# How to Implement *A Story of Units*

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# Introduction

The Common Core Learning Standards (CCLS) in mathematics contain challenging new expectations for students and teachers. Meeting these standards demands great mathematics content, a strong emphasis on meaningful assessment and data-driven instruction, and exciting, learning-friendly classroom experiences.

*A Story of Units* interweaves these three components into a clearly sequenced and comprehensive educational program. Coupled with effective and replicable professional development, *A Story of Units* provides New York teachers with the knowledge and tools necessary to implement the important instructional shifts required by the CCLS.

First, *A Story of Units* tells the unfolding story of mathematics as expressed in the standards—lesson by lesson, throughout each grade and over the student's entire school career. The story draws together what are, at first glance, disparate ideas of different quantities as it emphasizes key themes: the creation and manipulation of, as well as the relationship between units.

Second, there is a major focus on meaningful assessment. Well-designed materials quickly and accurately identify student misconceptions and misunderstandings around content. Because they promote self-monitoring and self-improvement, assessment tools also serve as learning devices for students and are thus essential for creating effective student-teacher partnerships. The teacher that used to say, "I taught the standard to my students and I hope they understand it," can now assert, "My students and I collaborated on that standard until they understood it. Here, let them show you how they've mastered it."

The third key feature of *A Story of Units* is an engaging lesson structure that helps teachers lead students through fast-paced practice, encourage perseverance, and foster thoughtful development of understanding. The structure helps the teachers focus their energy on engaging students in the mathematical story through the introduction of challenging problems that call for quantitative and creative thinking. At the same time, they provide puzzle-solving tools and models and help students identify patterns so common that students' understanding of subjects like algebra extends naturally from their knowledge of numbers.

# I. The Common Core Approach to Mathematics

*A Story of Units* has the Common Core Learning Standards as its foundation and has been developed in close consultation with seminal documents, also drafted by the Common Core State Standards Writing Team, entitled Progressions for the Common Core State Standards in Mathematics. Actually a series of papers, these "progressions documents" lay out the structure of the mathematics and research in cognitive development that was the frame upon which the Common Core State Standards were built. Teachers should be aware that the authors of *A Story of Units* studied these documents deeply and that the resulting curriculum is not only built entirely around the Common Core Learning Standards but the progressions documents as well.

The curricular design for *A Story of Units* is based on the principle that mathematics is most effectively taught as a logical, engaging story. At the elementary level, the story's main character is the basic building block of arithmetic, or the unit. Themes like measurement, place value, and fractions run throughout the storyline, and each is given the amount of time proportionate to its role in the overall story. The story climaxes when students learn to add, subtract, multiply, and divide fractions; and solve multistep word problems with multiplicative and additive comparisons.

Few U.S. textbooks paint mathematics as a dynamic, unfolding tale. They instead prioritize teaching procedures and employ a spiraling approach, in which topics are partially taught and then returned to—sometimes years later—with the unrealistic expectation that students will somehow connect the dots. But teaching procedures as skills without a rich context is ineffective. Students can too easily forget procedures and will fail if they do not have deeper, more concrete knowledge from which they can draw.

## The Significance of the Unit

Even as new concepts are introduced, the overarching theme remains: defining the basic building block, the unit. Studying, relating, manipulating, and converting the unit allows students to add, subtract, complete word problems, multiply, divide, and understand concepts like place value, fractions, measurements, area, and volume. Students learn that unit-based procedures are transferable and can thus build upon their knowledge in new ways. The following progressions demonstrate how the curriculum moves from the introductory structures of addition, through place value and multiplication, to operations with fractions and beyond.

**Numbers through 10**

Initially in Pre-Kindergarten and Kindergarten, one object is one unit. *"Let's count the frogs! The frog is one unit: 1 frog, 2 frogs, 3 frogs..."* Students then relate numbers to each other and to 5 and 10. For example, in building a growth pattern (a stair-shaped structure) of unit cubes representing each number to 10 and having a color change at 5, students see that 7 is 1 unit more than 6, 1 less than 8, 2 more than 5, and 3 less than 10. That is, it can be broken into 1 and 6, 2 and 5. 7 can be a unit to manipulate (*"I can break it apart into 3 and 4."*), to be formed (*"I can add 1 more to 6."*), and related (*"It needs 3 more to be 10."*).

**Addition and Subtraction**

For example, in order to add 8 and 6, students form a unit of ten and add the remainder: $8 + 6 = 8 + (2 + 4) = (8 + 2) + 4 = 10 + 4 = 14$. They extend that skill by adding 18 + 6, 80 and 60, 800 kg + 600 kg, and 8 ninths + 6 ninths. This idea is easily transferable to more complex "units." Adding mixed units (e.g., 2 dogs 4 puppies + 3 dogs 5 puppies) means adding like units just as in 2 tens 4 ones + 3 tens 5 ones, 2 feet 4 inches + 3 feet 5 inches, 2 hours 4 minutes + 3 hours 50 minutes, etc.

**Place Value and the Standard Algorithms**

With regard to this overarching theme, the place value system is an organized, compact way to write numbers using *place value units* that are powers of 10: ones, tens, hundreds, etc. Explanations of all standard algorithms hinge upon the manipulation of these place value units and the relationships between them (e.g., 10 tens = 1 hundred).

**Multiplication**

One of the earliest, easiest methods of forming a new unit is by creating groups of another unit. Kindergarten students take a stick of 10 linker cubes and break it into twos. *"How many cubes are in your stick?" "10!" "Break it and make twos." "Count your twos with me, "1 two, 2 twos, 3 twos… We made 5 twos!"* Groups of 4 apples, for example, can be counted: 1 four, 2 fours, 3 fours, etc. Relating the new unit to the original unit develops the idea of multiplication: 3 groups of 4 apples equal 12 apples (or 3 fours are 12). Manipulating the new unit brings out other relationships: 3 fours + 7 fours = 10 fours or $(3 \times 4) + (7 \times 4) = (3 + 7) \times 4$.

**Fractions**

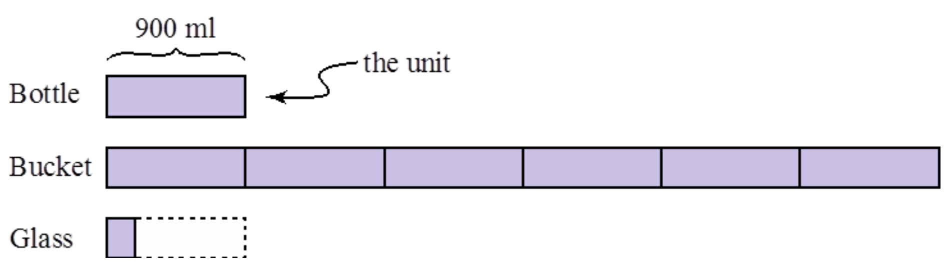
Forming fractional units is exactly the same as the procedure for multiplication, but the "group" can now be the amount when a whole unit is subdivided equally: A segment of length 1 can be subdivided into 4 segments of equal length, each representing the unit 1 fourth. The new unit can then be counted and manipulated just like whole numbers: 3 fourths + 7 fourths = 10 fourths.

**Word Problems**

Forming units to solve word problems is one of the most powerful examples of this overarching theme. The following situation states:

*Each bottle holds 900 ml of water.*
*A bucket holds 6 times as much water as a bottle.*
*A glass holds 1/5 as much water as a bottle.*

We can use the bottle capacity to form a unit pictorially and illustrate the other quantities with relationship to that unit:




The unit can then be used to answer word problems about this situation, such as, *"How much more does the bucket hold than 4 bottles?"* (2 units or 1800 ml)

Once the units are established and defined, the task is simply manipulating them via arithmetic. With this repetition of prior experiences, the student realizes that he has seen this before.

## How *A Story of Units* Aligns with the Instructional Shifts

*A Story of Units* is structured around the essential instructional shifts needed to implement the CCLS. These principles, articulated as six shifts (focus, coherence, fluency, deep understanding, application and dual intensity) by the New York State Department of Education and reorganized as three (focus, coherence and rigor) by Student Achievement Partners and the Publishers Criteria, help educators understand what is required to implement the necessary changes. *A Story of Units* reflects the grouping of three instructional shifts most essential to teaching the CCLS: focus, coherence and rigor.[1] Rigor involves the additional shifts of fluency, and conceptual understanding, and application—and all three are done with a dual intensity emphasis on practicing and understanding. All three instructional shifts are required to teach the CCLS.[2]

### Shift 1: Focus—"…focus deeply on only the concepts that are prioritized in the standards..."

*A Story of Units* follows the focus of the CCLS by relating every arithmetic idea back to understanding the idea of a unit: What the definition of the unit is in particular cases (whole numbers, fractions, decimals, measurements, etc.), commonalities between all units (they can be added, subtracted, multiplied, etc.), and the unique features of some units (e.g., a rectangle's area units, as opposed to its length units, can be calculated quickly by multiplying length measurements of the rectangle).

It is the study of the commonalities between units that drives the focus of *A Story of Units*, so that the concepts learned are prioritized by the CCLS. The commonalities form the interconnectedness of the math concepts and enable students to more easily transfer their mathematical skills and understanding across grades. Perhaps surprisingly, it is also the focused study of the commonalities between types of units that makes the contrast between different types of units more pronounced. That is, by understanding the commonalities between types of units, students develop their ability to compare and contrast the types of units. The focus drives an understanding of the commonalities and the differences in the ways that arithmetic can be used to manipulate numbers.

Evidence of focus is further seen in the integral use of the NY Math Content Emphases to focus on the major work of the grade level. Each module begins with the Focus Grade Level Standards clearly stating the clusters of standards that are emphasized in the material. As noted in the Publishers Criteria, approximately three-quarters of the work is on the major clusters where students should be most fluent. Supporting clusters are interwoven as connecting components in core understanding while additional clusters introduce other key ideas.

### Shift 2: Coherence—"Principals and teachers carefully connect the learning within and across grades so that… students can build new understanding onto foundations built in previous years."

*A Story of Units* is not a collection of topics. Rather, the modules and topics in the curriculum are woven

[1] See www.corestandards.org, Publishers' Criteria for the CCSS in Mathematics, Grades K-8.
[2] Originally, www.engageNY.org listed 6 instructional shifts that were later combined into the 3 shifts described here. The conversion can be found at: http://schools.nyc.gov/NR/rdonlyres/B6435B3C-0FB5-4C2E-B932-083D1A5F8585/0/ShiftsCrosswalkMath.doc

through the progressions of the CCLS. *A Story of Units* carefully prioritizes and sequences the standards with a deliberate emphasis on mastery of major cluster standards outlined in the CCLS. This meticulous sequencing enables students, upon completion of each module, to transfer their mathematical knowledge and understanding to new, increasingly challenging concepts.

Module overview charts show how topics are aligned with standards to create an instructional sequence that is organized precisely to build on previous learning and to support future learning.

The teaching sequence chart for each topic outlines the instructional path by stating the learning objectives for each lesson. The sequence of problems in the material is structured to help teachers analyze the mathematics for themselves and to help them with differentiated instruction: As students advance from simple concepts to more complex concepts, the different problems provide opportunities for teachers to either (a) break problems down for students struggling with a next step, or (b) stretch problems out for those hungry for greater challenges.

Coherence is further supported through the use of a finite set of concrete and pictorial models. As a result, students develop increasing familiarity with this limited set of consistently used models over the years. In second grade, for example, students use number disks to represent place value; that model remains constant through the third, fourth, and fifth grades. As new ideas are introduced, the consistent use of the same model leads to more rapid and deeper understanding of new concepts.

## Shift 3: Rigor—"…pursue, with equal intensity, three aspects of rigor in the major work of each grade: conceptual understanding, procedural skill and fluency, and applications."

The three-pronged nature of rigor undergirds a main theme of the Publishers' Criteria. Fluency, deep understanding, and application with equal intensity **must drive** instruction for students to meet the Standards' rigorous expectations.

**Fluency—"Students are expected to have speed and accuracy with simple calculations; teachers structure class time and/or homework time for students to memorize, through repetition, core functions..."**

Fluency represents a major part of the instructional vision that shapes *A Story of Units*; it is a daily, substantial, and sustained activity. One or two fluencies are required for each grade level and fluency suggestions are included in most lessons. Implementation of effective fluency practice is supported by the lesson structure. (See Fluency in the Lesson Structure below.)

Fluency tasks are strategically designed for the teacher to easily administer and assess. A variety of suggestions for fluency activities—including mental math activities, interactive drills, quick and efficient games with dice, spinners, and cards, and concept worksheets—are offered. Throughout the school year, such activities can be used with new material to strengthen skills and enable students to see their accuracy and speed increase measurably each day.

**Conceptual Understanding—"Students deeply understand and can operate easily within a math concept before moving on. They learn more than the trick to get the answer right. They learn the math."**

Conceptual understanding requires far more than performing discrete and often disjointed procedures to determine an answer. Students must not only learn mathematical content, they must also be able to access that knowledge from numerous vantage points and communicate about the process. In *A Story of Units*, students use writing and speaking to solve mathematical problems, reflect on their learning, and analyze their thinking. The lessons and homework require students to write their solutions to word problems several times a week. Thus, students learn to express their understanding of concepts and articulate their thought processes through writing. Similarly, students participate in daily debriefs and learn to verbalize the patterns and connections between the current lesson and their previous learning, in addition to listening to and debating their peers' perspectives. The goal is to interweave the learning of new concepts with reflection time into students' everyday math experience.

At the module level, *sequence is everything*. Standards within the module and modules across the year carefully build to ensure that students have the requisite understanding to fully access new learning goals and integrate them into their developing schemas of understanding. The very deliberate progression of the material follows the critical instructional areas outlined in the introduction of the CCLS for each grade.

**Application—"Students are expected to use math and choose the appropriate concept for application even when they are not prompted to do so."**

*A Story of Units* is designed to help students understand how to choose and apply mathematics concepts to solve problems. To achieve this, the modules include mathematical tools and diagrams that aid problem solving, interesting problems that encourage students to think quantitatively and creatively, and opportunities to model situations using mathematics. The goal is for students come to see mathematics as connected to their environment, to other disciplines, and to the mathematics itself. A range of problems are presented within modules, topics, and lessons that serve multiple purposes:

- Single-step word problems that help students understand the meaning of a particular concept.

- Multi-step word problems that support and develop instructional concepts and allow for cross-pollination of multiple concepts into a single problem.

- Brainteasers and puzzles, or other non-routine problems that may be given anytime during the school day. These are meant to engage students in constructive play that encourages perseverance without performance or test-related anxiety.

- Exploratory tasks designed to break potential habits of "rigid thinking." For example, asking students to draw at least 3 different triangles with a 15-inch perimeter encourages them to think of triangles other than equilaterals. Geometry problems with multiple solution paths and mental math problems that can be solved in many ways are further examples.

The problems sets are designed so that there is a healthy mix of PARCC Type I, II, and III tasks.[3] The three

[3] Examples may be found at: http://parcconline.org/samples/mathematics/grade-3-mathematics

types of tasks outlined in PARCC are:

- Type I: Such tasks include computational problems, fluency exercises, conceptual problems, and applications, including 1-step and multi-step word problems.
- Type II: These tasks require students to demonstrate reasoning skills, justify their arguments, and critique the reasoning of their peers.
- Type III: For these problems, students must model real-world situations using mathematics and demonstrate more advanced problem-solving skills.

**Dual Intensity— "Students are practicing and understanding. There is more than a balance between these two things in the classroom—both are occurring with intensity."**

*A Story of Units* achieves this goal through a balanced approach to lesson structure (see section on Lesson Structure below). Each lesson is structured to incorporate at least 10-20 minutes of fluency activities, while the remaining time is devoted to developing conceptual understanding and/or applications.

New conceptual understanding paves the way for new types of fluency. *A Story of Units* starts each grade with a variety of relevant fluency choices from the previous grade. As the year progresses and new concepts are taught, the range of choices grows. Teachers can—and are expected to—adapt their lessons to provide the intense practice with the fluencies that their students most need. Thus, *A Story of Units* doesn't wait months to spiral back to a concept. Rather, once a concept is learned, it is immediately spiraled back into the daily lesson structure through fluency and applications.

## How *A Story of Units* Aligns with the Standards for Mathematical Practice

Like the Instructional Shifts, each Standard for Mathematical Practice is integrated into the design of *A Story of Units*.

1. **Make sense of problems and persevere in solving them.**

   An explicit way in which the curriculum integrates this standard is through its commitment to consistently engaging students in solving multi-step problems. Purposeful integration of a variety of problem types that range in complexity naturally invites children to analyze givens, constraints, relationships, and goals. Problems require students to organize their thinking through drawing and modeling, which necessitates critical self-reflection on the actions they take to problem-solve. On a more foundational level, concept sequence, activities, and lesson structure present information from a variety of novel perspectives. The question, "How can I look at this differently?" undergirds the organization of the curriculum, each of its components, and the design of every problem.

2. **Reason abstractly and quantitatively.**

   The use of tape diagrams is one way in which *A Story of Units* provides students with opportunities to reason abstractly and quantitatively. For example, consider the following problem:

   > *A cook has a bag of rice that weighs 50 pounds. The cook buys another bag of rice that weighs 25 pounds more than the first bag. How many pounds of rice does the cook have?*

   To solve this problem, a student uses a tape diagram to abstractly represent the first bag of rice. To make a tape diagram for the second bag, the student reasons to decide whether the next bar is bigger, smaller, or the same size—and then must decide by how much. Once the student has drawn the models on paper, the fact that these quantities are presented as bags in the problem becomes irrelevant as children shift their focus to manipulating the units to get the total. The unit has—appropriately—taken over the thought process necessary for solving the problem.

   Quantitative reasoning also permeates the curriculum as students focus in on units. Consider the problem *6 sevens plus 2 sevens is equal to 8 sevens.* The unit being manipulated in this sequence is sevens.

3. **Construct viable arguments and critique the reasoning of others.**

   Time for "debriefing" is included in every daily lesson plan and represents one way in which the curriculum integrates this standard. During debriefs, teachers lead students in discussions or writing exercises that prompt children to analyze and explain their work, reflect on their own learning, and make connections between concepts. In addition to debriefs, "turn and talks" and "rally robins" are woven throughout lessons to create ongoing, frequent opportunities for students to develop this mathematical practice. Students use drawings, models and numeric representations, and precise language to make their learning and thinking understood by others.

4. **Model with mathematics.**

   A first grade student represents "3 students were playing. Some more came. Then there were 10. How many students came?" with the number sentence $3 + \square = 10$. A fourth grade student represents a drawing of 5 halves of apples with an expression and writes $5 \times \frac{1}{2}$. Both these students are modeling with mathematics. This is happening daily in word problems. Students write both "situation equations" and "solution equations" when solving word problems. In doing so, they are modeling MP.2.

5. **Use appropriate tools strategically.**

   Building students' independence with the use of models is a key feature of *A Story of Units*, and our approach to empowering students to use strategic learning tools is systematic. Models are introduced and used continuously, so that eventually students use them automatically. The depth of familiarity that students have with the models not only ensures that they naturally become a part of students' schema, but also facilitates a more rapid and deeper understanding of new concepts as they are introduced.

Aside from models, tools are introduced in Kindergarten and reappear throughout the curriculum in every concept. For example, rulers are tools that students in Kindergarten use to create straight edges that organize their work and evenly divide their papers, and they will continue to use them through Grade 5. Worksheets are another form of tool: in daily fluency practice they become instrumental as students use them to recognize their own progress and proficiency.

6. **Attend to precision.**

   In every lesson of every module across the curriculum students are manipulating, relating, and converting units and are challenged not only to use units in these ways, but also to specify which unit they are using. Literally *anything* that can be counted by can be a unit: There might be 3 frogs, 6 apples, 2 fours, 5 tens, 4 fifths, 9 cups, or 7 inches. Students use precise language to describe their work: *"We used a paper clip as a unit of length."* Understanding of the unit is fundamental to their precise, conceptual manipulation. For example, 27 times 3 is not simply 2 times 3 and 7 times 3, rather it should be thought of as 2 tens times 3 and 7 ones times 3. Specificity and precision with the unit is paramount to conceptual coherence and unity.

7. **Look for and make use of structure.**

   There are several ways in which *A Story of Units* weaves this standard into the content of the curriculum. One way is through daily fluency practice. Sprints, for example, are fluency activities that are intentionally patterned. Students analyze the pattern of the Sprint and use its discovery to assist them with automaticity. For example, *"Is the pattern adding one, or adding ten? How does knowing the pattern help me get faster?"*

   An example from a Pre-Kindergarten lesson explicitly shows how concepts and activities are organized to guide students in identification and use of structure. In this lesson, the student is charged with the problem of using connecting cubes to make stairs for a bear to get up to his house. Students start with one cube to make the first stair. To make the second stair, students place a second cube next to the first, but quickly realize that the two "stairs" are equal in height. In order to carry the bear upward, another cube must be added to the second stair so that it becomes higher than the first. This problem uses the growth pattern of the "number stairs" to help students compare number size and develop number sense.

8. **Look for and express regularity in repeated reasoning.**

   Mental math is one way in which *A Story of Units* brings this standard to life. It begins as early as first grade, when students start to make tens. Making ten becomes both a general method and a pathway for quickly manipulating units through addition and subtraction. For example, to mentally solve 12 + 3, students identify the 1 ten and add 10 + (2 + 3). Isolating or using ten as a reference point becomes a form of repeated reasoning that allows students to quickly and efficiently manipulate units.

In summary, the Instructional Shifts and the Standards for Mathematical Practice help establish the mechanism for thoughtful sequencing and emphasis on key topics in *A Story of Units*. It is evident that these pillars of the CCLS combine to support the curriculum with a structural foundation for the content. Consequently, *A Story of Units* is artfully crafted to engage teachers and students alike while providing a powerful avenue for teaching and learning mathematics.

# II. The Common Core Approach to Assessment

Assessments provide an opportunity for students to show their learning accomplishments in addition to offering students a pathway to monitor their progress, celebrate successes, examine mistakes, uncover misconceptions, and engage in self-reflection and analysis. A central goal of the assessment system as a whole is to make students aware of their strengths and weaknesses and to give them opportunities to try again, do better and, in doing so, enjoy the experience of seeing their intelligent, hard work pay off as their skill and understanding increases. Furthermore, the data collected as a result of the assessments represent an invaluable tool in the hands of teachers and provides them with specific data about student understanding to direct their instruction. As such, in *A Story of Units*, assessment becomes a regular part of the class routine in the form of daily, mid-module and end-of-module appraisal. Both the mid-module tasks and the end-of-module tasks are designed to allow for quick teacher scoring that make it possible for them to implement instructionally relevant, actionable feedback to students and to monitor resulting student progress to determine the effectiveness of their instruction and make any needed adjustments. These mid-module and end-of-module tasks should be used in combination with instructionally embedded tasks, teacher-developed quizzes and other formative assessment strategies in order to realize the full benefits of data-driven instruction.

## Daily Assessments

### Problem Sets

As part of the concept development (see Lesson Structure below), students may be asked to complete a problem set. These assignments are done either independently or with teacher guidance, and can be graded in class as part of the lesson. The problem sets often include fluency pertaining to the Concept Development, as well as conceptual and application word problems. The primary goal of the problem set is for students to apply the conceptual understanding(s) learned during the lesson.

### Exit Tickets

Exit tickets are a critical element of the lesson structure (see Lesson Structure below). These quick assessments contain specific questions about what was learned that day. The purpose of the exit ticket is two-fold: to teach students to grow accustomed to being held individually accountable for the work they have done after one day's instruction, and to provide the teacher with valuable evidence of the efficacy of that day's work—which is indispensable for planning purposes.

### Homework

Similar in content and format to the problem sets, the homework gives students additional practice on the skills they learn in class each day. The idea is not to introduce brand-new concepts or ideas, but to build student confidence with the material learned in class. Having already worked similar problems in class, the homework gives them a chance to check their understanding and confirm that they can do the problems independently.

## Mid-Module Assessment Task

A mid-module assessment task will be provided for each module. These tasks are specifically tailored to address approximately the first half of the learning student outcomes for which the module is designed. Careful articulation in a rubric provides guidance in understanding common pre-conceptions or misconceptions of students for discrete portions of knowledge or skill on their way to proficiency for each standard and to prepare them for PARCC assessments. Typically, these tasks are one class period in length and are independently completed by the student without assistance. They should be new to the students and are not preceded by analogous problems. Teachers may use these tasks either formatively or summatively.

## End-of-Module Assessment Task

A summative end-of-module assessment task will also be administered for each module. These tasks are specifically designed based on the standards addressed in order to gauge students' full range of understanding of the module as a whole and to prepare them for PARCC assessments. Some items will test understanding of specific standards while others are synthesis items that assess either understanding of the broader concept addressed in the module or the ability to solve problems by combining knowledge, skills, and understanding. Like the mid-module tasks, these tasks are one class period in length and are independently completed by the student without assistance. They also should be new to the students and not preceded by analogous problems.

Problems on these mid-module and end-of-module assessment tasks will be similar to problems found on The Illustrative Mathematics Project, PARCC sample tasks, and problems that foreshadow what students may encounter in middle school via the Mathematics Assessment Project.

## Rigor in the Assessments

Each assessment utilizes a combination of problems that build from simple to complex and are designed to encourage students to demonstrate procedural skill, fluency, and conceptual understanding. Multiple-choice questions will be representative of PARCC Type I and II tasks. In order to check for conceptual understanding and procedural competence, the distractors for such questions are written to illuminate common student errors and misconceptions.

Application problems including multi-step word problems are *always* part of the assessments. Constructed response questions, as well as the application problems from the assessment at the end of each module, will typically involve more complex tasks that require students to explain their process for solving a problem—i.e., Type II and III tasks. For these problems, answers alone are insufficient. Students must be able to thoroughly explain their thought processes. Possible student work may include tape diagrams, number sentences, area models, paragraphs, etc. In any case, the rubrics for these items will include elements on judging the thoroughness and correctness of the student's explanation.

# III. The Common Core Approach to Differentiating Instruction

The Common Core State Standards for Mathematics require that "all students must have the opportunity to learn and meet the same high standards if they are to access the knowledge and skills necessary in their post-school lives."

The writers of *A Story of Units* agree and feel strongly that accommodations cannot be just an extra set of resources for particular students. Instead, scaffolding must be folded into the curriculum in such a way that it is part of its very DNA. Said another way, faithful adherence to the modules IS the primary scaffolding tool.

The modules that make up *A Story of Units* propose that the components of excellent math instruction do not change based on the audience. That said, there are specific resources included within this curriculum to highlight strategies that can provide critical access for all students.

Researched-based Universal Design for Learning (UDL) has provided a structure for thinking about how to meet the needs of diverse learners. Broadly speaking, that structure asks teachers to consider multiple means of representation; multiple means of action and expression; and multiple means of engagement. Charts at the end of this section offer suggested scaffolds, utilizing this framework, for English Language Learners, Students with Disabilities, Students Performing above Grade Level, and Students Performing below Grade Level. UDL offers ideal settings for multiple entry points for students and minimizes instructional barriers to learning. Teachers will note that many of the suggestions on a chart will be applicable to other students and overlapping populations.

Additionally, individual lessons contain marginal notes to teachers (in text boxes) highlighting specific UDL information about scaffolds that might be employed with particular intentionality when working with students. These tips are strategically placed in the lesson where the teacher might use the strategy to the best advantage.

It is important to note that the scaffolds/accommodations integrated into *A Story of Units* might change how a learner accesses information and demonstrates learning; they do not substantially alter the instructional level, content, or performance criteria. Rather, they provide students with choices in how they access content and demonstrate their knowledge and ability.

We encourage teachers to pay particular attention to the manner in which knowledge is sequenced in *A Story of Units* and to capitalize on that sequence when working with special student populations. Most lessons contain a suggested teaching sequence that moves from simple to complex, starting, for example, with an introductory problem for a math topic and building up inductively to the general case encompassing multi-faceted ideas. By breaking down problems from simple to complex, teachers can locate specific steps that students are struggling with or stretch out problems for students who desire a challenge.

Throughout *A Story of Units*, teachers are encouraged to give classwork utilizing a "time frame" rather than a "task frame." Within a given time frame, all students are expected to do their personal best, working at their maximum potential. "*Students, you have 10 minutes to work independently*." Bonus questions are always ready for accelerated students. The teacher circulates and monitors the work, error-correcting effectively and wisely. Some students will complete more work than others. Neither above nor below grade level students are overly praised or penalized. Personal success is what we are striving for.

Another vitally important component for meeting the needs of all students is the constant flow of data from student work. *A Story of Units* provides daily tracking through "exit tickets" for each lesson as well as mid- and end-of-module assessment tasks to determine student understanding at benchmark points. These tasks should accompany teacher-made test items in a comprehensive assessment plan. Such data flow keeps teaching practice firmly grounded in student learning and makes incremental forward movement possible. A culture of "precise error correction" in the classroom breeds a comfort with data that that is non-punitive and honest. When feedback is provided with emotional neutrality, students understand that making mistakes is part of the learning process. "*Students, for the next five minutes I will be meeting with Brenda, Scott, and Jeremy. They did not remember to rename the remainder in the tens place as 10 ones in their long division on Question 7*." Conducting such sessions then provides the teacher the opportunity to quickly assess if students need to start at a simpler level or just need more monitored practice now that their eyes are opened to their mistake.

Good mathematics instruction, like any successful coaching, involves demonstration, modeling, and lots of intelligent practice. In math, just as in sports, skill is acquired incrementally; as the student acquires greater skill, more complexity is added and proficiency grows. The careful sequencing of the mathematics and the many scaffolds that have been designed into *A Story of Units* makes it an excellent curriculum for meeting the needs of all students, including those with special and unique learning modes.

## Scaffolds for English Language Learners

English language learners provide a variety of experiences that can add to the classroom environment. Their differences do not translate directly to shortfalls in knowledge base but rather present an opportunity to enrich the teaching and learning. The following chart provides a bank of suggestions within the Universal Design for Learning framework to aid English language learners in your class. Variations on these accommodations are elaborated within lessons, demonstrating how and when they might be used.

**Provide Multiple Means of Representation**

- Introduce essential terms and vocabulary prior to the mathematics instruction.
- Clarify, compare, and make connections to math words in discussion, particularly during and after practice.
- Highlight critical vocabulary in discussion. For example, show a picture of 'half.'
- Couple teacher-talk with "math-they-can-see," such as models.
- Let students use models and gestures to calculate and explain. For example, a student searching to define "multiplication" may model groups of 6 with drawings or concrete objects and write the number sentence to match.
- Teach students how to ask questions (such as, "Do you agree?" and "Why do you think so?") to extend "think-pair-share" conversations. Model and post conversation "starters" such as: "I agree because…." "Can you explain how you solved it?" "I noticed that…" "Your solution is different from/ the same as mine because…" "My mistake was to…"
- Connect language (such as 'tens') with concrete and pictorial experiences (such as money and fingers).

**Provide Multiple Means of Action and Expression**

- Know, use, and make the most of student cultural and home experiences. Build on the student's background knowledge.
- Check for understanding frequently (e.g., 'show') to benefit those who may shy away from asking questions.
- Couple teacher-talk with illustrative gestures. Vary your voice to guide comprehension. Speak dynamically with expression. Make eye-to-eye contact and speak slowly and distinctly.
- Vary the grouping in the classroom, such as sometimes using small group instruction to help ELLs learn to negotiate vocabulary with classmates and other times using native language support to allow a student to find full proficiency of the mathematics first.
- Provide sufficient wait time to allow the student to process the meanings in the different languages.
- Listen intently in order to uncover the math content in the students' speech.
- Keep teacher-talk clear and concise.
- Point to visuals while speaking, using your hands to clearly indicate the image that corresponds to your words.
- Get students up and moving, coupling language with motion such as, "Say 'right angle' and show me a right angle with your legs." "Make groups of 5 right now!"
- Celebrate improvement. Intentionally highlight student math success frequently.

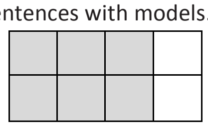
**Provide Multiple Means of Engagement**

- Provide a variety of ways to respond: oral, choral, student boards, concrete models (e.g., fingers), pictorial models (e.g., ten-frame), pair share, small group share.
- Treat "everyday" and first language and experiences as resources, not as obstacles. Be aware of gerunds such as "denominator" in English and "denominador" in Spanish.
- Provide oral options for assessment rather than multiple-choice.
- Cultivate a math discourse of synthesis, analysis, and evaluation, rather than simplified language.
- Support oral or written response with sentence frames, such as "______ is ___ hundreds,__ tens, and ___

| | |
|---|---|
| | ones." <br>▪ Ask questions to probe what students mean as they attempt expression in a second language.<br>▪ Scaffold questioning to guide connections, analysis, and mastery.<br>▪ Let students choose the language they prefer for arithmetic computation and discourse. |

## Scaffolds for Students with Disabilities[4]

Individualized education programs (IEP)s or Section 504 Accommodation Plans should be the first source of information for designing instruction for students with disabilities. The following chart provides an additional bank of suggestions within the Universal Design for Learning framework for strategies to use with these students in your class. Variations on these scaffolds are elaborated at particular points within lessons with text boxes at appropriate points, demonstrating how and when they might be used.

| | |
|---|---|
| Provide Multiple Means of Representation | ▪ Teach from simple to complex, moving from concrete to representation to abstract at the student's pace.<br>▪ Clarify, compare, and make connections to math words in discussion, particularly during and after practice.<br>▪ Partner key words with visuals (e.g., photo of "ticket") and gestures (e.g., for "paid"). Connect language (such as 'tens') with concrete and pictorial experiences (such as money and fingers). Couple teacher-talk with "math-they-can-see," such as models. Let students use models and gestures to calculate and explain. For example, a student searching to define "multiplication" may model groups of 6 with drawings or concrete objects and write the number sentence to match.<br>▪ Teach students how to ask questions (such as "Do you agree?" and "Why do you think so?") to extend "think-pair-share" conversations. Model and post conversation "starters," such as: "I agree because…" "Can you explain how you solved it?" "I noticed that…" "Your solution is different from/ the same as mine because…" "My mistake was to…"<br>▪ Couple number sentences with models. For example, for equivalent fraction sprint, present 6/8 with:<br><br>▪ Enlarge sprint print for visually impaired learners.<br>▪ Use student boards to work on one calculation at a time.<br>▪ Invest in or make math picture dictionaries or word walls. |
| Provide Multiple Means of Action and Expression | ▪ Provide a variety of ways to respond: oral; choral; student boards; concrete models (e.g., fingers), pictorial models (e.g., ten-frame); pair share; small group share. For example: Use student boards to adjust "partner share" for deaf and hard-of-hearing students. Partners can jot questions and answers to one another on slates. Use vibrations or visual signs (such as clap, rather than a snap or "show") to elicit responses from deaf/hard of hearing students.<br>▪ Vary choral response with written response (number sentences and models) on student boards to ease linguistic barriers. Support oral or written response with sentence frames, such as "______ is ____ hundreds, ____ tens, and ____ ones.<br>▪ Adjust oral fluency games by using student and teacher boards or hand signals, such as showing the sum with fingers. Use visual signals or vibrations to elicit responses, such as hand pointed downward means |

[4] Students with disabilities may require Braille, large print, audio, or special digital files. Please visit the website, www.p12.nysed.gov/specialed/aim, for specific information on how to obtain student materials that satisfy the National Instructional Materials Accessibility Standard (NIMAS) format.

count backwards in "Happy Counting."

- Adjust wait time for interpreters of deaf and hard-of-hearing students.
- Select numbers and tasks that are "just right" for learners.
- Model each step of the algorithm before students begin.
- Give students a chance to practice the next day's sprint beforehand. (At home, for example.)
- Give students a few extra minutes to process the information before giving the signal to respond.
- Assess by multiple means, including "show and tell" rather than written.
- Elaborate on the problem-solving process. Read word problems aloud. Post a visual display of the problem-solving process. Have students check off or highlight each step as they work. Talk through the problem-solving process step-by-step to demonstrate thinking process. Before students solve, ask questions for comprehension, such as, "What unit are we counting? What happened to the units in the story?" Teach students to use self-questioning techniques, such as, "Does my answer make sense?"
- Concentrate on goals for accomplishment within a time frame as opposed to a task frame. Extend time for task. Guide students to evaluate process and practice. Have students ask, "How did I improve? What did I do well?"
- Focus on students' mathematical reasoning (i.e., their ability to make comparisons, describe patterns, generalize, explain conclusions, specify claims, and use models), not their accuracy in language.

**Provide Multiple Means of Engagement**

- Make eye-to-eye contact and keep teacher-talk clear and concise. Speak clearly when checking answers for sprints and problems.
- Check frequently for understanding (e.g., 'show'). Listen intently in order to uncover the math content in the students' speech. Use non-verbal signals, such as "thumbs-up." Assign a buddy or a group to clarify directions or process.
- Teach in small chunks so students get a lot of practice with one step at a time.
- Know, use, and make the most of Deaf culture and sign language.
- Use songs, rhymes, or rhythms to help students remember key concepts, such as "Add your ones up first/Make a bundle if you can!"
- Point to visuals and captions while speaking, using your hands to clearly indicate the image that corresponds to your words.
- Incorporate activity. Get students up and moving, coupling language with motion, such as "Say 'right angle' and show me a right angle with your legs," and "Make groups of 5 right now!" Make the most of the fun exercises for activities like sprints and fluencies. Conduct simple oral games, such as "Happy Counting." Celebrate improvement. Intentionally highlight student math success frequently.
- Follow predictable routines to allow students to focus on content rather than behavior.
- Allow "everyday" and first language to express math understanding.
- Re-teach the same concept with a variety of fluency games.
- Allow students to lead group and pair-share activities.
- Provide learning aids, such as calculators and computers, to help students focus on conceptual understanding.

## Scaffolds for Students Performing Below Grade Level

The following chart provides a bank of suggestions within the Universal Design for Learning framework for accommodating students who are below grade level in your class. Variations on these accommodations are elaborated within lessons, demonstrating how and when they might be used.

| | |
|---|---|
| Provide Multiple Means of Representation | ▪ Model problem-solving sets with drawings and graphic organizers (e.g., bar or tape diagram), giving many examples and visual displays.<br>▪ Guide students as they select and practice using their own graphic organizers and models to solve.<br>▪ Use direct instruction for vocabulary with visual or concrete representations.<br>▪ Use explicit directions with steps and procedures enumerated. Guide students through initial practice promoting gradual independence. "I do, we do, you do."<br>▪ Use alternative methods of delivery of instruction such as recordings and videos that can be accessed independently or repeated if necessary.<br>▪ Scaffold complex concepts and provide leveled problems for multiple entry points. |
| Provide Multiple Means of Action and Expression | ▪ First use manipulatives or real objects (such as dollar bills), then make transfer from concrete to pictorial to abstract.<br>▪ Have students restate their learning for the day. Ask for a different representation in the restatement. 'Would you restate that answer in a different way or show me by using a diagram?'<br>▪ Encourage students to explain their thinking and strategy for the solution.<br>▪ Choose numbers and tasks that are "just right" for learners but teach the same concepts.<br>▪ Adjust numbers in calculations to suit learner's levels. For example, change 429 divided by 2 to 400 divided by 2 or 4 divided by 2. |
| Provide Multiple Means of Engagement | ▪ Clearly model steps, procedures, and questions to ask when solving.<br>▪ Cultivate peer-assisted learning interventions for instruction (e.g., dictation) and practice, particularly for computation work (e.g., peer modeling). Have students work together to solve and then check their solutions.<br>▪ Teach students to ask themselves questions as they solve: Do I know the meaning of all the words in this problem?; What is being asked?; Do I have all of the information I need?; What do I do first?; What is the order to solve this problem? What calculations do I need to make?<br>▪ Practice routine to ensure smooth transitions.<br>▪ Set goals with students regarding the type of math work students should complete in 60 seconds.<br>▪ Set goals with the students regarding next steps and what to focus on next. |

## Scaffolds for Students Performing Above Grade Level

The following chart provides a bank of suggestions within the Universal Design for Learning framework for accommodating students who are above grade level in your class. Variations on these accommodations are elaborated within lessons, demonstrating how and when they might be used.

| | |
|---|---|
| Provide Multiple Means of Representation | ▪ Teach students how to ask questions (such as, "Do you agree?" and "Why do you think so?") to extend "think-pair-share" conversations. Model and post conversation "starters," such as: "I agree because…" "Can you explain how you solved it?" "I noticed that…" "Your solution is different from/ the same as mine because…" "My mistake was to…"<br>▪ Incorporate written reflection, evaluation, and synthesis.<br>▪ Allow creativity in expression and modeling solutions. |
| Provide Multiple Means of Action and Expression | ▪ Encourage students to explain their reasoning both orally and in writing.<br>▪ Extend exploration of math topics by means of challenging games, puzzles, and brain teasers. Offer choices of independent or group assignments for early finishers.<br>▪ Encourage students to notice and explore patterns and to identify rules and relationships in math. Have students share their observations in discussion and writing (e.g., journaling).<br>▪ Foster their curiosity about numbers and mathematical ideas. Facilitate research and exploration through discussion, experiments, internet searches, trips, etc.<br>▪ Have students compete in a secondary simultaneous competition, such as skip-counting by 75s, while peers are completing the sprint.<br>▪ Let students choose their mode of response: written, oral, concrete, pictorial, or abstract.<br>▪ Increase the pace. Offer two word problems to solve, rather than one. Adjust difficulty level by increasing the number of steps (e.g., change a one-step problem to a two-step problem). Adjust difficulty level by enhancing the operation (e.g., addition to multiplication), increasing numbers to millions, or decreasing numbers to decimals/fractions.<br>▪ Let students write word problems to show mastery and/or extension of the content. |
| Provide Multiple Means of Engagement | ▪ Push student comprehension into higher levels of Bloom's Taxonomy with questions such as: "What would happen if…?" "Can you propose an alternative…?" "How would you evaluate…?" "What choice would you have made…?" Ask "Why?" and "What if?" questions.<br>▪ Celebrate improvement in completion time (e.g., Sprint A completed in 45 seconds and Sprint B completed in 30 seconds).<br>▪ Make the most of the fun exercises for practicing skip-counting.<br>▪ Accept and elicit student ideas and suggestions for ways to extend games.<br>▪ Cultivate student persistence in problem-solving and do not neglect their need for guidance and support. |

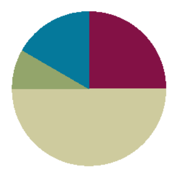
# IV. The Common Core Approach to Lesson Structure

Through a balanced approach to lesson design, *A Story of Units* supports the development of an increasingly complex understanding of the mathematical concepts and topics within the Common Core Learning Standards. Fluency, concept development, and application, all components of instructional rigor demanded by the CCLS (see Instructional Shifts above), are layered to help teachers guide students through the mathematics. Each lesson is structured to incorporate fluency activities along with the development of conceptual understanding, procedural skills, and problem solving. These components are taught through the deliberate progression of material, from concrete to pictorial to abstract. Lesson components and stages of instruction within components are designed to help students reach higher and higher levels of understanding.

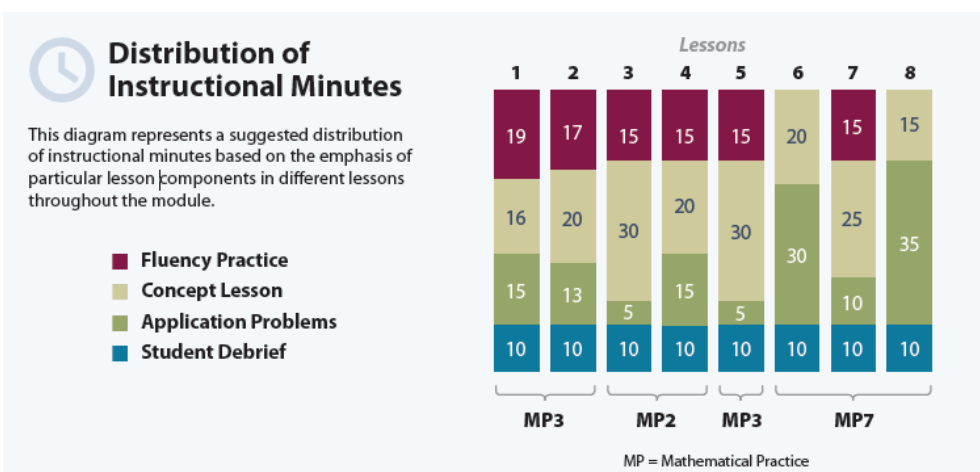
This type of balanced approached to lesson design naturally surfaces patterns and connections between concepts, tools, strategies, and real-world applications. To create the opportunity for teachers and students to explore these connections and establish critical routines for intense work within each component, we suggest starting the school year by consistently implementing the following daily lesson structure for at least the first two months.

**Suggested Lesson Structure**

| | |
|---|---|
| Fluency Practice | (15 minutes) |
| Concept Development | (30 minutes) |
| Application Problems | (5 minutes) |
| Student Debrief | (10 minutes) |
| **Total Time** | **(60 minutes)** |



The time spent on each rigor component in a daily lesson varies between lessons and is guided by the rigor emphasized in the standard(s) the lesson is addressing. For example, if the word "fluently" is used within the text of a standard, then a lesson involving that standard will often have more time devoted to Fluency Practice and Concept Development than to Application Problems. If the word "understand" is used in the text of a standard then the Concept Development component is more likely to be weighted heavily. Likewise, the phrase "real-world problems" in the text of a standard will lead to a lesson that concentrates more on application problems. The Student Debrief component concludes the lesson with yet another carefully orchestrated opportunity for students to engage in one or more Mathematical Practice standards.



The time devoted to each of the four components for each lesson in a module can be viewed at-a-glance using the distribution of instructional minutes chart. The actual order of Fluency Practice, Concept Development, Application Problems, and Student Debrief in each lesson is determined by the content and pedagogy of the lesson. Fluency is usually first—by beginning class with animated, adrenaline-rich fluency, students are more alert when presented with the Concept Development and Application Problems. The placement of an application problem may go before or after the Concept Development. Many times placing a word problem before the Concept Development can provide important structure to understanding a new concept. Other times a word problem shows the value and usefulness of a just-learned concept. The Student Debrief usually wraps up the ideas learned for the day and is often the most powerful component of the lesson structure (see its description below). The suggested lesson structure is modeled throughout Module 1 of each grade.

However, we realize the above lesson structure consisting of four major components is not the only way to meet the demands for quality instruction. In later modules there may be a pedagogical need for the structure to loosen up. The lesson might open with exploration, investigation, concept development, or a word problem followed by fluency, even possibly omitting fluency periodically. Another structure might look like:

Exploration and Application Problems leading into the Concept Development (35 minutes)
Student Debrief (10 minutes)
Fluency Practice (15 minutes)

The instructional shifts require that the three components of rigor be taught with equal intensity. Although this does not mean an "equal amount of time for each component per day," any alternative lesson structure needs to meet the high expectations demanded by the CCLS for all three components of rigor in the major work of each grade. If teachers choose to begin the year with an alternate structure to the one above, our recommendation is that it stays consistent for the first months of school. Regardless, a method for signaling the beginning of math class should be in place. Students should be aware that lessons have been carefully and thoughtfully crafted.

## Fluency Practice

Fluency is designed to promote automaticity by engaging students in practice in ways that get their adrenaline flowing. Automaticity is critical so that students avoid using up too many of their attention resources with lower-level skills when they are addressing higher-level problems. The automaticity prepares students with the computational foundation to enable deep understanding in flexible ways.

We suggest that for the first three weeks of school, teachers include *at minimum* 10 minutes of daily fluency work. It is generally high-paced and energetic, celebrating improvement and focusing on recognizing patterns and connections within the material. Early in the year, we want students to see their skills grow significantly on both the individual and class levels. Like opening a basketball practice with team drills and exercises, both personal and group improvements are exciting and prepare the players for application in the game setting.

Fluency Progressions accompany each module to assist with planning this work. These materials are banks of activities that teachers may either select and use, or study as they create their own. The bank of fluency activities for each lesson is intentionally organized so that activities revisit previously-learned material to develop automaticity, anticipate future concepts, and strategically preview or build skills for the day's Concept Development. It is important to provide ample opportunities to review familiar fluencies as needed, as well as to begin developing automaticity with new ones. The suggestions may or may not be ideal for the students in a given class. As teachers work with the materials, they should adjust them in consideration of their particular students' needs.

Like drills in sports, the value of structured Fluency Practice should be immediately recognizable to students. Abraham Lincoln is quoted as saying, "Give me six hours to chop down a tree and I will spend the first four sharpening the axe." Becoming proficient and staying proficient at math can be compared to doing the same in sports: Use it or lose it.

## Concept Development

The Concept Development constitutes the major portion of instruction and generally comprises at least 20 minutes of the total lesson time. It is the primary lesson component, in which new learning is introduced. Intentional sequencing of standards and topics within modules ensures that students have the requisite understanding to fully access new learning goals and integrate them into their developing schemas. Many Concept Developments articulate the standards and topics through a deliberate progression of material, from concrete to pictorial to abstract. This structure compliments and supports an increasingly complex understanding of concepts.

However, not every lesson will move in this exact order. Sometimes, the concrete level has been covered in a prior grade level. Sometimes, the lesson will move from the abstract to the pictorial or concrete. For example, in *"Draw a picture of 4 + 4 + 4*," and *"With your number disks (or bundles), show me 248 + 100,"* students begin with the "abstract" number problem and illustrate the idea pictorially or concretely.

Our goal is always to strengthen students' understanding of numbers rather than to teach manipulatives. The concrete is important when conceptual understanding is weak. Manipulatives are expensive in terms of instructional minutes. However, omitting them can sometimes result in elongating the learning process if students have failed to miss a critical connection. When necessary, it is better to take the time to re-establish meaning at the concrete and/or pictorial levels and move as efficiently as possible to the abstract.

To provide a basis for daily planning, teaching sequence charts are included at the beginning of each topic to organize content into coherent sequences toward achieving mastery. The lessons that follow further organize material according to concrete, pictorial, and abstract. These structures appear in various orders and combinations depending on our suggestion for delivery of the particular material. Concept Development elaborates on the "how-to" of delivery through models, sample vignettes, and dialogue, all meant to give teachers a snapshot of what the classroom might look and sound like at each step of the way. Teachers' word choice may be different from that in the vignettes, and they should use what works from the suggested talking points, along with their knowledge of their students' needs, as they write their own.

## Application Problems

*A Story of Units* is designed to help students understand how to choose and apply the correct mathematics concept to solve real world problems. To achieve this, lessons use tools and models, problems that cause students to think quantitatively and creatively, and patterns that repeat so frequently that students come to see them as connected to their environment and other disciplines. A range of problems presented within concepts serve multiple purposes: single-step word problems help children to understand the meaning of new ideas, and multi-step word problems support and develop instructional concepts.

There are clearly different possible choices for delivery of instruction when engaging in problem-solving. The beginning of the year is characterized by establishing routines that encourage hard, intelligent work through guided practice rather than exploration. This practice serves to model the behaviors that students will need to work independently later. It is also wise to clearly establish a very different tone of work on application problems from that of the "quick answers" of fluency. "*We did some fast math, now let's slow down and take more time with these problems.*"

Lessons provide application problems that directly relate to the Concept Development. The following steps of Read-Draw-Write (RDW) are suggested: Read the problem, draw and label, write a number sentence, and write a word sentence. The more students participate in reasoning through problems with a systematic approach, the more they internalize those behaviors and thought processes.

Sample vignettes within this section of the lessons provide examples of guided practice in solving a word problem using the suggested steps. Like the vignettes in the Concept Development, these narratives are meant to allow teachers to imagine what the classroom might sound like. Teachers' choices of words may differ! Often, we clarify our own choice of words by listening to and studying those of others. Teachers should take what works from the samples provided as they write their own outlines based on knowledge of student needs.

Notes to the teacher in this section remind them to guide students toward making connections with other parts of the lesson and to other concepts without being too explicit. The goal is a deliberate design that allows students to discover those connections and recognize and verbalize them. *"Oh! This is just like the 10 frame cards!" "Oh, this is just like the envelopes problem." "In what way?"* The sample vignettes demonstrate encouraging students to articulate those observations, and remind teachers to revisit them in the "focus" when the lesson's objective becomes eminently clear (in focus) to the students.

In addition to the problems written in the Application Problems section, each topic closes with another bank of problems for further application or homework. Teachers should use these as supplemental resources for their planning. The banks include problems directly related to the concepts learned, problems that anticipate future concepts, and brainteasers or puzzles.

## Student Debrief

Rather than stating the objective of the lesson to the students at its beginning, we wait until the dynamic action of the lesson has taken place. Then we reflect back on it with the students to analyze the learning that occurred. We want *them* to articulate the focus of the lesson. In the Student Debrief we develop students' metacognition by helping them make connections between parts of the lesson, concepts, strategies, and tools on their own. We draw out or introduce key vocabulary by helping students appropriately name the learning they describe.

The goal is for students to see and hear multiple perspectives from their classmates and mentally construct a multifaceted image of the concepts being learned. Through questions that help make these connections explicit and dialogue that directly engages students in the Standards for Mathematical Practice, they articulate those observations so that the lesson's objective becomes eminently clear (in focus) to them.

Like the other lesson components, the Student Debrief section includes sample dialogue or suggested lists of questions to invite the reflection and active processing of the totality of the lesson experience. The purpose of these talking points is to guide teachers' planning for eliciting the level of student thinking necessary to achieve this. Rather than ask all of the questions provided, teachers should use those that resonate most as they consider what will best support students in reaching self-articulation of the focus from the lesson's multiple perspectives.

Sharing and analyzing high-quality student work is a consistent feature of the vignettes in this section. This technique encourages students to engage in the mathematical practices, which they come to value and respect as much as or more than speed in calculation. What the teacher values, the students will too: Sharing and analyzing high quality work gives teachers the opportunity to model and then demand authentic student work and dialogue.

Conversation constitutes a primary medium through which learning occurs in the Student Debrief. Teachers can prepare students by establishing routines for talking early in the year. For example, "pair-sharing" is an invaluable structure to build for this and other components of the lesson. During the Debrief, teachers should circulate as students share, noting which partnerships are bearing fruit, and which need support. They might join struggling communicators for a moment to give them sentence stems. Regardless of the scaffolding techniques that a teacher decides to use, all students should emerge clear enough on the lesson's focus to either give a good example or make a statement about it.

"Exit Tickets" close the Student Debrief component of each lesson. These short, formative assessments are meant to provide quick glimpses of the day's major learning for students and teachers. Through this routine, students grow accustomed to showing accountability for each day's learning and produce valuable data for the teacher that becomes an indispensable planning tool.

## Standards for Mathematical Practice

The Standards for Mathematical Practice are seamlessly woven into each lesson through various components of delivery that require the level of thinking and behaviors that the practices embody. Here are some examples:

- Carefully crafted fluency activities engage students in looking for and making use of structure, as well as looking for and expressing regularity in repeated reasoning.
- The read, draw, write sequence upon which problem-solving is based naturally provides opportunities for students to select appropriate tools, model word problems using mathematics, and reason abstractly and quantitatively.
- Concept Development consistently invites students to make sense of problems and persevere in solving them as they grapple with new learning through increasingly complex concrete, pictorial, and abstract applications.
- Each lesson's Student Debrief, as well as ongoing debrief embedded within each lesson component, requires students to construct viable arguments and critique the reasoning of others. Questioning and dialogue throughout the lessons ensures that students are not only engaging in the standards, but also that they are explicitly aware of and reflecting on those behaviors.

Each topic has the potential to integrate most (if not all) mathematical practice standards into the lessons. So that teachers are able to see the Standards for Mathematical Practice expressed just as clearly as the writers, at least one mathematical practice standard is chosen per topic to be exemplified with more detail and explanation. The mathematical practice standard chosen is annotated in the left margin and is announced in the "Mathematical Practices Brought to Life" section at the beginning of the topic. For example, a "MP.7" indicated in the left margin alerts the teacher that the purpose of the questions and/or vignette is to develop students' ability to look for and make sense of structure. As teachers give lessons, they should modify the materials or develop their own talking points. The annotation supports planning and delivery that emphasizes implementation of the Standards for Mathematical Practice. It signifies additional importance for lesson delivery and also indicates where teachers can find ways to model the standards for the plans they author.

# V. The Common Core Approach to Mathematical Models

*A Story of Units* is a curriculum written by teachers for teachers to help every student build mastery of the Common Core Learning Standards for Mathematics. The theme of the story—creating, manipulating, and relating units—glues seemingly separate ideas into a coherent whole throughout each grade and over the years.

As noted earlier in The Common Core Approach to Instructional Shifts, coherence is supported in *A Story of*

*Units* through the use of a finite set of concrete and pictorial models. Students build increasing dexterity with these models through persistent use within and across levels of curriculum. The repeated appearance of familiar models helps to build the imperative vertical links between the topics of one grade level and the next. In addition, the depth of awareness that students have with the models not only ensures that they naturally become a part of the students' schema, but also facilitates a more rapid and multifaceted understanding of new concepts as they are introduced.

This information is designed to support teachers as they engage students in meaningful mathematical learning experiences aligned to the Common Core Learning Standards. This support is provided through the following information:

- The grade levels for which the model is most appropriate
- A description and example of the model
- A collection of instructional strategies for using the model presented in order of the natural progression of the concept(s)

The following categories indicate the primary application area for each model. However, as previously stated, models appear repeatedly across grades and topics. Therefore, instructional strategies will include examples spanning several levels of the curriculum.

**Numbers Through 10**

- Number Towers
- Number Path
- Number Bond

**Place Value and Standard Algorithms**

- Bundles
- Place Value Chart
- Base-Ten Blocks
- Money
- Number Disks (with Place Value Chart)

**Fractions**

- Number Line
- Area Model

**Addition and Subtraction**

- Ten-Frame

**Multiplication**

- Array and Area Model
- Rekenrek

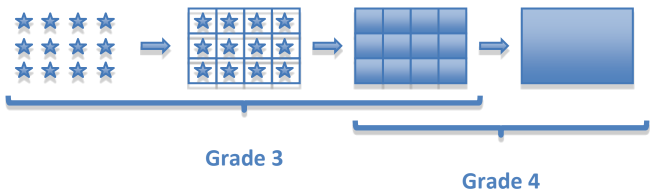
**Word Problems**

- Tape Diagram

## Array and Area Models

**Grade Level** **1 – 5**

### Description

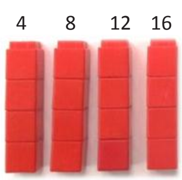



An array is an arrangement of a set of objects organized into equal groups in rows and columns. Arrays help make counting easy. Counting by equal groups is more efficient than counting objects one by one. The ten-frame is an array used in Kindergarten. Students count objects in arrays in Kindergarten and Pre-Kindergarten. (PK.CC.4) The rectangular array is used to teach multiplication and leads to understanding area. (3.OA.3)

Arrays reinforce the meaning of multiplication as repeated addition (e.g., $3 \times 4 = 4 + 4 + 4$), and the two meanings of division—that $12 \div 3$ can indicate how many will be in each group if I make 3 equal groups and that it can also indicate how many groups I can make if I put 3 in each group. Further using arrays reinforces the relationship between multiplication and division.

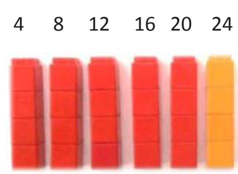
### Instructional Strategies

- Use number towers to depict multiplication problems in the shape of an array.

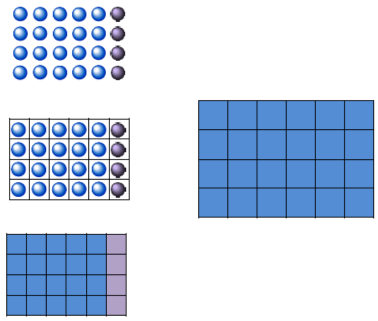



$4 \times 4 =$ ___.
$16 \div 4 =$ __.

5 fours + 1 four = 6 fours
$20 + 4 = 24$
$6 \times 4$ is 4 more than $5 \times 4$.



- Use the rectangular grid to model multiplication and division.



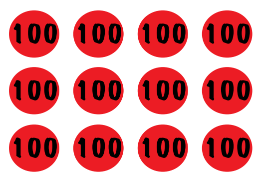
4 x 6 = ___
6 x 4 = ___

4 x ___ = 24
___ x 6 = 24

24 ÷ 4 = __
24 ÷ 6 = __

- Multiply units with arrays.

Multiplying hundreds:
4 hundreds × 3 = 12 hundreds
400 × 3 = 1200

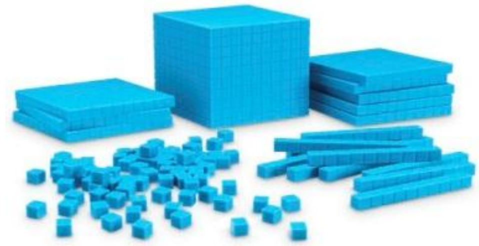



400 x 3 =

## Base-Ten Blocks

**Grade Level** **K – 2**

**Description**



Base-ten blocks (also referred to as Dienes blocks) include thousands "cubes," hundreds "flats," tens "rods," and ones. Base-ten blocks are a proportional representation of units of ones, tens, hundreds, and thousands and are useful for developing place value understanding. This is a "pre-grouped" model for base-ten that allows for more efficient modeling of larger quantities through the thousands. However, because this place value model requires students to more abstractly consider the 10 to 1 relationship of the various blocks, care must be taken to ensure that students attend to the "ten-ness" of the pieces that are now traded rather than bundled or un-bundled.

Base-ten blocks are introduced after students have learned the value of hundreds, tens, and ones and have had repeated experiences with composing and decomposing groups of 10 ones or groups of 10 tens with bundles.

### Instructional Strategies

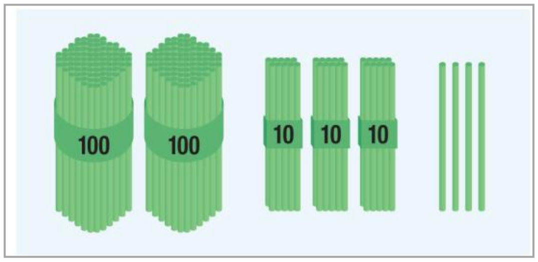
Please note: Instructional strategies for base-ten blocks are similar to those of bundles and place value disks. Therefore, see "Bundles," "Money," and "Place Value Disks" for other teaching ideas.

- Represent quantities on the mat and write in standard, expanded, and word form.
- Play "More" and "Less" games. Begin with an amount on a mat. At a predetermined signal (e.g., teacher claps or rings a bell), students add (or subtract) a quantity (2, 5, 10, or other) to the blocks on the mat.
- Give student equivalent representation riddles to be solved with base-ten pieces. For example, I have 29 ones and 2 hundreds. What number am I?
- Model addition, subtraction, multiplication, and division.
- Use blocks and mats as a support for teaching students to record the standard algorithms for all four operations.

## Bundles

**Grade Level** **K – 2**

**Description**

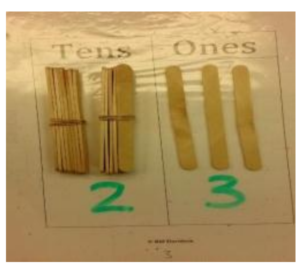



Bundles are discrete groupings of place value units (tens, hundreds, thousands), usually made by students/teachers placing a rubber band or chenille stem around straws, popsicle sticks, or coffee stirrers. Linking cubes may also be used in this fashion. Ten straws (or cubes) are bundled (or linked) into 1 unit of ten, 10 tens are bundled into 1 unit of a hundred, and so on. These student-made groupings provide the necessary conceptual foundation for children to be successful with pre-grouped, proportional, and non-proportional base-ten materials. (See Base-Ten Blocks and Number Disks.)

Understanding tens and ones is supported in Kindergarten as students learn to compose and decompose tens and ones by "bundling" and "unbundling" the materials. Numbers 11-19 are soon seen as 1 ten (a bundled set of 10 ones) and some extra ones.

By Grade 2, students expand their skill with and understanding of units by bundling units of ones, tens, and hundreds up to one thousand with sticks. These larger units are discrete and can be counted: "1 hundred, 2 hundred, 3 hundred, etc." Bundles also help students extend their understanding of place value to 1000. (2.NBT.1) Repeated bundling experiences help students to internalize the pattern that 10 of one unit make 1 of the next larger unit. Expanded form, increased understanding of skip-counting (2.NBT.2), and fluency in counting larger numbers are all supported by the use of this model.

Bundles are also useful in developing conceptual understanding of renaming in addition and subtraction. The mat below shows 2 tens and 3 ones. To solve 23 – 9, one bundle of ten is "unbundled" to get 1 ten and 13 ones in order to take away 9 ones.




**Instructional Strategies**

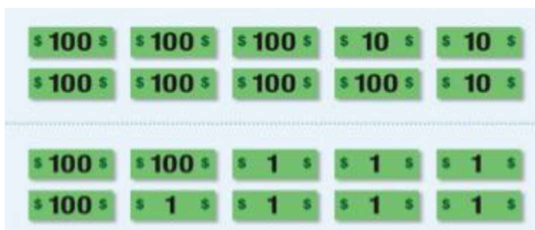
- Represent various quantities with bundles and "singles."
- Count school days. Each day a single straw/stick is added to the ones pocket and counted. Sticks are bundled when 10 days have passed and moved to the tens pocket. Have a "100th Day" celebration.

- Bundles may also be used to count down to a significant event (e.g.., the last day of school), unbundling as necessary.
- Play "Race to Zero" with a partner. Students start with a quantity between 30 and 40 in bundles. Roll two dice to determine what can be taken away from the starting quantity (unbundling as necessary). First partner to reach zero is the winner. (This game may also be played as an addition game.)
- Count in unit form (2 tens, 8 ones; 2 tens, 9 ones; 3 tens, etc.).
- Represent quantities on place value mats to be added or subtracted.

## Money

**Grade Level** **2**

**Description**



Dollar bills (1s, 10s, and 100s) are non-proportional units that are used to develop place value understanding. That is, bills are an abstract representation of place value because their value is not proportionate to their size. Ten bills can have a value of $10 or $1000 but appear identical aside from their printed labels. Bills can be "traded" (e.g., 10 ten-dollar bills for 1 hundred-dollar bill) to help students learn equivalence of the two amounts.

As with other place value models, students can use bills to model numbers up to three digits, to read numbers formed with the bills, and to increase fluency in skip-counting by tens and hundreds.

The picture above shows that the arrangement of the $100s, $10s, and $1s can be counted in this manner:

The first frame, S: 100, 200, 300, 400, 500, 600, 700, 710, 720, 730.

The second frame, S: 100, 200, 300, 301, 302, 303, 304, 305, 306, 307.

The transition from a discrete unit of a "bundle" to proportional materials such as base-ten blocks to a non-proportional unit of a bill is a significant leap in a student's place value learning trajectory.

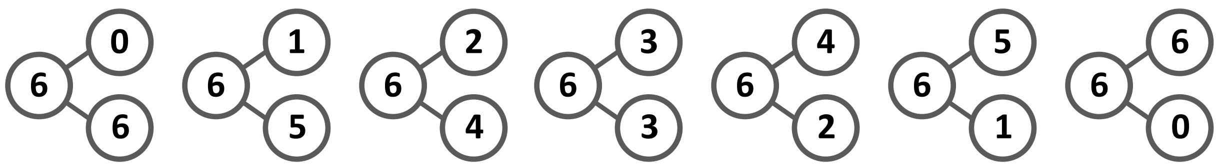
### Instructional Strategies

- Skip-count up and down by $10 between 45 and 125.
  (45, 55, 65, 75, 85, 95, 105, 115, 125).
- Practice "making change" by counting on from an amount up to a specified total.
- "More" and "Less" games may also be played with money (See Base-Ten Blocks).
- Play equivalency games. How many $5 bills in a $10 bill? A $20 bill? A $100 bill? etc.

## Number Bond

**Grade Level** K – 5

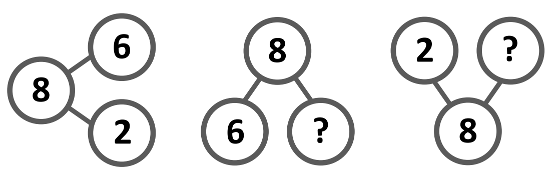
### Description




The number bond is a pictorial representation of part-part-whole relationships and shows that within a part-whole relationship, smaller numbers (the parts) make up larger numbers (the whole). The number bond may be presented as shown, using smaller circles (or squares) for the parts to distinguish the part from the whole. As students become more comfortable using number bonds, they may be presented using the same size shape for parts and whole.

Number bonds of 10 have the greatest priority because students will use them for adding and subtracting across 10. Students move towards fluency in Grade 1 with numbers to 10 building on the foundation laid in Kindergarten. They learn to decompose numbers to ten with increasing fluency. (1.OA.6) Students learn the meaning of addition as "putting together" to find the whole or total and subtraction as "taking away" to find a part.

Notice in the diagrams below that the orientation of the number bond does not change its meaning and function. (6 + 2 = 8, 2 + 6 = 8, 8 – 6 = 2, 8 – 2 = 6)




### Instructional Strategies

- Make bonds with a specified whole using concrete objects. Students place all the objects into the "parts" circles of the bond using various combinations. These can be recorded pictorially (students draw objects in the bonds), abstractly (children write numerals in the bonds), or a combination of these representations as appropriate.
- Generate number stories for each number from 5 to 10 from pictures and situations.
- Develop fluency: Show all the possible ways to make ___, for all the numbers from 1 to 10.
- Present bonds in which the whole and one part are visible (using concrete, pictorial, and eventually abstract representations). Students solve for the other part by bonding, counting on, or subtracting.
- Transition students from number bonds to tape diagrams by drawing both representations for number stories.
- Use number bonds as a support for mental math techniques such as "Make 10" (see grade specific

examples below).

- Use number bonds to see part-whole fraction and decimal relationships.

**Grade 1 Example**

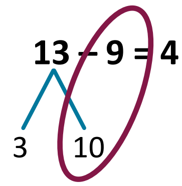
Decompose 13 into 10 and 3.

Subtract 9 from the 10.

$10 - 9 = 1$

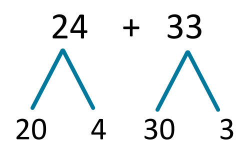
Then add 1 + 3.

$1 + 3 = 4$, so $13 - 9 = 4$




**Grade 2 Example**

Solve 24 +33 mentally.
Use bonds to show your thinking.




$(20 + 30) + (4 + 3) = 57$

**Grade 4 Example 1**

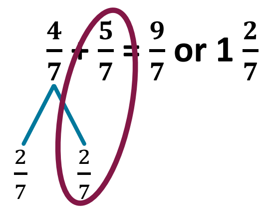
Decompose $\frac{4}{7}$ into $\frac{2}{7}$ and $\frac{2}{7}$.

Add $\frac{2}{7}$ to $\frac{5}{7}$ to make 1 whole.

$\frac{2}{7} + \frac{5}{7} = \frac{7}{7}$

Then add $\frac{7}{7}$ to $\frac{2}{7}$.

$\frac{7}{7} + \frac{2}{7} = \frac{9}{7}$ or $1\frac{2}{7}$



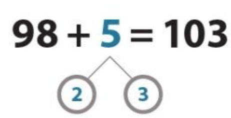

**Grade 4 Example 2**

T: 98 + 5 = 100 + ____?

S: 98 + 2 + 3 = 100 + 3.

T: 98 + 5 is ____?

S: 103.

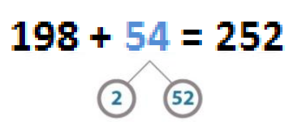



T: 198 + 54 = 200 + ____?

S: 198 + 54 = 200 + 52.

T: 198 + 54 is ____?

S: 252.

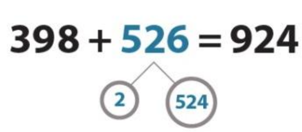



T: 398 + 526 = 400 + ____?

S: 398 + 2 + 524 = 400 + 524.

T: 398 + 526 is ____?

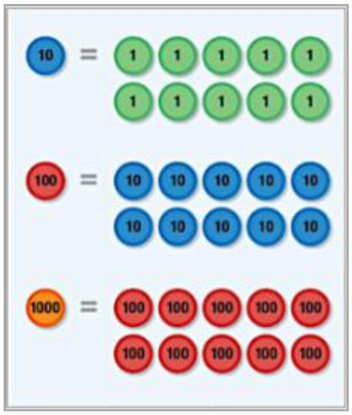
S: 924.

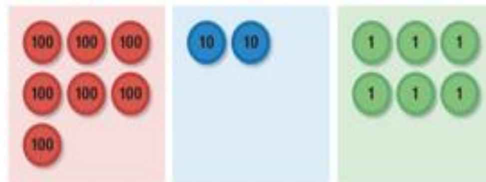


## Number Disks

**Grade Level** **2 – 5**

**Description**




Place Value Chart with Number Disks

Number disks are non-proportional units used to further develop place value understanding. Like money, the value of the disk is determined by the value printed on it, not by its size. Number disks are used by students through Grade 5 when modeling algorithms and as a support for mental math with very large whole numbers. Whole number place value relationships modeled with the disks are easily generalized to decimal numbers and operations with decimals.

### Instructional Strategies

- Play pattern games: "What is 100 less than 253?" Students simply remove a 100 disk and state and/or record their new number.
- Play partner games: Partner A hides the disks from Partner B within a file folder. Partner A says, "I am looking at the number 241. I will make 10 less (physically removing a 10 disk). What is 10 less than 241?" Partner B writes the answer on his personal board/notebook and then states a full response: "10 less than 241 is 231." Partner A removes the folder and the partners compare the written response with the disks.
- Perform all four operations with both whole numbers and decimals on mats.
- Use materials to bridge to recording the standard algorithms for all four operations with both whole numbers and decimals.

## Number Line

**Grade Level** K – 5

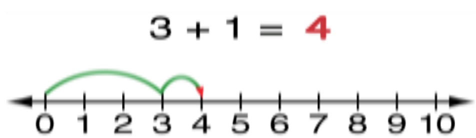
**Description**



The number line is used to develop a deeper understanding of whole number units, fraction units, measurement units, decimals, and negative numbers. Throughout Grades K-5, the number line models measuring units.

### Instructional Strategies

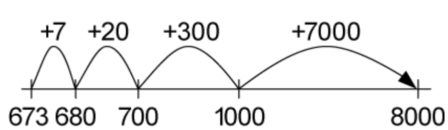
- Measure lengths in meters and centimeters.
- Counting on: Have students place their finger on the location for the first addend, and count on from there to add the second addend.



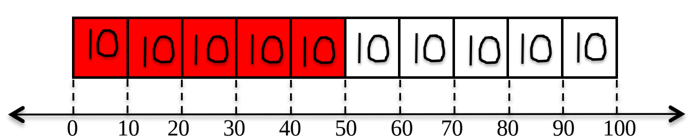

- Have students use a "clock" made from a 24 inch ribbon marked off at every 2 inches to skip-count by fives.

- Compute differences by counting up.

**8000 – 673 = 7, 327**

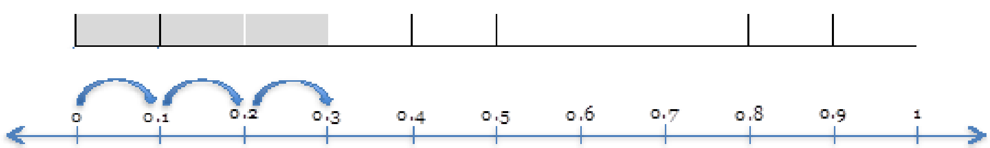



- Multiplying by 10; students visualize how much 5 10's is, and relate it to the number line.



- Rounding to the nearest ten or hundred; e.g., students identify which 'hundreds' come before 820, "820 is between 800 and 900."


- Model tenths in unit, expanded, fraction, and decimal form.




3 tenths $= 3 \times \frac{1}{10} = \frac{3}{10} = 0.3$

- Create and analyze line plots.


## Number Path

**Grade Level** **PK – 1**

**Description**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** |

The number path can be thought of as a visual (pictorial) representation of the number tower (see description below) and is foundational to understanding and using the number line. It also serves as a visual representation of 1:1 correspondence and the concept of whole numbers (one number, one space, and each being equal in size). The color change at 5 helps to reinforce the 5 and 10 benchmarks. The number path also serves as an early precursor to measurement concepts and a support for cardinal counting. (If a student places 7 objects in each of the 7 spaces on the path, they must realize that there are 7 objects, not 10. Simply because the path goes up to 10 does not mean there are 10 objects.)

## Instructional Strategies

- Sort, classify, and count up to 5 with meaning and then work on extending "How Many" questions up to 10.
- Match amounts to numerals.
- Write numerals 1 to 5.
- Extend the meaning of 6, 7, and 8 with numerals (6 is 5 and 1, 7 is 5 and 2, 8 is 5 and 3.)
- Become fluent with numbers to 10 and practice "before" and "after," as well as relationships of "1 more/less" and "2 more/less"
- Order numbers from 1 to 10.
- Play number order games (e.g., Partner A closes eyes while Partner B covers a number with a penny—then Partner A has to guess the hidden number).
- Fold the number path so that only small sections are visible. Students show 4, 5, 6, 7; teacher says "4, 5, hmm, 7 what number is missing?"
- Play "I Wish I Had" games (e.g., "I wish I had 7, but I only have 5." Student answers by placing a finger on 5 and then counting on to say "2" – the amount needed to make the target number.)
- Match ordered sets with numerals on the number path.

## Number Towers

**Grade Level** **PK – 3**

**Description**

Number towers, also known as number stairs, are representations of quantity constructed by joining together interlocking cubes such as Unifix ©. In the beginning of the Story, they are used to help younger children quite literally build their knowledge of cardinality by erecting towers of various numbers. Number towers are then used to teach concepts of "more/less" globally and the patterns of "1 more/less" and "2 more/less" specifically. This model leads to an understanding of comparison and the word "than," not only in the context of "more than" and "less than," but also in the context of "taller than," "shorter than," heavier than," "longer than," etc.

Children are encouraged to build towers for quantities 1 through 5 in one color. Quantities beyond 5 are added on in a second color. This color change provides support for several important developmental milestones. First, it facilitates children's understanding of 5 as a benchmark, which provides an important beginning to their ability to subitize. Second, it allows students to see relationships such as "5 needs 2 more to be 7;" "5 is 1 less than 6;" and "5 and 4 is 9, which is 1 less than 10." Finally, it encourages students to count on from 5 rather than starting at 1 to count quantities of 6, 7, 8, 9, and 10.

Such comparisons lead to looking at the parts that make up a number. ("3 is less than 7. 3 and 4 make 7.") These concepts are foundational to students' understanding of part/whole models (see Number Bonds). This, in turn, leads naturally to discussions of addition and subtraction, fact fluencies (+1, +2, +3, -1, -2, -3), and even the commutative property (flip the tower; 3 + 4 or 4 + 3—does the whole change?), which are explored in Kindergarten and Grade 1.

In Grades 2 and 3, as students prepare for and study multiplication and division, each unit in the number stair can be ascribed a value other than 1. For example: "Each of our cubes is equal to three. What is the value of the stair with five cubes?"

3 3 3 3 3

Further, the use of number stairs can be extended to help children understand more complex properties like the distributive property. "Each of our cubes is equal to three. Make a stair with five cubes. Now add two more cubes. The stair with 7 cubes is 2 more threes. So, 5 threes is 15, 2 threes is 6, and together 7 threes is 15 + 6 or 21."

5 threes + 2 threes = (5 + 2) threes

3 3 3 3 3 3 3

## Instructional Strategies

- Sort, classify, and count up to 5 with meaning and then begin extending How Many questions up to 10.
- Build a series of towers from 1 to 10, and then use the towers to relate quantities, e.g., "5 is before 6." "6 is after 5." "5 + 1 more is 6." "6 is more than 5." "6 is 1 more than 5." "5 is 1 less than 6." "5 and 2 make 7." "5 + 2 = 7."
- Build a tower that shows 6.
- Build a specific tower and count the cubes. (Cardinality)
- Partners roll dice, each build a different tower and state which has more (less).
- Build a tower while stating the ''one more" relationship (e.g., 4, 1 more is 5).
- Deconstruct the tower while stating the ''one less" relationship (e.g., 7, one less is 6).
- Count on from 5 (e.g., to count 7, students use the color change to say "5, 6, 7"instead of starting from 1). The color change at 5 may be presented to students as a shortcut by having students slide their finger over a group of 5 as they count. (Subitizing)
- Count up from numbers other than 0 and 1.
- Count down from numbers other than 10 to numbers other than 0 and 1.
- Compare numbers within 1 and 10.

## Place Value Chart

**Grade Level** 2 – 5

**Description**

| | | |
|---|---|---|
| | | |

**Place Value Chart Without Headings**
(Used with labeled materials such as disks)

| Hundreds | Tens | Ones |
|---|---|---|
| | | |

**Place Value Chart with Headings**
(Used with unlabeled materials such as base-ten blocks or bundles)

The place value chart is a graphic organizer that students can use (beginning in Grade 1 with tens and ones through Grade 5 with decimals) to see the coherence of place value and operations between different units.

### Instructional Strategies

- Have students build numbers on mats. Place value cards may be used to show the expanded form of a number that is represented on the place value chart.


- Count the total value of ones, tens, and hundreds with any discrete, proportional or non-proportional material such as bundles, base-ten blocks or number disks.
- Model and use language to tell about 1 more/less, 10 more/less on the place value chart with disks when there is change in the hundreds unit.

- Complete a pattern counting up and down.
- Model addition and subtraction using base-ten blocks or number disks.
- Use the mat and place value materials as a support for learning to record the standard algorithms for addition, subtraction, multiplication, and division.

## Rekenrek

**Grade Level** **PK – 5**

**Description**

20-Bead Rekenrek

100-Bead Rekenrek

The Rekenrek has a 5 and 10 structure, with a color change at 5 (eliciting the visual effect of grouping 5 and grouping 10). The 20-bead Rekenrek consists of 2 rows of 10 beads, allowing students to see numbers to 10 either as a number line on one row or a ten-frame (5 beads on two rows). A 100-bead Rekenrek has 10 rows of 10 beads. Other names for the Rekenrek are “Calculating Frame,” “Slavonic Abacus,” “Arithmetic Rack,” or “Math Rack.”

### Instructional Strategies

**Grades PK – 1**

- Count up and down in short sequences (1, 2, 3, 2, 3, 4, 3, 2,…, simulate the motion of a roller-coaster).
- Think of 7 as “2 more than 5.”
- See “inside” numbers (subitize – “instantly see how many”).
- Count in unit form (1 ten 1, 1 ten 2, 1 ten 3… 2 tens 1, 2 tens 2, etc.).
- Skip-count with complexity such as counting by 10’s on the 1’s (3, 23, 33, 43, …).
- Group numbers in 5’s and 10’s. Compare Rekenrek to ten-frame.
- Build fluency with doubles.
- Make 10.
- Add across 10; subtract from 10.
- Build numbers 11-20.
- Show different strategies for adding 7 + 8 (5 + 5 + 2 + 3, 7 + 7 + 1, 10 + 5, 8 + 8 - 1).
- Compose and decompose numbers.
- Solve addition and subtraction story problems (e.g., putting together, taking away, part-part-whole and comparison).

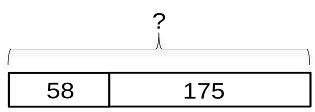
**Grades 2 – 5**

- Show fluency with addition and subtraction facts.
- Find complements of numbers up to 10, 20, 30, …100.
- Skip count by 2, 3, 4, 5, 6, 7, 8, and 9 within 100.
- Identify doubles plus one and doubles minus 1.
- Model rectangular arrays to build conceptual understanding of multiplication.
- Demonstrate the distributive property. Think of 3 x 12 as 3 x 10 plus 3 x 2.

## Tape Diagram

**Grade Level** **1 – 5**

**Description**




*Rachel collected 58 seashells. Sam gave her 175 more.*
*How many seashells did she have then?*

Tape diagrams, also called bar models, are pictorial representations of relationships between quantities used to solve word problems. Students begin using tape diagrams in 1st grade, modeling simple word problems involving the four operations. It is common for students in 3rd grade to express that they don't need the tape diagram to solve the problem. However, in Grades 4 and 5, students begin to appreciate the tape diagram as it enables students to solve increasingly more complex problems.

At the heart of a tape diagram is the idea of *forming units*. In fact, forming units to solve word problems is one of the most powerful examples of the unit theme and is particularly helpful for understanding fraction arithmetic.

The tape diagram provides an essential bridge to algebra and is often called "pictorial algebra."

Like any tool, it is best introduced with simple examples and in small manageable steps so that students have time to reflect on the relationships they are drawing. For most students, structure is important. RDW (read, draw, write) is a process used for problem solving:

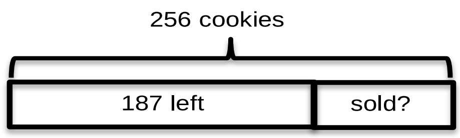
- Read a portion of the problem.
- Create or adjust a drawing to match what you've read. Label your drawing.
- Continue the process of reading and adjusting the drawing until the entire problem has been read and represented in the drawing.
- Write and solve an equation.
- Write a statement.

There are two basic forms of the tape diagram model. The first form is sometimes called the part-whole model; it uses bar segments placed end-to-end (Grade 3 Example below depicts this model), while the second form, sometimes called the comparison model, uses two or more bars stacked in rows that are typically left justified. (Grade 5 Example below depicts this model.)

Rather than talk to students about the 2 forms, simply model the most suitable form for a given problem and allow for flexibility in the students' modeling. Over time, students will develop their own intuition for which model will work best for a given problem. It is helpful to ask students in a class, 'Did anyone do it differently?' and allow students to see more than one way of modeling the problem, then perhaps ask, "Which way makes it easiest for you to visualize this problem?"

Grade 3 Example

*Sarah baked 256 cookies. She sold some of them. 187 were left. How many did she sell?*

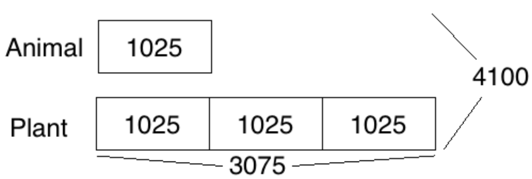



256 – 187 = $\boxed{\phantom{00}}$

Sarah sold $\boxed{\phantom{00}}$ cookies.

Grade 5 Example

*Sam has 1025 animal stickers. He has 3 times as many plant stickers as animal stickers. How many plant stickers does Sam have? How many stickers does Sam have altogether?*



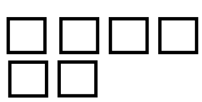

1 unit = 1025
3 units = 3075
2 units = 2050
4 units = 4100

1. He has __3075__ plant stickers.
2. He has __4100__ stickers altogether.

## Instructional Strategies

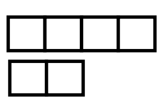
- Modeling two discrete quantities with small individual bars where each individual bar represents one unit. (This serves as an initial transition from the Unifix© cube model to a pictorial version.)

Bobby's candy bars
Molly's candy bars

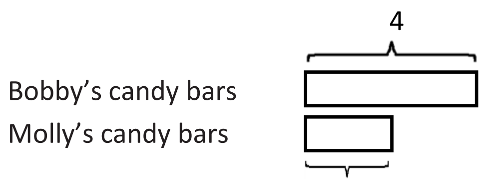


- Modeling two discrete quantities with incremented bars where each increment represents one unit.

Bobby's candy bars
Molly's candy bars



- Modeling two quantities (discrete or continuous) with non-incremented bars.

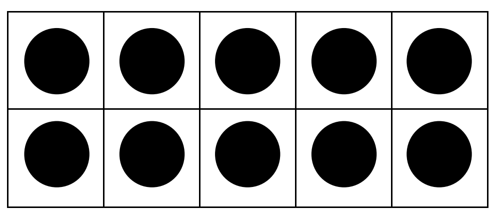



- Modeling a part-part-whole relationship where the bars represent known quantities, the total is unknown.
- Modeling a part-part-whole relationship with one part unknown.
- Modeling addition and subtraction comparisons.
- Modeling with equal parts in multiplication and division problems.
- Modeling with equal parts in fraction problems.

## Ten-Frame

**Grade Level PK – 3**

### Description



A ten-frame is a 2 by 5 grid (array) used to develop an understanding of concepts such as 5-patterns, combinations to 10, and adding and subtracting within 20. The frame is filled beginning on the top row, left to right, then proceeding to the bottom row building left to right. This pattern of filling supports subitizing by building on the 5 benchmark, as well as providing a pattern for placing disks on place value mats in later grades. Concrete counters as well as pictorial dots may be used to represent quantities on the frame.

In Kindergarten and in early Grade 1 a double ten-frame can be used to establish early foundations of place value (e.g., 13 is 10 and 3 or 1 ten and 3 ones) and can also be used on place value mats to support learning to add double digit numbers with regrouping. The "completion of a unit" on the ten-frame in early grades empowers students in later grades to understand a "make 100 (or 1000)" strategy, to add 298 and 37 (i.e., 298 + 2 + 35), and to more fully understand addition and subtraction of measurements (e.g., 4 ft. 8 in. + 5 in).

### Instructional Strategies

- "Flash" a ten-frame for 3-5 seconds then ask students to re-create what was filled/not filled on their own personal ten-frame. (Students may also tell how many they saw or match the "flash" with a numeral card.)
- Use "flash" technique, but ask students to tell 1 more or less than the number flashed.

- Roll dice and build the number on the ten-frame.
- Partner games: Partner 1 rolls a die and builds the number on the frame. Partner 2 rolls and adds that number to the frame (encouraging "10" and "leftovers" or using two ten-frames to represent the sum).
- Play Crazy Mixed Up Numbers. Have children represent a number on the ten-frame, then give various directions for changing the frame (e.g., start with 4 – "two more" – "one less" – "one fewer" – "double it" – "take away three"). This activity has the added benefit of providing the teacher with the opportunity to observe how students count – who clears the mat and starts over each time and who is counting on and/or subtracting.
- Write number stories about the filled and "unfilled" parts of the ten-frame.
- Counting in unit form:

| Regular | Unit Form |
|---|---|
| eleven<br>twelve<br>thirteen<br>twenty<br>twenty-six | 1 ten one<br>1 ten two<br>1 ten three<br>2 tens<br>2 tens six |

- Represent a number between 5 and 10 on the frame with one color counter. Have students add a quantity between 6 and 9 (represented by a second color) to it (e.g., 7 + 6). Encourage students to "fill the frame" and re-state the problem as 10 + 3.