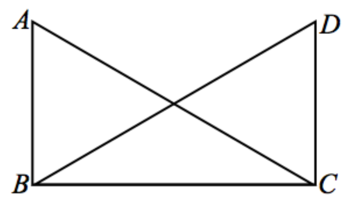
Opening Exercise

Describe the additional piece of information needed for each pair of triangles to satisfy the SAS triangle congruence criteria.

1. Given: $AB = DC$

   ____

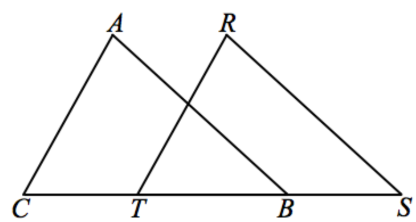
   Prove: $\triangle ABC \cong \triangle DCB$



2. Given: $AB = RS$
   $\overline{AB} \parallel \overline{RS}$

   ____

   Prove: $\triangle ABC \cong \triangle RST$



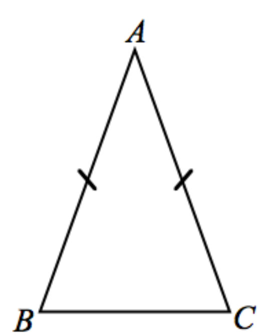
Exploratory Challenge

Today we examine a geometry fact that we already accept to be true. We are going to prove this known fact in two ways: (1) by using transformations and (2) by using SAS triangle congruence criteria.

Here is isosceles triangle ABC. We accept that an isosceles triangle, which has (at least) two congruent sides, also has congruent base angles.

Label the congruent angles in the figure.

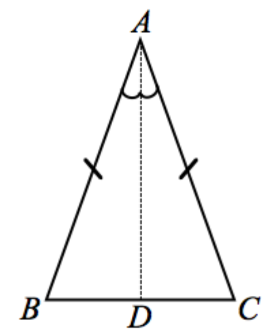
Now we will prove that the base angles of an isosceles triangle are always congruent.



Prove Base Angles of an Isosceles are Congruent: Transformations

Given: Isosceles $\triangle ABC$, with $AB = AC$

Prove: $\text{m}\angle B = \text{m}\angle C$



Construction: Draw the angle bisector $\overrightarrow{AD}$ of $\angle A$, where D is the intersection of the bisector and $\overline{BC}$. We need to show that rigid motions will map point B to point C and point C to point B.

Let r be the reflection through $\overleftrightarrow{AD}$. Through the reflection, we want to demonstrate two pieces of information that map B to point C and vice versa: (1) $\overrightarrow{AB}$ maps to $\overrightarrow{AC}$, and (2) $AB = AC$.

Since A is on the line of reflection, $\overleftrightarrow{AD}$, $r(A) = A$. Reflections preserve angle measures, so the measure of the reflected angle $r(\angle BAD)$ equals the measure of $\angle CAD$; therefore, $r\left(\overrightarrow{AB}\right) = \overrightarrow{AC}$. Reflections also preserve lengths of segments; therefore, the reflection of $\overline{AB}$ will still have the same length as $\overline{AB}$. By hypothesis, $AB = AC$, so the length of the reflection will also be equal to AC. Then $r(B) = C$. Using similar reasoning, we can show that $r(C) = B$.

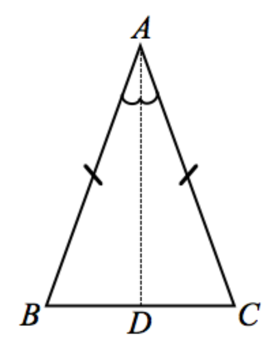
Reflections map rays to rays, so $r\left(\overrightarrow{BA}\right) = \overrightarrow{CA}$ and $r\left(\overrightarrow{BC}\right) = \overrightarrow{CB}$. Again, since reflections preserve angle measures, the measure of $r(\angle ABC)$ is equal to the measure of $\angle ACB$.

We conclude that $\text{m}\angle B = \text{m}\angle C$. Equivalently, we can state that $\angle B \cong \angle C$. In proofs, we can state that "base angles of an isosceles triangle are equal in measure" or that "base angles of an isosceles triangle are congruent."

Prove Base Angles of an Isosceles are Congruent: SAS

Given: Isosceles $\triangle ABC$, with $AB = AC$

Prove: $\angle B \cong \angle C$

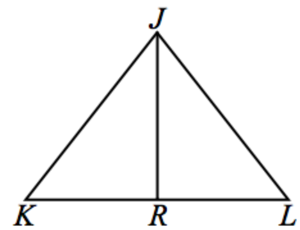


Construction: Draw the angle bisector $\overrightarrow{AD}$ of $\angle A$, where D is the intersection of the bisector and $\overline{BC}$. We are going to use this auxiliary line towards our SAS criteria.

Exercises

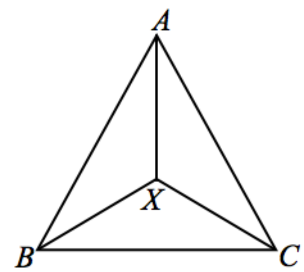
1. Given: $JK = JL$; $\overline{JR}$ bisects $\overline{KL}$

   Prove: $\overline{JR} \perp \overline{KL}$



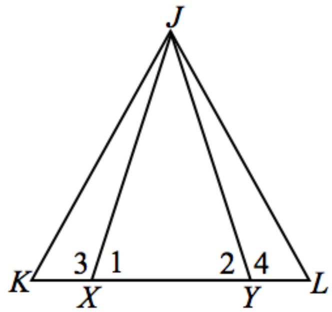
2. Given: $AB = AC$, $XB = XC$

   Prove: $\overline{AX}$ bisects $\angle BAC$

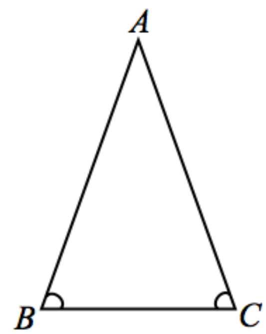


3. Given: $JX = JY$, $KX = LY$

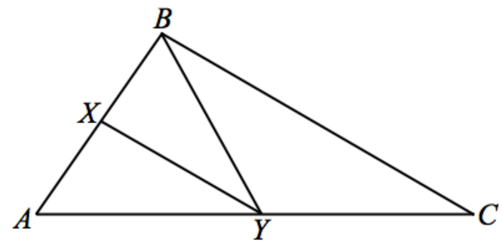
   Prove: $\triangle JKL$ is isosceles



4. Given: $\triangle ABC$, with $m\angle CBA = m\angle BCA$
   Prove: $BA = CA$
   (Converse of base angles of isosceles triangle)
   Hint: Use a transformation.



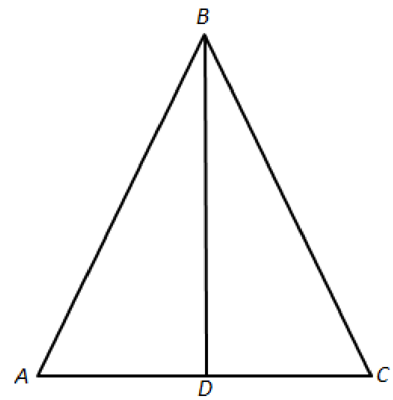
5. Given: $\triangle ABC$, with $\overline{XY}$ is the angle bisector of $\angle BYA$, and $\overline{BC} \parallel \overline{XY}$
   Prove: $YB = YC$



Problem Set

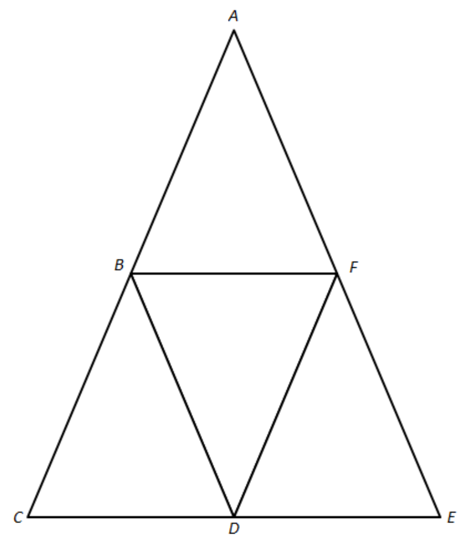
1. Given: $AB = BC$, $AD = DC$

   Prove: $\triangle ADB$ and $\triangle CDB$ are right triangles

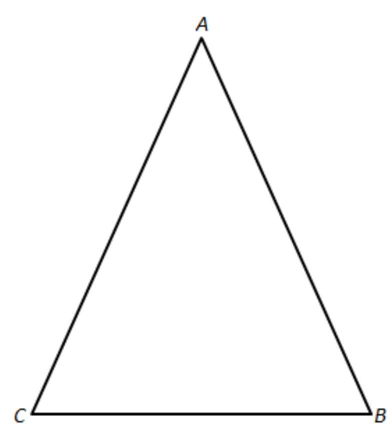


2. Given: $AC = AE$ and $\overline{BF} \parallel \overline{CE}$

   Prove: $AB = AF$



3. In the diagram, $\triangle ABC$ is isosceles with $\overline{AC} \cong \overline{AB}$. In your own words, describe how transformations and the properties of rigid motions can be used to show that $\angle C \cong \angle B$.

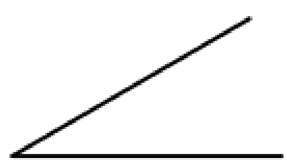


Lesson 24: Congruence Criteria for Triangles—ASA and SSS

Classwork

Opening Exercise

Use the provided 30° angle as one base angle of an isosceles triangle. Use a compass and straight edge to construct an appropriate isosceles triangle around it.



Compare your constructed isosceles triangle with a neighbor's. Does the use of a given angle measure guarantee that all the triangles constructed in class have corresponding sides of equal lengths?

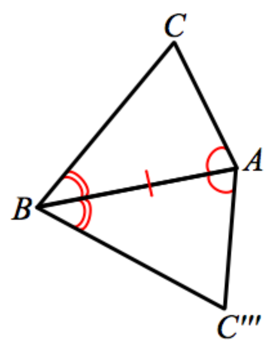
Discussion

Today we are going to examine two more triangle congruence criteria, Angle-Side-Angle (ASA) and Side-Side-Side (SSS), to add to the SAS criteria we have already learned. We begin with the ASA criteria.

Angle-Side-Angle Triangle Congruence Criteria (ASA): Given two triangles ABC and $A'B'C'$. If $\mathrm{m}\angle CAB = \mathrm{m}\angle C'A'B'$ (Angle), $AB = A'B'$ (Side), and $\mathrm{m}\angle CBA = \mathrm{m}\angle C'B'A'$ (Angle), then the triangles are congruent.

Proof:

We do not begin at the very beginning of this proof. Revisit your notes on the SAS proof, and recall that there are three cases to consider when comparing two triangles. In the most general case, when comparing two distinct triangles, we translate one vertex to another (choose congruent corresponding angles). A rotation brings congruent, corresponding sides together. Since the ASA criteria allows for these steps, we begin here.



In order to map $\triangle ABC'''$ to $\triangle ABC$, we apply a reflection r across the line AB. A reflection will map A to A and B to B, since they are on line AB. However, we will say that $r(C''') = C^*$. Though we know that $r(C''')$ is now in the same half-plane of line AB as C, we cannot assume that C''' maps to C. So we have $r(\triangle ABC''') = \triangle ABC^*$. To prove the theorem, we need to verify that C^* is C.

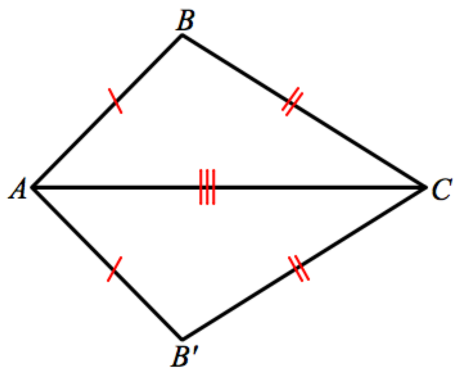
By hypothesis, we know that $\angle CAB \cong \angle C'''AB$ (recall that $\angle C'''AB$ is the result of two rigid motions of $\angle C'A'B'$, so must have the same angle measure as $\angle C'A'B'$). Similarly, $\angle CBA \cong \angle C'''BA$. Since $\angle CAB \cong r(\angle C'''AB) \cong \angle C^*AB$, and C and C^* are in the same half-plane of line AB, we conclude that $\overrightarrow{AC}$ and $\overrightarrow{AC^*}$ must actually be the same ray. Because the points A and C^* define the same ray as $\overrightarrow{AC}$, the point C^* must be a point somewhere on $\overrightarrow{AC}$. Using the second equality of angles, $\angle CBA \cong r(\angle C'''BA) \cong \angle C^*BA$, we can also conclude that $\overrightarrow{BC}$ and $\overrightarrow{BC^*}$ must be the same ray. Therefore, the point C^* must also be on $\overrightarrow{BC}$. Since C^* is on both $\overrightarrow{AC}$ and $\overrightarrow{BC}$, and the two rays only have one point in common, namely C, we conclude that $C = C^*$.

We have now used a series of rigid motions to map two triangles onto one another that meet the ASA criteria.

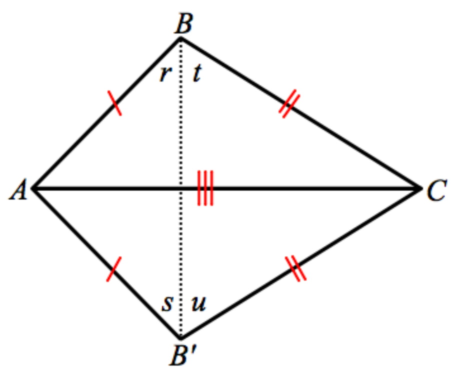
Side-Side-Side Triangle Congruence Criteria (SSS): Given two triangles ABC and $A'B'C'$. If $AB = A'B'$ (Side), $AC = A'C'$ (Side), and $BC = B'C'$ (Side) then the triangles are congruent.

Proof:

Again, we do not need to start at the beginning of this proof, but assume there is a congruence that brings a pair of corresponding sides together, namely the longest side of each triangle.



Without any information about the angles of the triangles, we cannot perform a reflection as we have in the proofs for SAS and ASA. What can we do? First we add a construction: Draw an auxiliary line from B to B', and label the angles created by the auxiliary line as r, s, t, and u.



Since $AB = AB'$ and $CB = CB'$, $\triangle\, ABB'$ and $\triangle\, CBB'$ are both isosceles triangles respectively by definition. Therefore, $r = s$ because they are base angles of an isosceles triangle ABB'. Similarly, $\mathrm{m}\angle t = \mathrm{m}\angle u$ because they are base angles of $\triangle\, CBB'$. Hence, $\angle ABC = \mathrm{m}\angle r + \mathrm{m}\angle t = \mathrm{m}\angle s + \mathrm{m}\angle u = \angle AB'C$. Since $\mathrm{m}\angle ABC = \mathrm{m}\angle AB'C$, we say that $\triangle\, ABC \cong \triangle\, AB'C$ by SAS.

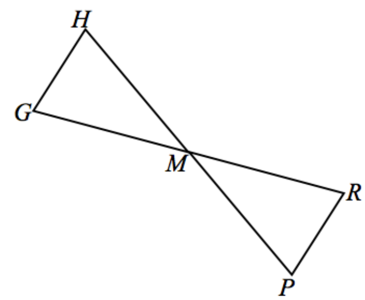
We have now used a series of rigid motions and a construction to map two triangles that meet the SSS criteria onto one another. Note that when using the Side-Side-Side triangle congruence criteria as a reason in a proof, you need only state the congruence and "SSS." Similarly, when using the Angle-Side-Angle congruence criteria in a proof, you need only state the congruence and "ASA."

Now we have three triangle congruence criteria at our disposal: SAS, ASA, and SSS. We will use these criteria to determine whether or not pairs of triangles are congruent.

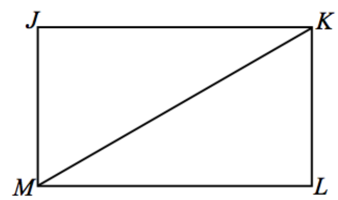
Exercises

Based on the information provided, determine whether a congruence exists between triangles. If a congruence exists between triangles or if multiple congruencies exist, state the congruencies and the criteria used to determine them.

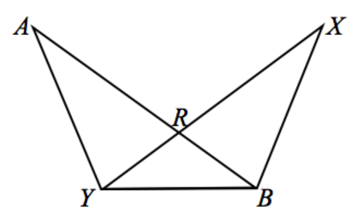
1. Given: M is the midpoint of $\overline{HP}$, $\text{m}\angle H = \text{m}\angle P$.



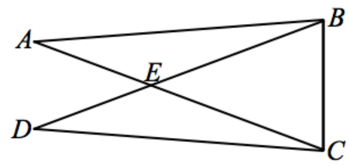
2. Given: Rectangle $JKLM$ with diagonal KM.



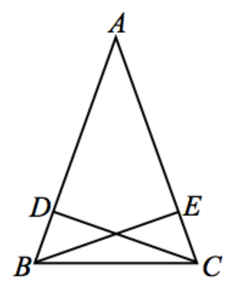
3. Given: $RY = RB$, $AR = XR$.



4. Given: $\text{m}\angle A = \text{m}\angle D$, $AE = DE$.



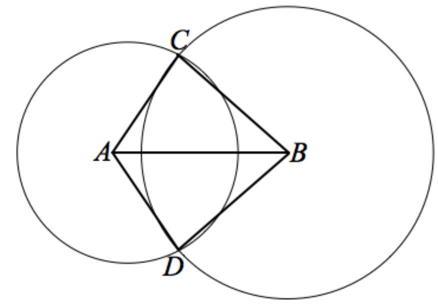
5. Given: $AB = AC,\ BD = \frac{1}{4}AB,\ CE = \frac{1}{4}AC$.



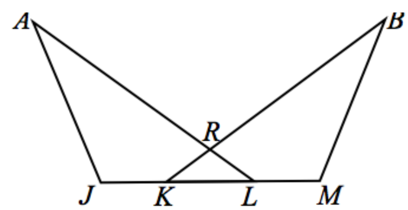
Problem Set

Use your knowledge of triangle congruence criteria to write proofs for each of the following problems.

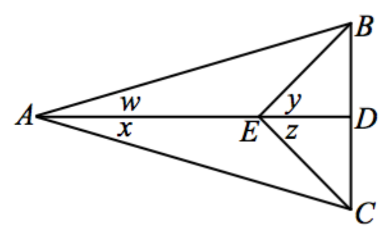
1. Given: Circles with centers A and B intersect at C and D.
   Prove: $\angle CAB \cong \angle DAB$.



2. Given: $\angle J \cong \angle M$, $JA = MB$, $JK = KL = LM$.
   Prove: $\overline{KR} \cong \overline{LR}$.



3. Given: $\mathrm{m}\angle w = \mathrm{m}\angle x$ and $\mathrm{m}\angle y = \mathrm{m}\angle z$.
   Prove: (1) $\triangle ABE \cong \triangle ACE$
   (2) $AB = AC$ and $\overline{AD} \perp \overline{BC}$



4. After completing the last exercise, Jeanne said, "We also could have been given that $\angle w \cong \angle x$ and $\angle y \cong \angle z$. This would also have allowed us to prove that $\triangle ABE \cong \triangle ACE$." Do you agree? Why or why not?

Lesson 25: Congruence Criteria for Triangles—AAS and HL

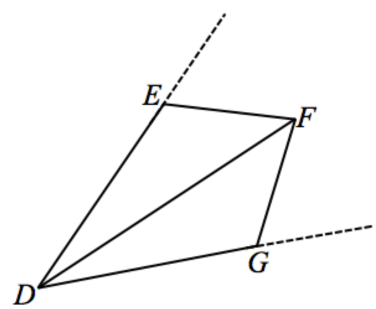
Classwork

Opening Exercise

Write a proof for the following question. Once done, compare your proof with a neighbor's.

Given: $DE = DG$, $EF = GF$

Prove: DF is the angle bisector of $\angle EDG$



Proof:

Exploratory Challenge

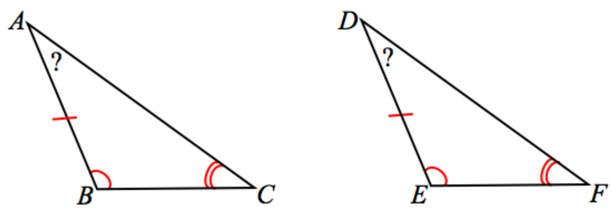
Today we are going to examine three possible triangle congruence criteria, Angle-Angle-Side (AAS), Side-Side-Angle (SSA), and Angle-Angle-Angle (AAA). Ultimately, only one of the three possible criteria will ensure congruence.

Angle-Angle-Side Triangle Congruence Criteria (AAS): Given two triangles ABC and $A'B'C'$. If $AB = A'B'$ (Side), $\mathrm{m}\angle B = \mathrm{m}\angle B'$ (Angle), and $\mathrm{m}\angle C = \mathrm{m}\angle C'$ (Angle), then the triangles are congruent.

Proof:

Consider a pair of triangles that meet the AAS criteria. If you knew that two angles of one triangle corresponded to and were equal in measure to two angles of the other triangle, what conclusions can you draw about the third angles of each triangle?

Since the first two angles are equal in measure, the third angles must also be equal in measure.



Given this conclusion, which formerly learned triangle congruence criteria can we use to determine if the pair of triangles are congruent?

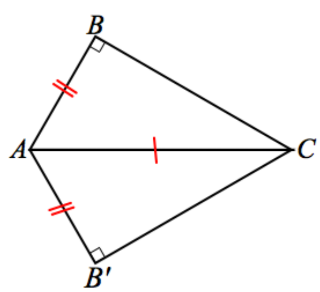
Therefore, the AAS criterion is actually an extension of the _____________ triangle congruence criterion.

Note that when using the Angle-Angle-Side triangle congruence criteria as a reason in a proof, you need only state the congruence and "AAS."

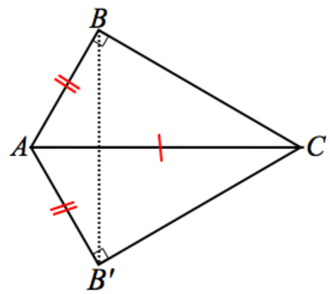
Hypotenuse-Leg Triangle Congruence Criteria (HL): Given two right triangles ABC and $A'B'C'$with right angles B and B', if $AB = A'B'$ (Leg) and $AC = A'C'$ (Hypotenuse), then the triangles are congruent.

Proof:

As with some of our other proofs, we will not start at the very beginning, but imagine that a congruence exists so that triangles have been brought together such that $A = A'$ and $C = C'$; the hypotenuse acts as a common side to the transformed triangles.



Similar to the proof for SSS, we add a construction and draw $\overline{BB'}$.

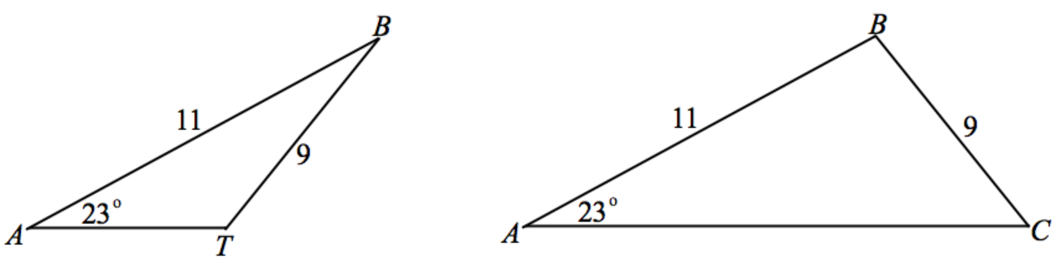


$\triangle ABB'$ is isosceles by definition, and we can conclude that base angles $\mathrm{m}\angle ABB' = \mathrm{m}\angle AB'B$. Since $\angle CBB'$ and $\angle CB'B$ are both the complements of equal angle measures ($\angle ABB'$ and $\angle AB'B$), they too are equal in measure. Furthermore, since $m\angle CBB' = \mathrm{m}\angle CB'B$, the sides of $\triangle CBB'$ opposite them are equal in measure: $BC = B'C'$.

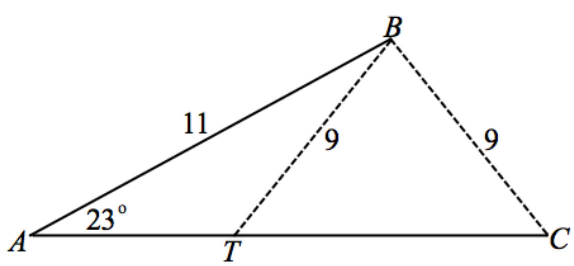
Then, by SSS, we can conclude $\triangle ABC \cong \triangle A'B'C'$. Note that when using the Hypotenuse-Leg triangle congruence criteria as a reason in a proof, you need only state the congruence and "HL."

Criteria that do not determine two triangles as congruent: SSA and AAA

<u>Side-Side-Angle (SSA)</u>: Observe the diagrams below. Each triangle has a set of adjacent sides of measures 11 and 9, as well as the non-included angle of 23°. Yet, the triangles are not congruent.



Examine the composite made of both triangles. The sides of lengths 9 each have been dashed to show their possible locations.



The triangles that satisfy the conditions of SSA cannot guarantee congruence criteria. In other words, two triangles under SSA criteria may or may not be congruent; therefore, we cannot categorize SSA as congruence criterion.

Angle-Angle-Angle (AAA): A correspondence exists between $\triangle ABC$ and $\triangle DEF$. Trace $\triangle ABC$ onto patty paper, and line up corresponding vertices.

Based on your observations, why isn't AAA categorizes as congruence criteria? Is there any situation in which AAA does guarantee congruence?

Even though the angle measures may be the same, the sides can be proportionally larger; you can have similar triangles in addition to a congruent triangle.

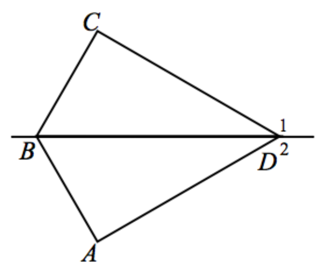
List all the triangle congruence criteria here: ______________________________

List the criteria that do not determine congruence here: ______________________________

Examples

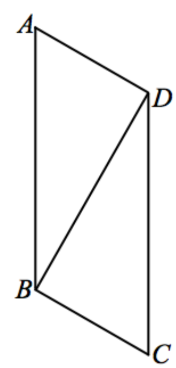
1. Given: $\overline{BC} \perp \overline{CD}$, $\overline{AB} \perp \overline{AD}$, $m\angle 1 = m\angle 2$

   Prove: $\triangle BCD \cong \triangle BAD$



2. Given: $AD \perp BD$, $BD \perp BC$, $AB = CD$

   Prove: $\triangle ABD \cong \triangle CDB$

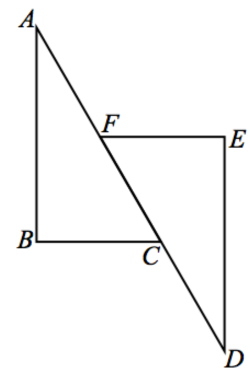


Problem Set

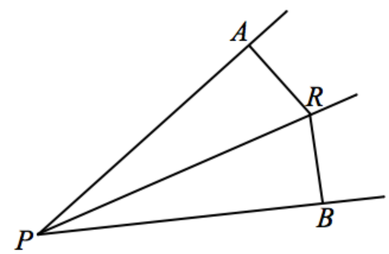
Use your knowledge of triangle congruence criteria to write proofs for each of the following problems.

1. Given: $\overline{AB} \perp \overline{BC},\ \overline{DE} \perp \overline{EF},\ \overline{BC} \parallel \overline{EF},\ AF = DC$

   Prove: $\triangle ABC \cong \triangle DEF$

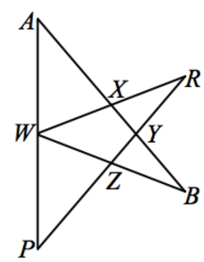


2. In the figure, $\overline{PA} \perp \overline{AR}$ and $\overline{PB} \perp \overline{RB}$ and R is equidistant from $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$. Prove that $\overline{PR}$ bisects $\angle APB$.



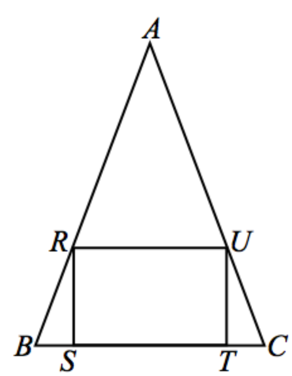
3. Given: $\angle A \cong \angle P,\ \angle B \cong \angle R$, W is the midpoint of $\overline{AP}$

   Prove: $\overline{RW} \cong \overline{BW}$



4. Given: $BR = CU$, rectangle $RSTU$

   Prove: $\triangle ARU$ is isosceles

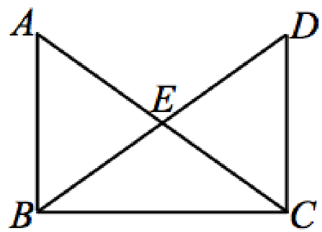


Lesson 26: Triangle Congruency Proofs

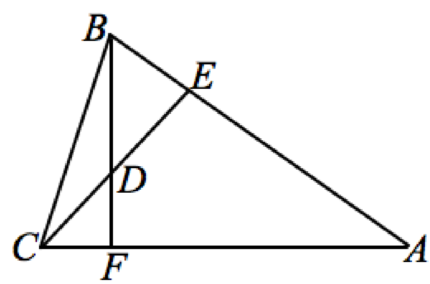
Classwork

Exercises 1–6

1. Given: $\overline{AB} \perp \overline{BC}$, $\overline{BC} \perp \overline{DC}$.
   $\overline{DB}$ bisects $\angle ABC$, $\overline{AC}$ bisects $\angle DCB$.
   $EB = EC$.
   Prove: $\triangle BEA \cong \triangle CED$.

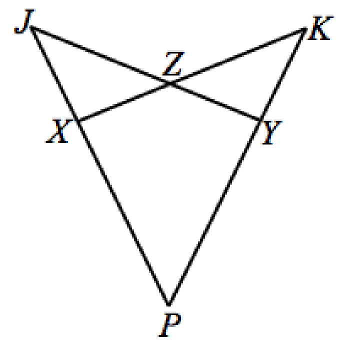


2. Given: $BF \perp AC$, $CE \perp AB$.
   $AE = AF$.
   Prove: $\triangle ACE \cong ABF$.



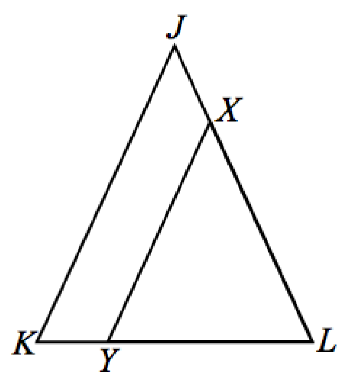
3. Given: $XJ = YK,\ PX = PY,\ \angle ZXJ = \angle ZYK.$

   Prove: $JY = KX.$



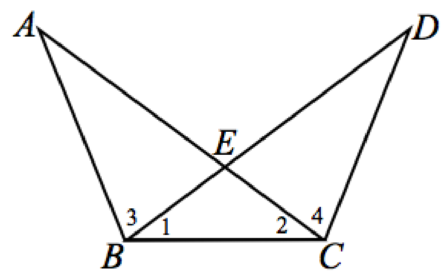
4. Given: $JK = JL,\ \overline{JK} \parallel \overline{XY}.$

   Prove: $XY = XL.$



5. Given: $\angle 1 \cong \angle 2,\ \angle 3 \cong \angle 4.$

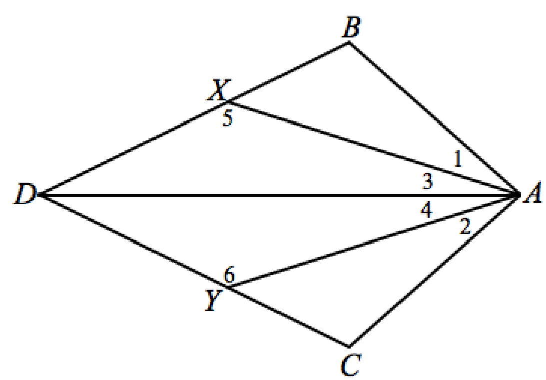
   Prove: $\overline{AC} \cong \overline{BD}.$



6. Given: $\mathrm{m}\angle 1 = \mathrm{m}\angle 2,\ \mathrm{m}\angle 3 = \mathrm{m}\angle 4,\ AB = AC.$

   Prove: (a) $\triangle ABD \cong \triangle ACD$.

   (b) $m\angle 5 = m\angle 6.$



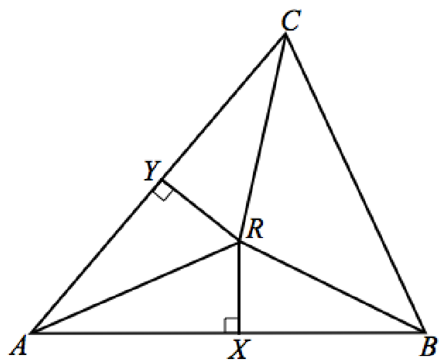
Problem Set

Use your knowledge of triangle congruence criteria to write a proof for the following:

In the figure $\overline{RX}$ and $\overline{RY}$ are the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$, respectively.

Prove: (a) $\triangle RAX \cong \triangle RAY$.

(b) $\overline{RA} \cong \overline{RB} \cong \overline{RC}$.

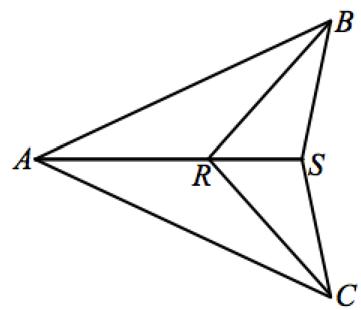


Lesson 27: Triangle Congruency Proofs

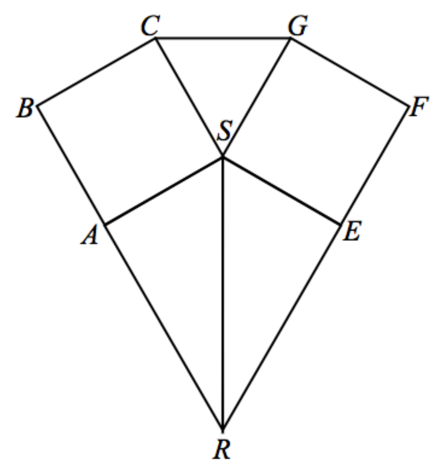
Classwork

Exercises

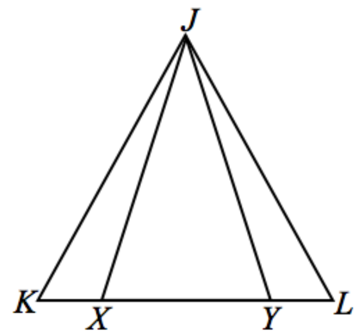
1. Given: $AB = AC$, $RB = RC$.
   Prove: $SB = SC$.



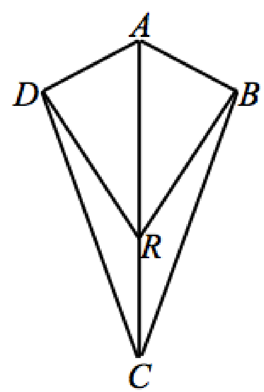
2. Given: Square $ABCS \cong$ Square $EFGS$,
   $\overleftrightarrow{RAB}$, $\overleftrightarrow{REF}$.
   Prove: $\triangle ASR \cong ESR$.



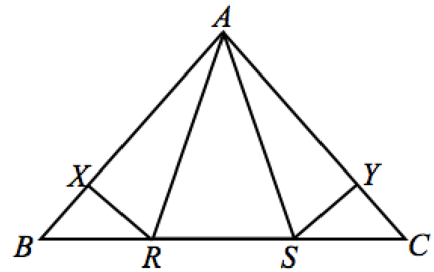
3. Given: $JK = JL, JX = JY$.
   Prove: $KX = LY$.



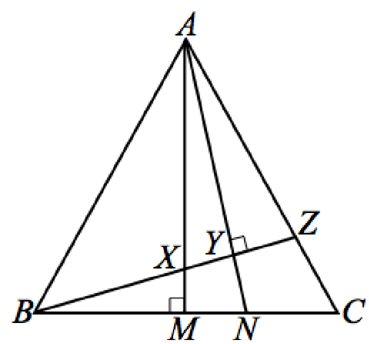
4. Given: $AD \perp DR, AB \perp BR$,
   $\overline{AD} \cong \overline{AB}$.
   Prove: $\angle DCR = \angle BCR$.



5. Given: $AR = AS, BR = CS$,
$RX \perp AB, SY \perp AC$.
Prove: $BX = CY$.



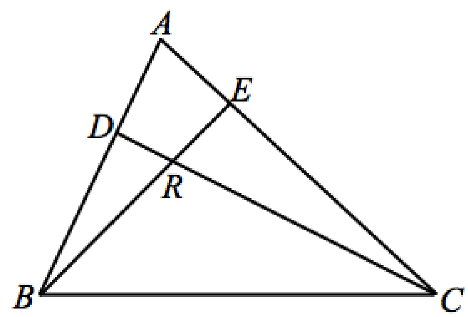
6. Given: $AX = BX, \angle AMB = \angle AYZ = 90°$.
Prove: $NY = NM$.



Problem Set

Use your knowledge of triangle congruence criteria to write a proof for the following:

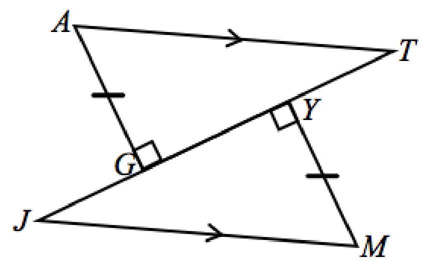
In the figure $\overline{BE} \cong \overline{CE}$, $DC \perp AB$, $BE \perp AC$, prove $\overline{AE} \cong \overline{RE}$.



Lesson 28: Properties of Parallelograms

Classwork

Opening Exercise



1. If the triangles are congruent, state the congruence.

2. Which triangle congruence criterion guarantees part 1?

3. $\overline{TG}$ corresponds with:

Discussion

How can we use our knowledge of triangle congruence criteria to establish other geometry facts? For instance, what can we now prove about the properties of parallelograms?

To date, we have defined a parallelogram to be a quadrilateral in which both pairs of opposite sides are parallel. However, we have assumed other details about parallelograms to be true too. We assume that:

- Opposite sides are congruent.
- Opposite angles are congruent.
- Diagonals bisect each other.

Let us examine why each of these properties is true.

Example 1

If a quadrilateral is a parallelogram, then its opposite sides and angles are equal in measure. Complete the diagram and develop an appropriate *Given* and *Prove* for this case. Use triangle congruence criteria to demonstrate why opposite sides and angles of a parallelogram are congruent.

Given: ______________________________

Prove: ______________________________

Construction: Label the quadrilateral $ABCD$, and mark opposite sides as parallel. Draw diagonal $\overline{BD}$.

Example 2

If a quadrilateral is a parallelogram, then the diagonals bisect each other. Complete the diagram and develop an appropriate *Given* and *Prove* for this case. Use triangle congruence criteria to demonstrate why diagonals of a parallelogram bisect each other. Remember, now that we have proved opposite sides and angles of a parallelogram to be congruent, we are free to use these facts as needed (i.e., $AD = CB$, $AB = CD$, $\angle A \cong \angle C$, $\angle B \cong \angle D$).

Given: ______________________________________

Prove: ______________________________________

Construction: Label the quadrilateral $ABCD$. Mark opposite sides as parallel. Draw diagonals AC and BD.

Now we have established why the properties of parallelograms that we have assumed to be true are in fact true. By extension, these facts hold for any type of parallelogram, including rectangles, squares, and rhombuses. Let us look at one last fact concerning rectangles. We established that the diagonals of general parallelograms bisect each other. Let us now demonstrate that a rectangle has congruent diagonals.

Example 3

If the parallelogram is a rectangle, then the diagonals are equal in length. Complete the diagram and develop an appropriate *Given* and *Prove* for this case. Use triangle congruence criteria to demonstrate why diagonals of a rectangle are congruent. As in the last proof, remember to use any already proven facts as needed.

Given: ______________________________

Prove: ______________________________

Construction: Label the rectangle $GHIJ$. Mark opposite sides as parallel, and add small squares at the vertices to indicate $90°$ angles. Draw diagonal GI and HJ.

Converse Properties: Now we examine the converse of each of the properties we proved. Begin with the property and prove that the quadrilateral is in fact a parallelogram.

Example 4

If the opposite angles of a quadrilateral are equal, then the quadrilateral is a parallelogram. Draw an appropriate diagram, and provide the relevant *Given* and *Prove* for this case.

Given: ________________________________

Prove: ________________________________

Construction: Label the quadrilateral $ABCD$. Mark opposite angles as congruent. Draw diagonal BD. Label $\angle A$ and $\angle C$ as $x°$. Label the four angles created by $\overline{BD}$ as $r°$, $s°$, $t°$, and $u°$.

Example 5

If the opposite sides of a quadrilateral are equal, then the quadrilateral is a parallelogram. Draw an appropriate diagram, and provide the relevant *Given* and *Prove* for this case.

Given: ______________________________

Prove: ______________________________

Label the quadrilateral $ABCD$, and mark opposite sides as equal. Draw diagonal $\overline{BD}$.

Example 6

If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram. Draw an appropriate diagram, and provide the relevant *Given* and *Prove* for this case. Use triangle congruence criteria to demonstrate why the quadrilateral is a parallelogram.

Given: ______________________________

Prove: ______________________________

Construction: Label the quadrilateral $ABCD$, and mark opposite sides as equal. Draw diagonals AC and BD.

Example 7

If the diagonals of a parallelogram are equal in length, then the parallelogram is a rectangle. Complete the diagram, and develop an appropriate *Given* and *Prove* for this case.

Given: ____

Prove: ____

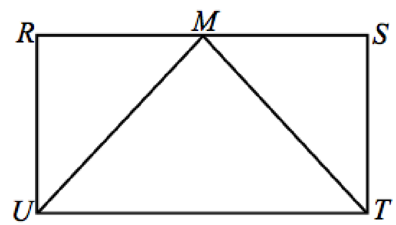
Construction: Label the quadrilateral $GHIJ$. Draw diagonals $\overline{GI}$ and $\overline{HJ}$.

Problem Set

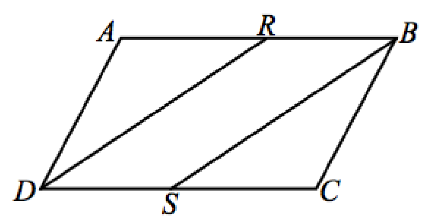
Use the facts you have established to complete exercises involving different types of parallelograms.

1. Given: $\overline{AB} \parallel \overline{CD}$, $AD = AB$, $CD = CB$.
   Prove: $ABCD$ is a rhombus

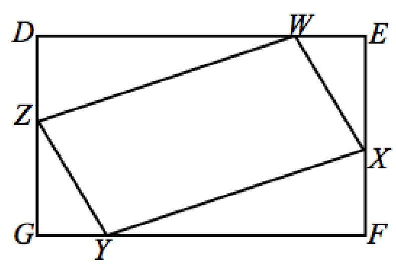
2. Given: Rectangle $RSTU$, M is the midpoint of RS.
   Prove: $\triangle UMT$ is isosceles.



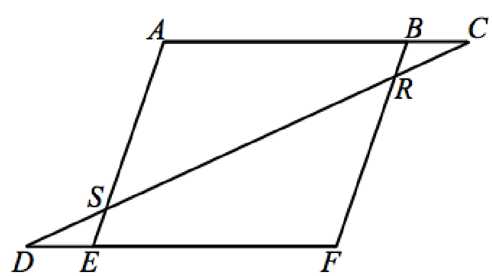
3. Given: $ABCD$ is a parallelogram, RD bisects $\angle ADC$, SB bisects $\angle CBA$.
   Prove: $DRBS$ is a parallelogram.



4. Given: $DEFG$ is a rectangle, $WE = YG$, $WX = YZ$.
   Prove: $WXYZ$ is a parallelogram.



5. Given: Parallelogram $ABFE$, $CR = DS$.
   Prove: $BR = SE$.



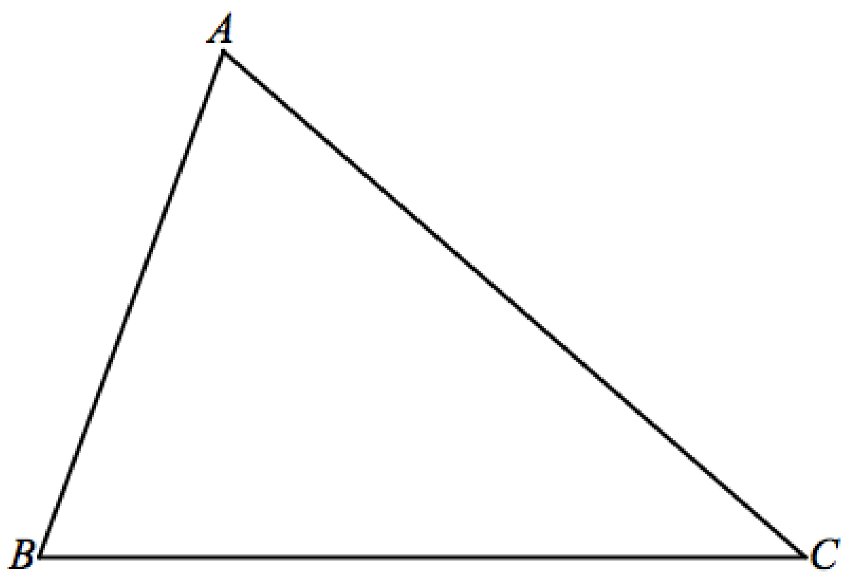
Lesson 29: Special Lines in Triangles

Classwork

Opening Exercise

Construct the midsegment of the triangle below. A midsegment is a line segment that joins the midpoints of two sides of a triangle or trapezoid. For the moment, we will work with a triangle.

1. Use your compass and straightedge to determine the midpoints of $\overline{AB}$ and $\overline{AC}$ as X and Y, respectively.
2. Draw midsegment $\overline{XY}$.



Compare $\angle AXY$ and $\angle ABC$; compare $\angle AYX$ and $\angle ACB$. Without using a protractor, what would you guess is the relationship between these two pairs of angles? What are the implications of this relationship?

Discussion

Note that though we chose to determine the midsegment of $\overline{AB}$ and $\overline{AC}$, we could have chosen any two sides to work with. Let us now focus on the properties associated with a midsegment.

The midsegment of a triangle is parallel to the third side of the triangle and half the length of the third side of the triangle.

We can prove these properties to be true. You will continue to work with the figure from the Opening Exercise.

Given: $\overline{XY}$ is a midsegment of $\triangle ABC$

Prove: $\overline{XY} \parallel \overline{BC}$ and $XY = \frac{1}{2}BC$

Construct the following: In the Opening Exercise figure, draw triangle $\triangle YGC$ according to the following steps. Extend $\overline{XY}$ to point G so that $YG = XY$. Draw $\overline{GC}$.

(1) What is the relationship between XY and YG? Explain why. ______________________________

____

(2) What is the relationship between $\angle AYX$ and $\angle GYC$? Explain why. ______________________________

____

(3) What is the relationship between $\overline{AY}$ and $\overline{YC}$? Explain why. ______________________________

____

(4) What is the relationship between $\triangle AXY$ and $\triangle CGY$? Explain why. ______________________________

____

(5) What is the relationship between GC and AX? Explain why. ______________________________

____

(6) Since $AX = BX$, what other conclusion can be drawn? Explain why. ______________________________

____

(7) What is the relationship between $\angle AXY$ and $\angle YGC$? Explain why. ______________________________

____

(8) Based on (7), what other conclusion can be drawn about $\overline{AB}$ and $\overline{GC}$? Explain why. ______________________________

____

(9) What conclusion can be drawn about $BXGC$ based on (7) and (8)? Explain why. ______________________________

____

(10) Based on (9), what is the relationship between XG and BC? ______________________________

____

(11) Since $YG = XY$, $XG =$ _______ XY. Explain why. ______________________________

____

(12) This means $BC =$ ______ XY. Explain why. ______________________________

____

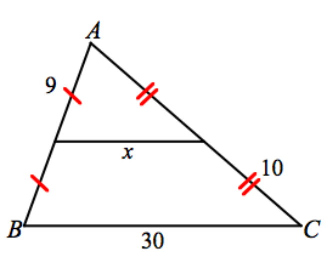
(13) Or by division, $XY =$_____ BC.

Note that steps (9) and (13) demonstrate our 'Prove' statement.

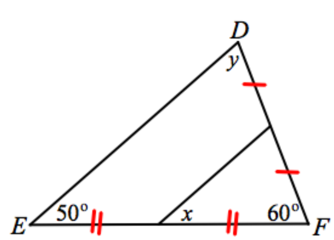
Exercises 1–4

Apply what you know about the properties of midsegments to solve the following examples.

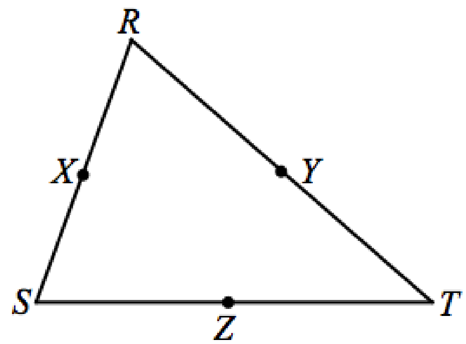
1. $x =$ ___________
   Perimeter of $\triangle ABC =$ __________



2. $x =$ _______
   $y =$ __________



3. In $\triangle RST$, the midpoints of each side have been marked by points X, Y, and Z.
   - Mark the halves of each side divided by the midpoint with a congruency mark. Remember to distinguish congruency marks for each side.
   - Draw midsegments XY, YZ, and XZ. Mark each midsegment with the appropriate congruency mark from the sides of the triangle.



   a. What conclusion can you draw about the four triangles within $\triangle RST$? Explain Why. ______________________

   ____

   b. State the appropriate correspondences among the four triangles within $\triangle RST$. ___________________________

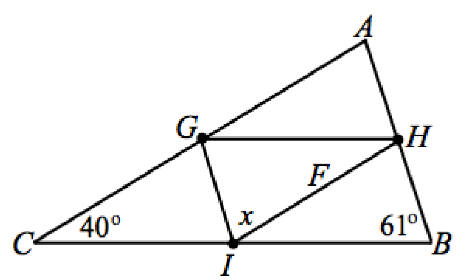
   ____

   c. State a correspondence between $\triangle RST$ and any one of the four small triangles. ________________________

   ____

4. Find x.

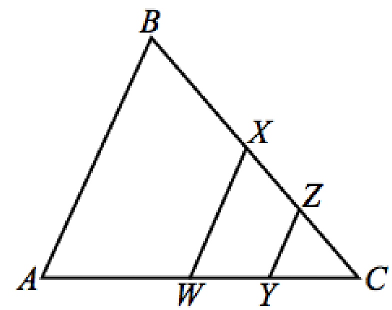
   $x =$ ________



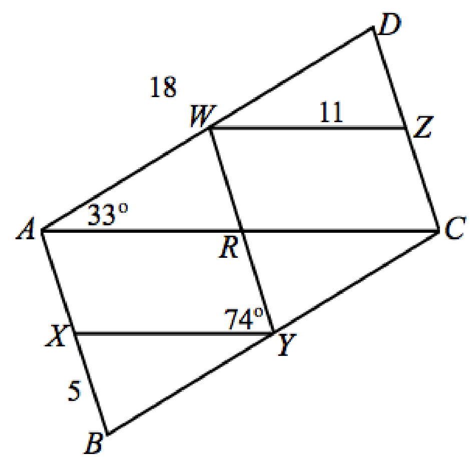
Problem Set

Use your knowledge of triangle congruence criteria to write proofs for each of the following problems.

1. $\overline{WX}$ is a midsegment of $\triangle ABC$, and $\overline{YZ}$ is a midsegment of $\triangle CWX$. $BX = AW$.
   a. What can you conclude about $\angle A$ and $\angle B$? Explain why.
   b. What is the relationship in length between $\overline{YZ}$ and $\overline{AB}$?



2. W, X, Y, and Z are the midpoints of $\overline{AD}, \overline{AB}, \overline{BC}$, and $\overline{CD}$ respectively. $AD = 18$, $WZ = 11$, and $BX = 5$. $m\angle WAC = 33°$, $m\angle RYX = 74°$.
   a. $\angle DZW =$ ___________
   b. Perimeter of $ABYW =$ ___________
   c. Perimeter of $ABCD =$ ___________
   d. $m\angle WAX =$ ___________
      $m\angle B =$ ___________
      $m\angle YCZ =$ ___________
      $m\angle D =$ ___________
   e. What kind of quadrilateral is $ABCD$?

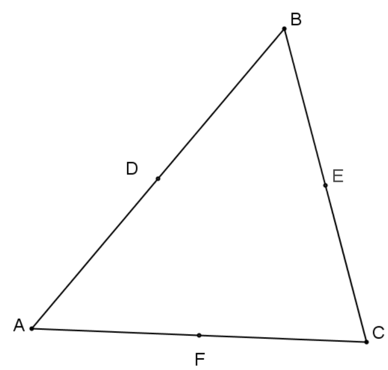


Lesson 30: Special Lines in Triangles

Classwork

Opening Exercise

In $\triangle ABC$ at the right, D is the midpoint of $\overline{AB}$; E is the midpoint of $\overline{BC}$, and F is the midpoint of $\overline{AC}$. Complete each statement below.



$\overline{DE}$ is parallel to ________ and measures ________ the length of ________.

$\overline{DF}$ is parallel to ________ and measures ________ the length of ________.

$\overline{EF}$ is parallel to ________ and measures ________ the length of ________.

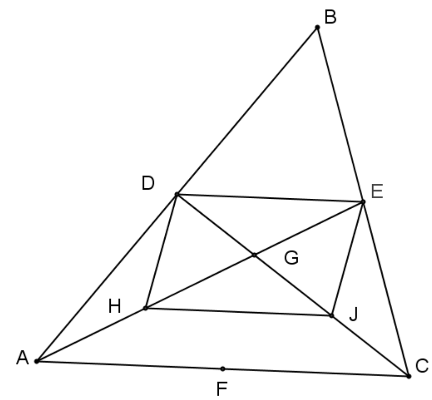
Discussion

In the previous two lessons, we proved that (a) the midsegment of a triangle is parallel to the third side and half the length of the third side and (b) diagonals of a parallelogram bisect each other. We use both of these facts to prove the following assertion:

All medians of a triangle are ____________________. That is, the three medians of a triangle (the segments connecting each vertex to the midpoint of the opposite side) meet at a single point. This point of concurrency is called the ______________________, or the center of gravity, of the triangle. The proof will also show a length relationship for each median: The length from the vertex to the centroid is ________________ the length from the centroid to the midpoint of the side.

Example 1

Provide a valid reason for each step in the proof below.



Given: $\triangle ABC$ with D, E, and F the midpoints of sides AB, BC, and AC, respecti

Prove: The three medians of $\triangle ABC$ meet at a single point.

(1) Draw AE and DC; label their intersection as point G.

(2) Construct and label the midpoint of AG as point H and the midpoint of GC as point J.

(3) $DE \parallel AC$,

___

(4) $HJ \parallel AC$,

___

(5) $DE \parallel HJ$,

___

(6) $DE = \frac{1}{2}AC$ and $HJ = \frac{1}{2}AC$,

___

(7) $DEJH$ is a parallelogram,

___

(8) $HG = EG$ and $JG = DG$,

___

(9) $AH = HG$ and $CJ = JG$,

___

(10) $AH = HG = GE$ and $CJ = JG = GD$,

___

(11) $AG = 2GE$ and $CG = 2GD$,

____

(12) We can complete steps (1)–(11) to include the median from B; the third median, $\overline{BF}$, passes through point G, which divides it into two segments such that the longer part is twice the shorter.

(13) The intersection point of the medians divides each median into two parts with lengths in a ratio of 2:1; therefore, all medians are concurrent at that point.

The three medians of a triangle are concurrent at the ____________________, or the center of gravity. This point of concurrency divides the length of each median in a ratio of ______________; the length from the vertex to the centroid is __________________ the length from the centroid to the midpoint of the side.

Example 2

In the figure to the right, $DF = 4$, $BF = 16$, $AG = 30$. Find each of the following measures.

a. $FC =$ ____________________

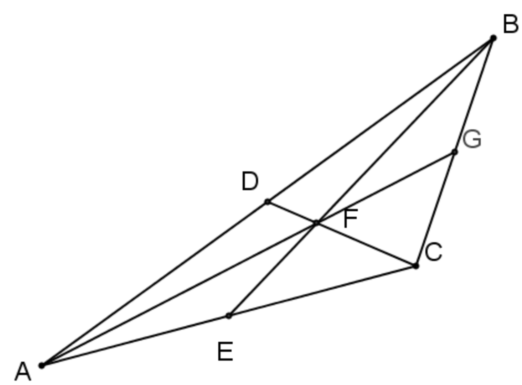
b. $DC =$ ____________________

c. $AF =$ ____________________

d. $BE =$ ____________________

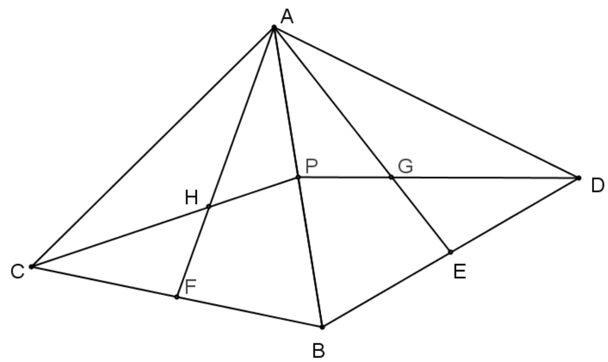
e. $FG =$ ____________________

f. $EF =$ ____________________



Example 3

In the figure to the right, $\triangle ABC$ is reflected over $\overline{AB}$ to create $\triangle ABD$. Points P, E, and F are midpoints of $\overline{AB}$, $\overline{BD}$, and $\overline{BC}$, respectively. If $AH = AG$, prove that $PH = GP$.



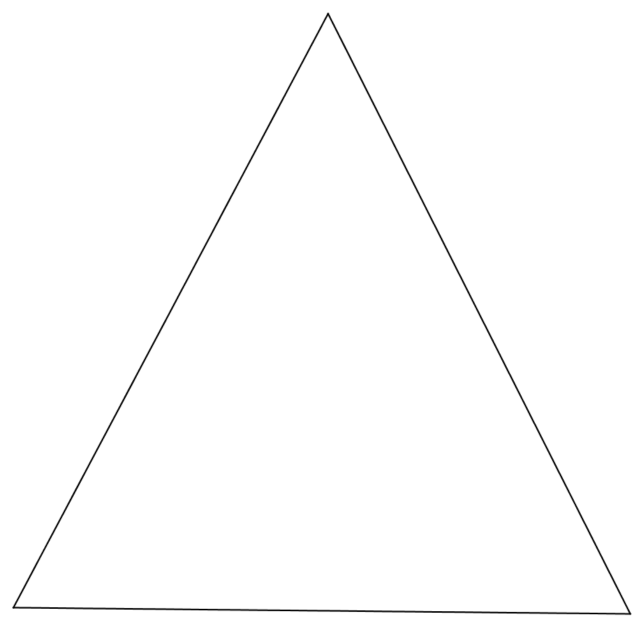
Problem Set

Ty is building a model of a hang glider using the template below. To place his supports accurately, Ty needs to locate the center of gravity on his model.

1. Use your compass and straightedge to locate the center of gravity on Ty's model.

2. Explain what the center of gravity represents on Ty's model.

3. Describe the relationship between the longer and shorter sections of the line segments you drew as you located the center of gravity.



Lesson 31: Construct a Square and a Nine-Point Circle

Classwork

Opening Exercise

With a partner, use your construction tools and what you learned in Lessons 1–5 to attempt the construction of a square. Once you are satisfied with your construction, write the instructions to perform the construction.

Exploratory Challenge

Now, we are going to construct a nine-point circle. What is meant by the phrase "nine-point circle"?

___

___

Steps to construct a nine-point circle:

1. Draw a triangle $\triangle ABC$.
2. Construct the midpoints of the sides $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, and label them as L, M, and N, respectively.
3. Construct the perpendicular from each vertex to the opposite side of the triangle (each is called an altitude).
4. Label the intersection of the altitude from C to $\overline{AB}$ as D, the intersection of the altitude from A to $\overline{BC}$ as E, and of the altitude from B to $\overline{CA}$ as F.
5. The altitudes are concurrent at a point, label it H.
6. Construct the midpoints of $\overline{AH}$, $\overline{BH}$, $\overline{CH}$ and label them X, Y, and Z, respectively.
7. The nine points, L, M, N, D, E, F, X, Y, Z, are the points that define the nine-point circle.

Example

On a blank white sheet of paper, construct a nine-point circle using a different triangle than you used during the notes. Does the type of triangle you start with affect the construction of the nine-point circle?

Problem Set

Construct square $ABCD$ and square $GHIJ$ so that

a. Each side of $GHIJ$ is half the length of each $ABCD$.

b. $\overline{AB}$ contains $\overline{GH}$.

c. The midpoint of $\overline{AB}$ is also the midpoint of $\overline{GH}$.

Lesson 32: Construct a Nine-Point Circle

Classwork

Opening Exercise

During this unit we have learned many constructions. Now that you have mastered these constructions write a list of advice for someone who is about to learn the constructions you have learned for the first time. What did and did not help you? What tips did you wish you had at the beginning that would have made it easier along the way?

Exploratory Challenge 1

Yesterday, we began the nine-point circle construction. What did we learn about the triangle that we start our construction with? Where did we stop in the construction?

We will continue our construction today.

There are two constructions for finding the center of the nine-point circle. With a partner, work through both constructions.

Construction 1

1. To find the center of the circle, draw inscribed $\triangle LMN$.
2. Find the circumcenter of $\triangle LMN$, and label it as U.

Recall that the circumcenter of a triangle is the center of the circle that circumscribes the triangle, which in this case, is the nine-point circle.

Construction 2

1. Construct the circle that circumscribes $\triangle ABC$.
2. Find the circumcenter of $\triangle ABC$, which is the center of the circle that circumscribes $\triangle ABC$. Label its center CC.
3. Draw the segment that joins point H (the orthocenter from the construction of the nine-point circle in Lesson 31) to the point CC.
4. Find the midpoint of the segment you drew in Step 3, and label that point U.

Describe the relationship between the midpoint you found in Step 4 of the second construction and the point U in the first construction.

Exploratory Challenge 2

Construct a square $ABCD$. Pick a point E between B and C, and draw a segment from point A to a point E. The segment forms a right triangle and a trapezoid out of the square. Construct a nine-point circle using the right triangle.

Problem Set

Take a blank sheet of $8\frac{1}{2}$ inch by 11 inch white paper and draw a triangle with vertices on the edge of the paper. Construct a nine-point circle within this triangle. Then draw a triangle with vertices on that nine-point circle, and construct a nine-point circle within that. Continue constructing nine-point circles until you no longer have room inside your constructions.

Lesson 33: Review of the Assumptions

Classwork

Review Exercises

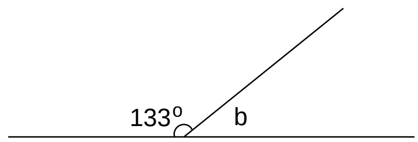
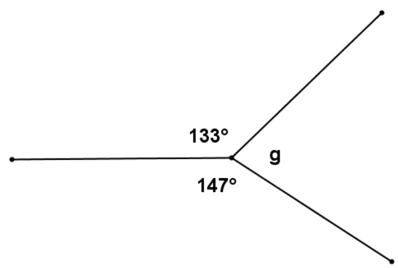
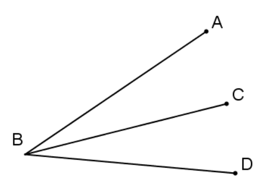
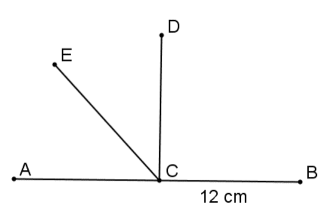
We have covered a great deal of material in Module 1. Our study has included definitions, geometric assumptions, geometric facts, constructions, unknown angle problems and proofs, transformations, and proofs that establish properties we previously took for granted.

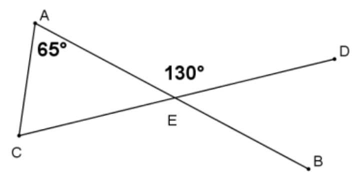
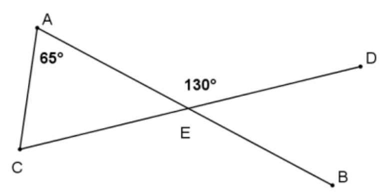
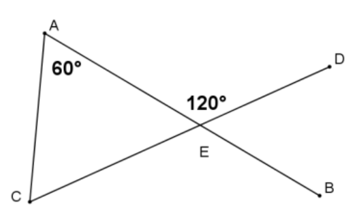
In the first list below, we compile all of the geometric assumptions we took for granted as part of our reasoning and proof-writing process. Though these assumptions were only highlights in lessons, these assumptions form the basis from which all other facts can be derived (e.g., the other facts presented in the table). College-level geometry courses often do an in-depth study of the assumptions.

The latter tables review the facts associated with problems covered in Module 1. Abbreviations for the facts are within brackets.

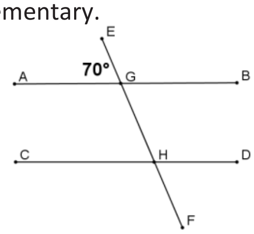
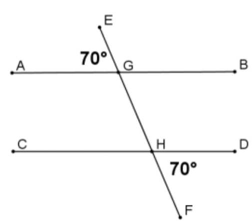
Geometric Assumptions (Mathematicians call these "Axioms.")

1. (Line) Given any two distinct points, there is exactly one line that contains them.
2. (Plane Separation) Given a line contained in the plane, the points of the plane that do not lie on the line form two sets, called half-planes, such that
   a. Each of the sets is convex,
   b. If P is a point in one of the sets and Q is a point in the other, then $\overline{PQ}$ intersects the line.
3. (Distance) To every pair of points A and B there corresponds a real number dist $(A, B) \geq 0$, called the distance from A to B, so that
   a. $\text{dist}(A, B) = \text{dist}(B, A)$.
   b. $\text{dist}(A, B) \geq 0$, and $\text{dist}(A, B) = 0 \Leftrightarrow A$ and B coincide.
4. (Ruler) Every line has a coordinate system.
5. (Plane) Every plane contains at least three non-collinear points.
6. (Basic Rigid Motions) Basic rigid motions (e.g., rotations, reflections, and translations) have the following properties:
   a. Any basic rigid motion preserves lines, rays, and segments. That is, for any basic rigid motion of the plane, the image of a line is a line, the image of a ray is a ray, and the image of a segment is a segment.
   b. Any basic rigid motion preserves lengths of segments and angle measures of angles.
7. (180° Protractor) To every $\angle AOB$, there corresponds a real number $\text{m}\angle AOB$, called the degree or measure of the angle, with the following properties:
   a. $0° < \text{m}\angle AOB < 180°$.
   b. Let $\overrightarrow{OB}$ be a ray on the edge of the half-plane H. For every r such that $0° < r < 180°$, there is exactly one ray $\overrightarrow{OA}$ with A in H such that $\text{m}\angle AOB = r°$.
   c. If C is a point in the interior of $\angle AOB$, then $\text{m}\angle AOC + \text{m}\angle COB = \text{m}\angle AOB$.
   d. If two angles $\angle BAC$ and $\angle CAD$ form a linear pair, then they are supplementary, e.g., $\text{m}\angle BAC + \text{m}\angle CAD = 180°$
8. (Parallel Postulate) Through a given external point, there is at most one line parallel to a given line.

Fact/Property	Guiding Questions/Applications	Notes/Solutions
Two angles that form a linear pair are supplementary.		
The sum of the measures of all adjacent angles formed by three or more rays with the same vertex is $360°$.		
Vertical angles have equal measure.	Use the fact that linear pairs form supplementary angles to prove that vertical angles are equal in measure.	
The bisector of an angle is a ray in the interior of the angle such that the two adjacent angles formed by it have equal measure.	In the diagram below, $\overline{BC}$ is the bisector of $\angle ABD$, which measures $64°$. What is the measure of $\angle ABC$? 	
The perpendicular bisector of a segment is the line that passes through the midpoint of a line segment and is perpendicular to the line segment.	In the diagram below, $\overline{DC}$ is the $\perp$ bisector of $\overline{AB}$, and $\overline{CE}$ is the angle bisector of $\angle ACD$. Find the measures of $\overline{AC}$ and $\angle ECD$. 	

The sum of the 3 angle measures of any triangle is $180°$.	Given the labeled figure below, find the measures of $\angle DEB$ and $\angle ACE$. Explain your solutions. 	
When one angle of a triangle is a right angle, the sum of the measures of the other two angles is $90°$.	This fact follows directly from the preceding one. How is simple arithmetic used to extend the angle sum of a triangle property to justify this property?	
An exterior angle of a triangle is equal to the sum of its two opposite interior angles.	In the diagram below, how is the exterior angle of a triangle property proved? 	
Base angles of an isosceles triangle are congruent.	The triangle in the figure above is isosceles. How do we know this?	
All angles in an equilateral triangle have equal measure. [equilat. $\triangle$]	If the figure above is changed slightly, it can be used to demonstrate the equilateral triangle property. Explain how this can be demonstrated. 	

The facts and properties in the immediately preceding table relate to angles and triangles. In the table below, we will review facts and properties related to parallel lines and transversals.

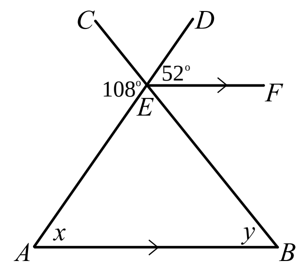
Fact/Property	Guiding Questions/Applications	Notes/Solutions
If a transversal intersects two parallel lines, then the measures of the corresponding angles are equal.	Why does the property specify parallel lines?	
If a transversal intersects two lines such that the measures of the corresponding angles are equal, then the lines are parallel.	The converse of a statement turns the relevant property into an *if and only if* relationship. Explain how this is related to the guiding question about corresponding angles.	
If a transversal intersects two parallel lines, then the interior angles on the same side of the transversal are supplementary.	This property is proved using (in part) the corresponding angles property. Use the diagram below ($\overline{AB} \parallel \overline{CD}$) to prove that $\angle AGH$ and $\angle CHG$ are supplementary. 	
If a transversal intersects two lines such that the same side interior angles are supplementary, then the lines are parallel.	Given the labeled diagram below, prove that $\overline{AB} \parallel \overline{CD}$. 	
If a transversal intersects two parallel lines, then the measures of alternate interior angles are equal.	1. Name both pairs of alternate interior angles in the diagram above. 2. How many different angle measures are in the diagram?	
If a transversal intersects two lines such that measures of the alternate interior angles are equal, then the lines are parallel.	Although not specifically stated here, the property also applies to alternate exterior angles. Why is this true?	

Problem Set

Use any of the assumptions, facts, and/or properties presented in the tables above to find x and y in each figure below. Justify your solutions.

1. x =

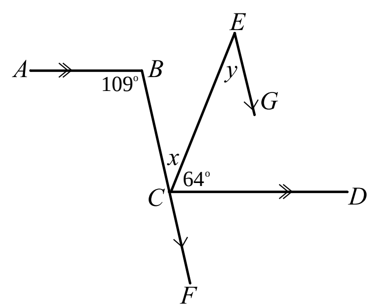
   y =



2. You will need to draw an auxiliary line to solve this problem.

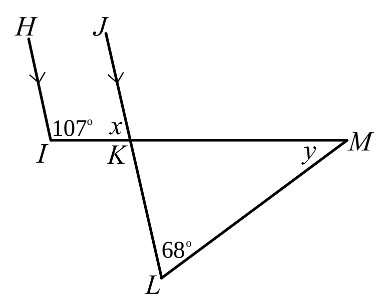
   x =

   y =

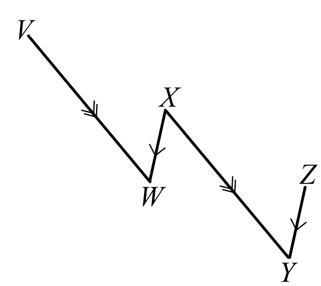


3. x =

   y =

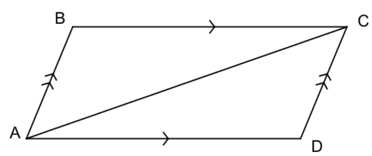
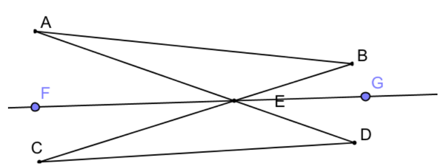
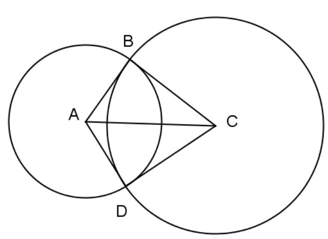


4. Given the labeled diagram at the right, prove that $\angle VWX \cong \angle XYZ$.



Lesson 34: Review of the Assumptions

Classwork

Assumption/Fact/Property	Guiding Questions/Applications	Notes/Solutions
Given two triangles $\triangle ABC$ and $\triangle A'B'C'$ so that $AB = A'B'$ (Side), $\text{m}\angle A = \text{m}\angle A'$ (Angle), $AC = A'C'$(Side), then the triangles are congruent. [SAS]	The figure below is a parallelogram $ABCD$. What parts of the parallelogram satisfy the SAS triangle congruence criteria for $\triangle ABD$ and $\triangle CDB$? Describe a rigid motion(s) that will map one onto the other. (Consider drawing an auxiliary line.)  	
Given two triangles $\triangle ABC$ and $\triangle A'B'C'$, if $\text{m}\angle A = \text{m}\angle A'$ (Angle), $AB = A'B'$ (Side), and $\text{m}\angle B = \text{m}\angle B'$ (Angle), then the triangles are congruent. [ASA]	In the figure below, $\triangle CDE$ is the image of the reflection of $\triangle ABE$ across line FG. Which parts of the triangle can be used to satisfy the ASA congruence criteria?  	
Given two triangles $\triangle ABC$ and $\triangle A'B'C'$, if $AB = A'B'$ (Side), $AC = A'C'$ (Side), and $BC = B'C'$ (Side), then the triangles are congruent. [SSS]	$\triangle ABC$ and $\triangle ADC$ are formed from the intersections and center points of circles A and C. Prove $\triangle ABC \cong \triangle ADC$ by SSS.  	

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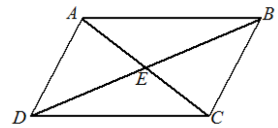
<td>Given two triangles, $\triangle ABC$ and $\triangle A'B'C'$, if $AB = A'B'$ (Side), $\mathrm{m}\angle B = \mathrm{m}\angle B'$ (Angle), and $\angle C = \angle C'$ (Angle), then the triangles are congruent.

[AAS]</td>
<td>The AAS congruence criterion is essentially the same as the ASA criterion for proving triangles congruent. Why is this true?

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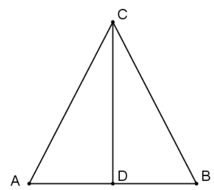
<td>Given two right triangles $\triangle ABC$ and $\triangle A'B'C'$ with right angles $\angle B$ and $\angle B'$, if $AB = A'B'$ (Leg) and $AC = A'C'$ (Hypotenuse), then the triangles are congruent.

[HL]</td>
<td>In the figure below, CD is the perpendicular bisector of AB and $\triangle ABC$ is isosceles. Name the two congruent triangles appropriately, and describe the necessary steps for proving them congruent using HL.

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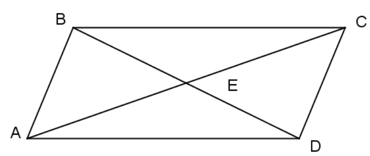
<td>The opposite sides of a parallelogram are congruent.</td>
<td rowspan="3">In the figure below, $BE \cong DE$ and $\angle CBE \cong \angle ADE$. Prove $ABCD$ is a parallelogram.

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<td>The opposite angles of a parallelogram are congruent.</td>
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<td>The diagonals of a parallelogram bisect each other.</td>
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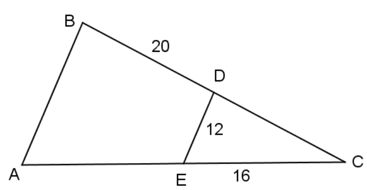
<td>The midsegment of a triangle is a line segment that connects the midpoints of two sides of a triangle; the midsegment is parallel to the third side of the triangle and is half the length of the third side.</td>
<td>$\overline{DE}$ is the midsegment of $\triangle ABC$. Find the perimeter of $\triangle ABC$, given the labeled segment lengths.

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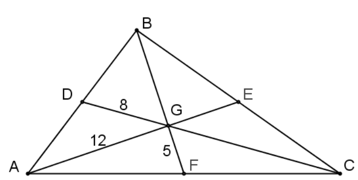
<td>The three medians of a triangle are concurrent at the centroid; the centroid divides each median into two parts, from vertex to centroid and centroid to midpoint in a ratio of 2: 1.</td>
<td>If $\overline{AE}$, $\overline{BF}$, and $\overline{CD}$ are medians of $\triangle ABC$, find the lengths of segments BG, GE, and CG, given the labeled ler

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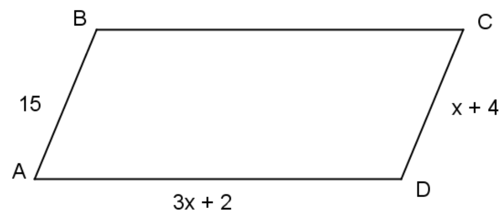
Problem Set

Use any of the assumptions, facts, and/or properties presented in the tables above to find x and/or y in each figure below. Justify your solutions.

1. Find the perimeter of parallelogram $ABCD$. Justify your solution.

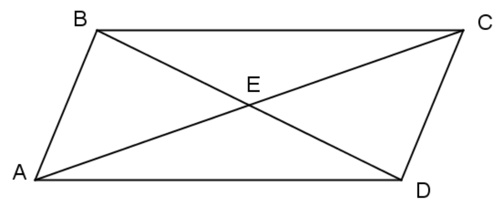


2. $AC = 34$

   $AB = 26$

   $BD = 28$

   Given parallelogram $ABCD$, find the perimeter of $\triangle CED$. Justify your solution.

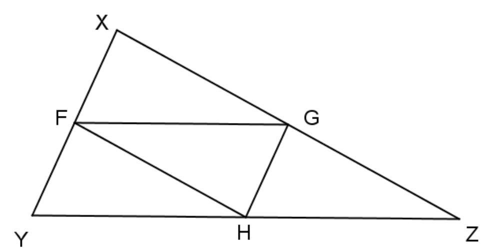


3. $XY = 12$

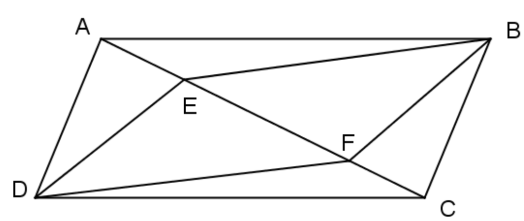
   $XZ = 20$

   $ZY = 24$

   F, G, and H are midpoints of the sides on which they are located. Find the perimeter of $\triangle FGH$. Justify your solution.



4. $ABCD$ is a parallelogram with $AE = CF$. Prove that $DEBF$ is a parallelogram.



5. C is the centroid of $\triangle RST$. $RC = 16$, $CL = 10$, $TJ = 21$

   $SC =$ ___________

   $TC =$ ___________

   $KC =$ ___________