Lesson 17: Fair Games

Student Outcomes

- Students use probability to learn what it means for a game to be fair.
- Students determine whether or not a game is fair.
- Students determine what is needed to make an unfair game fair.

Lesson Notes

The previous lesson focused on making fair decisions. The concept of fairness in statistics requires that one outcome is not favored over the other. In this lesson, students use probability to determine if a game is fair. The lesson begins with a class discussion of the meaning of *fair* in the context of games. When a fee is incurred to play a game, fair implies that the expected winnings are equal in value to the cost incurred by playing the game. If the game is not fair, students use the expected winnings to determine the cost to play the game. Later in the lesson, this idea is extended to warranties and using expected value to determine a fair price for coverage.

Classwork

Example 1 (2 minutes): What Is a Fair Game?

In the previous lesson, students used probability to determine if a fair decision was made (i.e., if one outcome was not favored over another). Discuss as a class the meaning of *fair* as it relates to a fee to play a game. Encourage students to share their ideas about fair games. Consider using the following during the discussion:

- Is a pay-to-play game fair only if the chance of winning and losing are equally likely?
- Does the amount you pay to play the game have an effect on whether the game is fair? How about the amount you can potentially win?

An alternative setting for the instant lottery game card described is to have six similar paper bags labeled A through F. Five of the bags each contain a $\$1.00$ bill, and one contains a $\$10.00$ bill. Randomly scratching off two disks on the card is the same as choosing two bags at random without replacement.

Make sure that students understand the game, and then have them complete Exercises 1–5.

Scaffolding:

- The word *fair* has multiple meanings and may confuse English language learners.
- In some instances, fair means without unjust advantage or cheating.
- In statistics, fair requires that one outcome is not favored over the other.
- A game is *fair* if the expected winnings are equal in value to the cost incurred to play the game.

Before they begin, consider posing the following question. Ask students to write or share their answers with a neighbor.

- How much would you be willing to pay in order to play this game? Explain your answer.
  - *Answers will vary. Student responses should be from* $\$2.00$ *to* $\$11.00$*. For example, I would pay* $\$4.00$ *to play the game. I could potentially win either* $\$2.00$ *or* $\$11.00$*, and* $\$4.00$ *seems like a reasonable amount given the outcomes.*

Example 1: What Is a Fair Game?

An instant lottery game card consists of six disks labeled A, B, C, D, E, F. The game is played by purchasing a game card and scratching off two disks. Each of five of the disks hides $\$1.00$, and one of the disks hides $\$10.00$. The total of the amounts on the two disks that are scratched off is paid to the person who purchased the card.

Scaffolding:

Note that the word *fair* is used differently in this lesson compared to the last:

- Lesson 16: A random number generator can be used to make a fair decision for who gets to choose a song to play at a school dance.
- Lesson 17: Paying $2.00 is a fair cost to play a carnival game where you have a $50/50$ chance of winning a stuffed animal.

Exercises 1–5 (7 minutes)

Have students work through the exercises with a partner, and then discuss answers as a class. The point of these exercises is for students to come to the conclusion that to justify the cost to play the game, the expected winnings should be equal to that cost, i.e., making the game fair.

As students are working, quickly check their work for Exercise 2 to be sure they are correctly identifying the number of ways for choosing the disks. When discussing the answers to Exercises 4 and 5 as a class, allow for multiple responses, but emphasize that the cost to play the game should be equal to the expected winnings for the game to be fair.

Exercises 1–5

1. **What are the possible total amounts of money you could win if you scratch off two disks?**

   If two $\$1.00$ disks are uncovered, the total is $\$2.00$. If one $\$1.00$ disk and the $\$10.00$ disk are uncovered, the total is $\$11.00$.

2. **If you pick two disks at random:**

   a. **How likely is it that you win $\$2.00$?**

   $$P(\text{win } \$2.00) = \frac{10}{15} = \frac{2}{3}$$

   b. **How likely is it that you win $\$11.00$?**

   $$P(\text{win } \$11.00) = \frac{5}{15} = \frac{1}{3}$$

   Following are two methods to determine the probabilities of getting $\$2.00$ and $\$11.00$.

   Method 1:

   List the possible pairs of scratched disks in a sample space, S, keeping in mind that two different disks need to be scratched and the order of choosing them does not matter. For example, you could use the notation AB that indicates disk A and disk B were chosen, in either order.

   $$S = \{AB, AC, AD, AE, AF, BC, BD, BE, BF, CD, CE, CF, DE, DF, EF\}$$

   There are 15 different ways of choosing two disks without replacement and without regard to order from the six possible disks.

   Identify the winning amount for each choice under the outcome in S. Suppose that disks A–E hide $\$1.00$, and disk F hides $\$10.00$.

Scaffolding:

- For advanced learners, consider posing the following question, and allow students to conjecture and devise a way to support their claim with mathematics.
- To play the game, you must purchase a game card. The price of the card is set so that the game is fair. How much should you be willing to pay for a game card if the game is to be a fair one? Explain using a probability distribution to support your answer.

Outcomes	AB	AC	AD	AE	AF	BC	BD	BE	BF	CD	CE	CF	DE	DF	EF
Winnings	$\$2$	$\$2$	$\$2$	$\$2$	$\$11$	$\$2$	$\$2$	$\$2$	$\$11$	$\$2$	$\$2$	$\$11$	$\$2$	$\$11$	$\$11$

Since each of the outcomes in S is equally likely, the probability of winning $\$2.00$ is the number of ways of winning $\$2.00$, namely 10, out of the total number of possible outcomes, 15. $P(\text{win } \$2.00) = \frac{10}{15} = \frac{2}{3}$.
Similarly, $P(\text{win } \$11.00) = \frac{5}{15} = \frac{1}{3}$.

Method 2:

Previous lessons studied permutations and combinations. Recall that counting when sampling was done without replacement and without regard to order involved combinations.

The number of ways of choosing two disks without replacement and without regard to order is ${}_6C_2 = \frac{6(5)}{2} = 15$. (${}_nC_k$ denotes the number of combinations of n items taken k at a time without replacement and without regard to order.)

To win $\$2.00$, two disks need to be chosen from the five $\$1.00$ disks. The number of ways of doing that is ${}_5C_2 = \frac{5(4)}{2} = 10$.

So the probability of winning $\$2.00$ is $\frac{10}{15} = \frac{2}{3}$.

To win $\$11.00$, one disk needs to be chosen from the five $\$1.00$ disks, and the $\$10.00$ disk needs to be chosen. The number of ways of doing that is $({}_5C_1)({}_1C_1) = 5(1) = 5$. So the probability of winning $\$11.00$ is $\frac{5}{15} = \frac{1}{3}$.

3. **Based on Exercise 3, how much should you expect to win on average per game if you played this game a large number of times?**

   The expected winning amount per play is $(2)\left(\frac{2}{3}\right) + (11)\left(\frac{1}{3}\right) = \$5.00$.

4. **To play the game, you must purchase a game card. The price of the card is set so that the game is fair. What do you think is meant by a fair game in the context of playing this instant lottery game?**

   Responses from students concerning what they think is meant by a fair game will no doubt vary. For example, the cost to play the game should be equal to the expected winnings.

5. **How much should you be willing to pay for a game card if the game is to be a fair one? Explain.**

   Responses will vary. In the context of this instant lottery game, the game is fair if the player is willing to pay $\$5.00$ (the expected winning per play) to purchase each game card.

Example 2 (2 minutes): Deciding Between Two Alternatives

Read through the example as a class, and answer any questions students may have about the game. Before having students complete the exercise, ask them if they were to encounter such a situation, would they actually play the game? Expect an interesting discussion that will involve risk taking. Those who would take the $10.00 rather than play the game are risk-adverse—perhaps even if the payoffs amounted to a higher expected value than the given situation of $12.00. (For example, three $1.00 bills, one $5.00 bill, and two $20.00 bills yield an expected value of $16.00.) Other students may be on the fence, perhaps tossing a fair coin to make their decision risk-neutral. And then there are the risk-seeking players who would play the game as long as the expected winnings exceeded the $10.00 amount that Mom was paying.

Ask students to respond to the following in writing and share their response with a neighbor:

- Would you play your mom's game? Explain why or why not.
  - *Answers will vary. Some students may be conservative and want to stick with keeping the* $10.00 *for completing their chore and not risk playing the game only to walk away with* $4.00. *Others may argue that it is worth the risk to play Mom's game and get* $25.00.

Example 2: Deciding Between Two Alternatives

You have a chore to do around the house for which your mom plans to pay you $\$10.00$. When you are done, your mom, being a mathematics teacher, gives you the opportunity to change the amount that you are paid by playing a game. She puts three $\$2.00$ bills in a bag along with two $\$5.00$ bills and one $\$20.00$ bill. She says that you can take the $\$10.00$ she offered originally or you can play the game by reaching into the bag and selecting two bills without looking. You get to keep these two bills as your payment.

Exercise 6 (5 minutes)

This exercise presents two alternatives. Students should make a rational decision between choosing to take Mom's $10.00 offer or to play Mom's game based on expected value. Clearly, the expected value of choosing Mom's $10.00 offer is just $10.00. The expected value of playing the game involves finding probabilities of outcomes and then calculating the expected value.

Mom's game is like the instant lottery of six disks A, B, C, D, E, F in Example 1, but instead of having two different dollar amounts, there are three. In Mom's game, three disks hide $2.00 bills, two disks hide $5.00 bills, and one disk hides a $20.00 bill.

Have students work in pairs or small groups to complete Exercise 6. As students are working, check their probability distributions. Then discuss the answers as a class. Although answers will vary as to whether or not students will play the game, the expected winnings ($12.00) should be used to justify their responses.

Scaffolding:

For students who struggle to answer the question, use the following to help guide them through the problem:

- What are the possible amounts of money that you might be paid if you play the game?
- Determine a probability distribution for the winnings.
- What is the expected value of this random variable?

MP.3 & MP.4

Exercise 6–7

6. **Do you think you should take your mom's original payment of $\$10.00$ or play the "bag" game? In other words, is this game a fair alternative to getting paid $\$10.00$? Use a probability distribution to help answer this question.**

Suppose that disks (bags) A, B, C hide $\$2.00$ *each, disks D and E hide* $\$5.00$ *each, and disk F hides* $\$20.00$.

Outcomes	*AB*	*AC*	*AD*	*AE*	*AF*	*BC*	*BD*	*BE*	*BF*	*CD*	*CE*	*CF*	*DE*	*DF*	*EF*
Winnings	$\$4$	$\$4$	$\$7$	$\$7$	$\$22$	$\$4$	$\$7$	$\$7$	$\$22$	$\$7$	$\$7$	$\$22$	$\$10$	$\$25$	$\$25$

You could win $\$4.00, \$7.00, \$10.00, \$22.00,$ *or* $\$25.00$.

Since each of the outcomes in S is equally likely, the probability of winning $\$4.00$ *is the number of ways of winning* $\$4.00$, *namely 3, out of the total number of possible outcomes,* 15. $P(win\ \$4.00) = \frac{3}{15}$. *Similarly,* $P(\text{win } \$7.00) =$ *the number of ways of winning* $7.00 *divided by the total number of possible outcomes, namely* $P(\text{win } \$7.00) = \frac{6}{15}$.

MP.3 & MP.4

The probability distribution for the winning amount per play is as follows:

Winning ($)	*Probability*
4	$\frac{3}{15}$
7	$\frac{6}{15}$
10	$\frac{1}{15}$
22	$\frac{3}{15}$
25	$\frac{2}{15}$

The expected winning amount per play is

$$(4)\left(\frac{3}{15}\right)+(7)\left(\frac{6}{15}\right)+(10)\left(\frac{1}{15}\right)+(22)\left(\frac{3}{15}\right)+(25)\left(\frac{2}{15}\right)=\$12.00.$$

The game is in your favor as its expected winning is $\$12.00$ compared to $\$10.00$, but answers will vary. The key is that students recognize that the expected payment for the game is greater than $\$10.00$ but that on any individual play, they could get less than $\$10.00$. In fact, the probability of getting less than $\$10.00$ is greater than the probability of getting $\$10.00$ or more.

Exercise 7 (5 minutes)

The game in Exercise 6 favors the player and not Mom. The purpose of this exercise it to have students explore how Mom's game can be changed so that it is fair on both sides, i.e., the expected winnings are equal to the cost to play ($\$10.00$). Have students work in a small group or with a partner to complete the exercise. There are multiple answers to this exercise and, if time allows, have groups share their answers with the class. Be sure that students support answers using expected value.

7. Alter the contents of the bag in Example 2 to create a game that would be a fair alternative to getting paid $\$10.00$. You must keep six bills in the bag, but you can choose to include bill-sized pieces of paper that are marked as $\$0.00$ to represent a $\$0.00$ bill.

Answers will vary. The easiest answer is to replace all six bills with $\$5.00$ bills. But other combinations are possible, such as three $\$0.00$ bills and three $\$10.00$ bills. If students come up with other possibilities, make sure they support their answer with an expected value calculation.

Example 3 (2 minutes): Is an Additional Year of Warranty Worth Purchasing?

Discuss the example with the class to make sure students understand the context. This example is an extension of the idea of a fair game: What cost would justify purchasing the warranty? That is, what is a fair price?

Example 3: Is an Additional Year of Warranty Worth Purchasing?

Suppose you are planning to buy a computer. The computer comes with a one-year warranty, but you can purchase a waranty for an additional year for $\$24.95$. Your research indicates that in the second year, there is a 1 in 20 chance of incurring a major repair that costs $\$180.00$ and a probability of 0.15 of a minor repair that costs $\$65.00$.

Exercises 8–9 (5 minutes)

Have students work through the exercises in a small group or with a partner. Students should use expected value to justify a fair price in Exercise 9.

Exercises 8–9

8. **Is it worth purchasing the additional year warranty? Why or why not?**

   The expected cost of repairs in the second year is $(0.05)(180) + (0.15)(65) = \$18.75$. *So, based on expected value,* $\$24.95$ *is too high.*

9. **If the cost of the additional year warranty is too high, what would be a fair price to charge?**

   A fair price for the waranty would be $\$18.75$.

Example 4 (5 minutes): Spinning a Pentagon

Lead a class discussion of Example 4, leading to the probabilities given below. Then have students complete Exercises 10 and 11. The probabilities of getting an odd sum or an even sum in spinning a regular pentagon can be found from the following matrix. The sums are the cell entries.

		SPIN 2				
		1	2	3	4	5
S	1	2	3	4	5	6
P	2	3	4	5	6	7
I	3	4	5	6	7	8
N	4	5	6	7	8	9
1	5	6	7	8	9	10

Scaffolding:

For advanced learners, consider providing the following extension to the lesson:

- Design your own fair game of chance. Use a probability distribution and expected value to explain why it is fair.

From the matrix, the probability of an odd sum is $\frac{12}{25}$ and the probability of an even sum is $\frac{13}{25}$.

Example 4: Spinning a Pentagon

Your math club is sponsoring a game tournament to raise money for the club. The game is to spin a fair pentagon spinner twice and add the two outcomes. The faces of the spinner are numbered $1, 2, 3, 4$, and 5. If the sum is odd, you win; if the sum is even, the club wins.

Exercises 10–11 (5 minutes)

The following exercises provide students with additional practice for determining how to make a game fair. Have students complete the exercises with a partner. If time is running short, work through the problems as a class.

Exercises 10–11

10. **The math club is trying to decide what to charge to play the game and what the winning payoff should be per play to make it a fair game. Give an example.**

    Answers will vary. If the game is to be fair, the expected amount of money taken in should equal the expected amount of payoff. Let x cents be the amount to play the game and y be the amount won by the player. The math club receives x cents whether or not the player wins. The math club loses y cents if the player wins. So, for the game to be fair, $x - y\left(\frac{12}{25}\right)$ must be 0. Any x and y that satisfy $25x - 12y = 0$ are viable. One example is for the math club to charge 12 cents to play each game with a payoff of 25 cents.

11. **What should the math club charge per play to make $\$0.25$ on average for each game played? Justify your answer.**

    Answers will vary. For the math club to clear 25 cents on average per game, the expected amount they receive minus the expected amount they pay out needs to equal 25, i.e., $x - y\left(\frac{12}{25}\right) = 25$, or $25x - 12y = 625$ where x and y are in cents. For example, if the math club charges 100 cents to play, then the player would receive 156.25 cents if she wins and the club would expect to clear 25 cents per game in the long run.

Closing (2 minutes)

- Ask students to summarize the main ideas of the lesson in writing or with a neighbor. Use this as an opportunity to informally assess comprehension of the lesson. The Lesson Summary below offers some important ideas that should be included.

Lesson Summary

- **The concept of fairness in statistics requires that one outcome is not favored over another.**
- **In a game that involves a fee to play, a game is fair if the amount paid for one play of the game is the same as the expected winnings in one play.**

Exit Ticket (5 minutes)

Name ______________________________ Date ____________

Lesson 17: Fair Games

Exit Ticket

A game is played with only the four kings and four jacks from a regular deck of playing cards. There are three "one-eyed" cards: the king of diamonds, the jack of hearts, and the jack of spades. Two cards are chosen at random without replacement from the eight cards. Each one-eyed card is worth $\$2.00$, and non-one-eyed cards are worth $\$0.00$. In the following table, *JdKs* indicates that the two cards chosen were the jack of diamonds and the king of spades. Note that there are 28 pairings. The one-eyed cards are highlighted.

JcJd	Jc**Jh**	Jc**Js**	JcKc	Jc**Kd**	JcKh	JcKs
Jd**Jh**	Jd**Js**	JdKc	Jd**Kd**	JdKh	JdKs	
JhJs	**Jh**Kc	**JhKd**	**Jh**Kh	**Jh**Ks		
JsKc	**JsKd**	**Js**Kh	**Js**Ks			
Kc**Kd**	KcKh	KcKs				
KdKh	**Kd**Ks					
KhKs						

a. What are the possible amounts you could win in this game? Write them in the cells of the table next to the corresponding outcome.

b. Find the the expected winnings per play.

c. How much should you be willing to pay per play of this game if it is to be a fair game?

Exit Ticket Sample Solutions

A game is played with only the four kings and four jacks from a regular deck of playing cards. There are three "one-eyed" cards: the king of diamonds, the jack of hearts, and the jack of spades. Two cards are chosen at random without replacement from the eight cards. Each one-eyed card is worth $\$2.00$, and non-one-eyed cards are worth $\$0.00$. In the following table, *JdKs* indicates that the two cards chosen were the jack of diamonds and the king of spades. Note that there are 28 pairings. The one-eyed cards are highlighted.

JcJd	Jc**Jh**	Jc**Js**	JcKc	Jc**Kd**	JcKh	JcKs
Jd**Jh**	Jd**Js**	JdKc	Jd**Kd**	JdKh	JdKs	
JhJs	**Jh**Kc	**JhKd**	**Jh**Kh	**Jh**Ks		
JsKc	**JsKd**	**Js**Kh	**Js**Ks			
Kc**Kd**	KcKh	KcKs				
KdKh	**Kd**Ks					
KhKs						

a. **What are the possible amounts you could win in this game? Write them in the cells of the table next to the corresponding outcome.**

JcJd	0	Jc**Jh**	2	Jc**Js**	2	JcKc	0	Jc**Kd**	2	JcKh	0	JcKs	0
Jd**Jh**	2	Jd**Js**	2	JdKc	0	Jd**Kd**	2	JdKh	0	JdKs	0		
JhJs	4	**Jh**Kc	2	**JhKd**	4	**Jh**Kh	2	**Jh**Ks	2				
JsKc	2	**JsKd**	4	**Js**Kh	2	**Js**Ks	2						
Kc**Kd**	2	KcKh	0	KcKs	0								
KdKh	2	**Kd**Ks	2										
KhKs	0												

b. **Find the the expected winnings per play.**

Ask your students to verify their accumulated counts using combinations. There are 10 pairing where neither card is one-eyed (${}_5C_2 = 10$); 15 pairings with one card one-eyed ($({}_5C_1)({}_3C_1) = 15$); and 3 pairings in which both cards are one-eyed (${}_3C_3 = 1$).

Expected winnings per play $= (0)\left(\frac{10}{28}\right) + (2)\left(\frac{15}{28}\right) + (4)\left(\frac{3}{28}\right) = \frac{42}{28}$, *or* $\$1.50$.

c. **How much should you be willing to pay per play of this game if it is to be a fair game?**

According to the expected value of part (b), a player should be willing to pay $\$1.50$ to play the game in order for it to be a fair game.

Problem Set Sample Solutions

1. **A game is played by drawing a single card from a regular deck of playing cards. If you get a black card, you win nothing. If you get a diamond, you win $\$5.00$. If you get a heart, you win $\$10.00$. How much would you be willing to pay if the game is to be fair? Explain.**

   The following represents a probability distribution:

Outcomes	*Probability*
0	$\frac{1}{2}$
5	$\frac{1}{4}$
10	$\frac{1}{4}$

   The expected winnings are $0\left(\frac{1}{2}\right)+5\left(\frac{1}{4}\right)+10\left(\frac{1}{4}\right)=3.75$. *For the game to be fair, the fee should be* $\$3.75$.

2. **Suppose that for the game described in Problem 1, you win a bonus for drawing the queen of hearts. How would that change the amount you are willing to pay for the game? Explain.**

   The fee to play the game should increase, as the expected winnings will increase. For example, if a bonus of $\$10.00$ *is earned for drawing the queen of hearts, the probability distribution will be*

Outcomes	*Probability*
0	$\frac{1}{2}$
5	$\frac{1}{4}$
10	$\frac{12}{52}$
20	$\frac{1}{52}$

   The expected winnings are $0\left(\frac{1}{2}\right)+5\left(\frac{1}{4}\right)+10\left(\frac{12}{52}\right)+20\left(\frac{1}{52}\right)\approx 3.94$. *For the game to be fair, the fee should be no more than* $\$3.94$, *which is an increase from* $\$3.75$.

3. **You are trying to decide between playing two different carnival games and want to only play games that are fair.**

One game involves throwing a dart at a balloon. It costs $\$10.00$ to play, and if you break the balloon with one throw, you win $\$75.00$. If you do not break the balloon, you win nothing. You estimate that you have about a 15% chance of breaking the balloon.

The other game is a ring toss. For $\$5.00$ you get to toss three rings and try to get them around the neck of a bottle. If you get one ring around a bottle, you win $\$3.00$. For two rings around the bottle, you win $\$15.00$. For three rings, you win $\$75.00$. If no rings land around the neck of the bottle, you win nothing. You estimate that you have about a 15% chance of tossing a ring and it landing around the neck of the bottle. Each toss of the ring is independent.

Which game will you play? Explain.

The following are probability distributions for each of the games:

Dart Outcomes	*Probability*
$0\ (\$0.00)$	0.85
$1\ (\$75.00)$	0.15

Ring Toss Outcomes	*Probability*
$0\ (\$0.00)$	0.85^3
$1\ (\$3.00)$	$3(0.15)(0.85)^2$
$2\ (\$15.00)$	$3(0.15)^2(0.85)$
$3\ (\$75.00)$	$(0.15)^3$

The expected winnings for the dart game are $0(0.85) + 75(0.15) = \$11.25$. The dart game is fair because the cost to play is less than the expected winnings.

The expected winnings for the ring toss game are $0 \cdot (0.85)^3 + 3 \cdot 3(0.15)(0.85)^2 + 15 \cdot 3(0.15)^2(0.85) + 75 \cdot (0.15)^3 \approx \$2.09$. For the ring toss outcomes of 1 and 2, there is a multiplier of 3 because there are three different ways to get 1 and 2 rings on a bottle in 3 tosses. The ring toss game is not fair because the cost to play is more than the expected winnings.

I would play the dart game.

4. **Invent a fair game that involves three fair number cubes. State how the game is played and how the game is won. Explain how you know the game is fair.**

Answers will vary. One example is given here:

The key is to determine probabilities correctly. Rolling three fair number cubes results in $(6)(6)(6) = 216$ possible ordered triples. Two events of interest could be "all same," i.e., 111, 222, 333, 444, 555, or 666. The probability of "all same" is $\frac{6}{216}$. Another event could be "all different." The number of ways of getting "all different" digits is the permutation, $(6)(5)(4) = 120$. Getting any other outcome would have probability $1 - \left(\frac{6}{216}\right) - \left(\frac{120}{216}\right) = \frac{90}{216}$.

Consider this game: Roll three fair number cubes. If the result is "all same digit," then you win $\$20.00$. If the result is "all different digits," you win $\$1.00$. Otherwise, you win $\$0.00$. It costs $\$1.11$ to play.

The game is approximately fair since the expected winnings per play $= (20)\left(\frac{6}{216}\right) + (1)\left(\frac{120}{216}\right) + (0)\left(\frac{90}{216}\right) \approx \$1.11$.

5. **Invent a game that is not fair that involves three fair number cubes. State how the game is played and how the game is won. Explain how you know the game is not fair.**

Answers will vary. One example is given here:

From Problem 4, any fee to play that is not $\$1.11$ results in a game that is not fair. If the fee to play is lower than $\$1.11$, then the game is in the player's favor. If the fee to play is higher than $\$1.11$, then the game is in the favor of whoever is sponsoring the game.