Lesson 14: Games of Chance and Expected Value

Student Outcomes

- Students use expected payoff to compare strategies for a simple game of chance.

Lesson Notes

This lesson uses examples from the previous lesson as well as some new examples that expand on previous work. The lesson begins with an example in which students informally examine ticket payout distributions for three different games and form a conjecture around which game would be the best to play. Students then use expected value to support or revise their conjecture and answer questions about strategy. Throughout the lesson, students compare presented situations and choose the best course of action based on expected value.

Classwork

Example 1 (5 minutes): Which Game to Play?

MP.1

Explain the context of Example 1. Consider reading the example out loud as needed and using visuals to demonstrate the context of the problem. Make sure that students understand that they are to recommend which game should be played. Students informally examine payout distributions of three different games and conjecture about which game would be the best to play and evaluate their conjecture in the subsequent exercises.

- Now that you have investigated how expected value can be used to determine the average gain (or loss) associated with a random process such as a carnival game, lottery ticket, etc., you will investigate how to use expected value to determine the best strategy for reaching a certain goal. You will also further investigate some of the games listed in the previous lesson.

Example 1: Which Game to Play?

As mentioned in the previous lesson, games of chance are very popular. Some towns, amusement parks, themed restaurants, etc., have arcades that contain several games of chance. In many cases, tickets are awarded as a form of currency so that players can obtain tickets and eventually exchange them for a large prize at a prize center located within the arcade.

Suppose you are asked to give advice to a child who is interested in obtaining a prize that costs $1,000$ tickets. The child can choose from the following three games: a spinning wheel, a fishing game (very similar to the duck pond game described in the previous lesson), and a slot machine–style game with cartoon characters. Again, each of these is a game of chance; no skill is involved. Each game costs $\$0.50$ per play. The child will play only one of these three games but will play the game as many times as it takes to earn $1,000$ tickets.

Below are the ticket payout distributions for the three games:

Spinning Wheel	
Tickets	Probability
1	0.51
2	0.35
5	0.07
10	0.04
100	0.03

Fishing Game	
Tickets	Probability
1	0.50
5	0.20
10	0.15
30	0.14
150	0.01

Slot Machine	
Tickets	Probability
1	0.850
2	0.070
10	0.060
100	0.019
500	0.001

Which game would you recommend to the child?

Scaffolding:

- For English language learners, consider providing a sentence frame for Exercise 1:

  I think ____ might be the best choice because ____.
- Ask advanced learners to imagine designing their own game and answer the following:
- Describe your game.
- What is the expected payout of your game? Why?

Exercises 1–3 (15 minutes)

In this set of exercises, students use expected value to explore their conjecture about the best game to play. Consider having students work in groups of three on the exercises. That way, they can share the work of calculating the three expected values in Exercise 1.

MP.3

Exercises 1–3

1. **At first glance of the probability distributions of the three games, without performing any calculations, which do you think might be the best choice and why?**

   Answers will vary. Sample response: I think the spinning wheel might be the best choice because the probabilities are relatively high for winning 10 and 100 tickets.

2. **Perform necessary calculations to determine which game to recommend to the child. Explain your choice in terms of both tickets and price. Is this the result you anticipated?**

Spinning Wheel		
Tickets	Probability	$X \cdot P(X)$
1	0.51	0.51
2	0.35	0.70
5	0.07	0.35
10	0.04	0.40
100	0.03	3.00
	Sum = $E(X)$	$= 4.96$ *tickets*

Fishing Game		
Tickets	Probability	$X \cdot P(X)$
1	0.50	0.50
5	0.20	1.00
10	0.15	1.50
30	0.14	4.20
150	0.01	1.50
	Sum = $E(X)$	$= 8.7$ *tickets*

Slot Machine		
Tickets	Probability	$X \cdot P(X)$
1	0.850	0.85
2	0.070	0.14
10	0.060	0.60
100	0.019	1.90
500	0.001	0.50
	Sum = $E(X)$	$= 3.99$ *tickets*

The fishing game would be the best game to play because, on average, the child will earn about 9 tickets for each play—but for the spinning wheel and slot machine games, she will only win about 5 or 4 tickets (respectively, on average) for each play.

The cost of the prize, if won by playing the fishing game, will average $\frac{1000}{8.7} \cdot \$0.50 \approx \$57.47$. If the spinning wheel or slot machine game is played, the average cost of the prize becomes $\frac{1000}{4.96} \cdot \$0.50 \approx \$100.81$ or $\frac{1000}{3.99} \cdot \$0.50 \approx \$125.31$, respectively.

Answers about prediction will vary.

MP.3

3. The child states that she would like to play the slot machine game because it offers a chance of winning 500 tickets per game, and that means she might only have to play twice to reach her goal, and none of the other games offer that possibility. Using both the information from the distributions above and your expected value calculations, explain to her why this might not be the best strategy.

While the child is correct in her statement, the chances of earning 500 tickets with 1 play at the slot machine are very small. The event would happen in the long run only 1 out of $1,000$ tries. The chances of earning $1,000$ tickets in only 2 tries is very small (probability $= 0.001 \cdot 0.001 = 0.000001 = \frac{1}{1{,}000{,}000}$ or 1 in 1 million). Given the rare chance of a high payout and the high chance of a low payout, the slot machine is not the way to go.

With the fishing game, she will earn about 9 tickets on average for each play—but for the spinning wheel and slot machine games, she will earn about half that many tickets on average with each play. That means that she may have to spend nearly twice the amount of money for the same expected number of tickets.

Example 2 (5 minutes): Insurance

Explain the context of Example 2. Make sure that students understand the difference between the two plans. As in the first example of the lesson, students should informally examine the given information and form a conjecture around which plan might be the best option for the company to offer.

Example 2

Insurance companies consider expected value when developing insurance products and determining the pricing structure of these products. From the perspective of the insurance company, the company "gains" each time it earns more money from a customer than it needs to pay out to the customer.

An example of this would be a customer paying a one-time premium (that's the cost of insurance) of $\$500.00$ to purchase a one-year, $\$10,000.00$ casualty policy on an expensive household item that ends up never being damaged, stolen, etc., in that one-year period. In that case, the insurance company gained $\$500.00$ from that transaction.

However, if something catastrophic did happen to the household item during that one-year period (such that it was stolen or damaged so badly that it could not be repaired, etc.), the customer could then ask the insurance company for the $\$10,000.00$ of insurance money per the agreement, and the insurance company would lose $\$9,500.00$ from the transaction.

Imagine that an insurance company is considering offering two coverage plans for two major household items that owners would typically want to insure (or are required by law to insure). Based on market analysis, the company believes that it could sell the policies as follows:

- **Plan A: Customer pays a one-year premium of $\$600.00$ and gets $\$10,000.00$ of insurance money if Item A is ever stolen or damaged so badly that it could not be repaired, etc., that year.**
- **Plan B: Customer pays a one-year premium of $\$900.00$ and gets $\$8,000.00$ of insurance money if Item B is ever stolen or damaged so badly that it could not be repaired, etc., that year.**

It is estimated that the chance of the company needing to pay out on a Plan A policy is 0.09%, and the chance of the company needing to pay out on a Plan B policy is 3.71%.

Which plan should the company offer?

Exercise 4 (12 minutes)

In this exercise, students must make sense of the problem and construct a viable plan for supporting their conjecture from Example 2. Students will first need to determine the probability distributions for each plan and then use the distributions to calculate expected value. Let students continue to work in groups of three on Exercise 4.

MP.3

Exercise 4

4. **The company can market and maintain only one of the two policy types, and some people in the company feel it should market Plan B since it earns the higher premium from the customer and has the lower claim payout amount. Assuming that the cost of required resources for the two types of policies is the same (for the advertising, selling, maintaining, etc., of the policies) and that the same number of policies would be sold for either Plan A or Plan B. In terms of earning the most money for the insurance company, do you agree with the Plan B decision? Explain your decision.**

Plan A

Gain	*Probability*	$X \cdot P(X)$
$\$600.00$	0.9991	$\$599.46$
$-\$10,000.00$	0.0009	$-\$9.00$
	Sum = $E(X)$	$= \$590.46$

Plan B

Gain	*Probability*	$X \cdot P(X)$
$\$900.00$	0.9629	$\$866.61$
$-\$8,000.00$	0.0371	$-\$296.80$
	Sum = $E(X)$	$= \$569.81$

The Plan A policy would be better for the company as it has the higher expected value (and it is assumed that the same number of policies would be sold regardless of plan). This means the company will earn more on average for every Plan A policy sold than it would for every Plan B policy sold—specifically, $\$20.65$ ***more per policy sold on average. This difference will really add up as more and more policies are sold. So, even though Plan B has a higher premium and a lower payout, a payout is much more likely to occur with Plan B, making it the less attractive option.***

Closing (3 minutes)

- The "duck pond" game and Maryland Lottery game from the previous lesson both had a negative expected value from the player's perspective. However, these games of chance are very popular. Why do you suppose that games of chance, which have a negative expected value, are still widely played even when people are aware that they will most likely lose money in the long run? (Hint: Think about the child who wanted to play the slot machine game to earn her 1,000 tickets in this lesson.)
    - *The idea of a "big payout" is an appealing idea to most people who play games of chance. This is why lottery sales tend to increase dramatically when the jackpots become larger. Also, effective advertising about a "big payout" whether it is at a carnival or in media may influence people. Many people do not consider the long-run behavior of playing games of chance; rather, they are hopeful that the short-term immediate outcomes of a game will be favorable for them.*
- Ask students to summarize the main ideas of the lesson in writing or with a neighbor. Use this as an opportunity to informally assess comprehension of the lesson. The Lesson Summary below offers some important ideas that should be included.

Lesson Summary

The application of expected value is very important to many businesses, lotteries, and others. It helps to determine the average gain or loss that can be expected for a given iteration of a probability trial.

By comparing the expected value, $E(X)$, for different games of chance (or situations that closely mirror games of chance), one can determine the most effective strategy to reach one's goal.

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 14: Games of Chance and Expected Value

Exit Ticket

In the previous lesson, you examined the Maryland Lottery's Pick 3 game where the chance of winning with a Front Pair bet is 0.01. (http://mdlottery.com/games/pick-3/payouts)

In that game, a successful bet of $\$1.00$ pays out $\$50.00$ for a net gain to the player of $\$49.00$.

Imagine that the state also offers a $\$1.00$ scratch-off lottery game with the following net gain distribution:

Scratch-Off Lottery

Net Gain	Probability
$-\$1.00$	0.9600
$\$9.00$	0.0389
$\$99.00$	0.0010
$\$999.00$	0.0001

If you had a friend who wanted to spend $\$1.00$ each day for several days on only 1 of these 2 lottery games, which game would you recommend? Explain.

Exit Ticket Sample Solutions

In the previous lesson, you examined the Maryland Lottery's Pick 3 game where the chance of winning with a Front Pair bet is 0.01. (http://mdlottery.com/games/pick-3/payouts/)

In that game, a successful bet of $\$1.00$ pays out $\$50.00$ for a net gain to the player of $\$49.00$.

Imagine that the state also offers a $\$1.00$ scratch-off lottery game with the following net gain distribution:

Scratch-Off Lottery

Net Gain	Probability
$-\$1.00$	0.9600
$\$9.00$	0.0389
$\$99.00$	0.0010
$\$999.00$	0.0001

If you had a friend who wanted to spend $\$1.00$ each day for several days on only 1 of these 2 lottery games, which game would you recommend? Explain.

For the Maryland Lottery Pick 3, $E(X) = 0.99 \cdot -\$1.00 + 0.01 \cdot \$49.00 = -\$0.99 + \$0.49 = -\$0.50$ (determined in previous lesson).

Scratch-Off Lottery

Net Gain	Probability	$X \cdot P(X)$
$-\$1.00$	0.9600	-0.9600
$\$9.00$	0.0389	0.3501
$\$99.00$	0.0010	0.0990
$\$999.00$	0.0001	0.0999
	Sum $= E(X)$	$= -\$0.411$

So, if a person were going to play a $\$1.00$ game over and over again, he should play the scratch-off game as the player's average loss per game is lower by about $\$0.09$. (Note: Both games have negative expected values; placing the $\$1.00$ in a bank savings account each day is a much better idea in terms of earning money.)

Problem Set Sample Solutions

1. **In the previous lesson, a duck pond game was described with the following payout distribution to its players:**

Event	Y	Probability of Y
Small	$-\$1.50$	0.60
Medium	$-\$0.50$	0.30
Large	$\$3.00$	0.10

where $Y =$ the *net* amount that a player won (or lost) playing the duck game one time.

This led to a situation where the people running the game could expect to gain $\$0.75$ on average per attempt.

Someone is considering changing the probability distribution as follows:

Event	Y	Probability of Y
Small	$-\$1.50$	0.70
Medium	$-\$0.50$	0.18
Large	$\$3.00$	0.12

Will this adjustment favor the players, favor the game's organizers, or will it make no difference at all in terms of the amount the organization can expect to gain on average per attempt?

The expected value for the adjusted distribution:

Event	Y	Probability of Y	$Y \cdot$ *Probability of* Y
Small	$-\$1.50$	0.70	$-\$1.05$
Medium	$-\$0.50$	0.18	$-\$0.09$
Large	$\$3.00$	0.12	$\$0.36$
		Sum:	$-\$0.78 = E(Y)$

So, the new distribution will favor the game's organizers. On average, the organizers will earn an additional $\$0.03$ for every game played.

2. **In the previous lesson's Problem Set, you were asked to make a model of a spinning wheel with a point distribution as follows:**

- **You gain 2 points 50% of the time.**
- **You lose 3 points 25% of the time.**
- **You neither gain nor lose any points 25% of the time.**

When $X =$ the number of points earned in a given spin, $E(X) = 0.25$ points.

Suppose you change the probabilities by "moving" 10% of the distribution as follows:

- **You gain 2 points 60% of the time.**
- **You lose 3 points 15% of the time.**
- **You neither gain nor lose any points 25% of the time.**

a. Without performing any calculations, make a guess as to whether or not this new distribution will lead to a player needing a fewer number of attempts than before on average to attain 5 or more points. Explain your reasoning.

Since the chance of winning 2 points is higher, the chance of losing 3 points is lower, and the chance of earning 0 points is unchanged, that probably means that a person can reach 5 points more quickly than before.

b. Determine the expected value of points earned from 1 game based on this new distribution. Based on your computation, how many spins on average do you think it might take to reach 5 points?

$E(X) = 2 \cdot 0.60 + (-3) \cdot 0.15 + 0 \cdot 0.25 = 0.75$ *points. The expected value tripled. Now it is expected to only take* $\frac{5}{0.75} = 6.67$ *(about 7) spins on average to reach 5 points.*

c. Does this value from part (b) support your guess in part (a)? (Remember that with the original distribution and its expected value of 0.25 points per play, it would have taken 20 spins on average to reach 5 points.)

Yes. The new distribution will only require 6.67 spins on average to reach 5 points. That is far fewer than the previous expected number of spins (20) needed on average to reach 5 points. This is in line with our conjecture back in part (a) that a person would be able to reach 5 points more quickly than before.

3. You decide to invest $\$1,000.00$ in the stock market. After researching, you estimate the following probabilities:

- Stock A has a 73% chance of earning a 20% profit in 1 year, an 11% chance of earning no profit, and a 16% chance of being worthless.
- Stock B has a 54% chance of earning a 75% profit in 1 year, a 23% chance of earning no profit, and a 23% chance of being worthless.

a. At first glance, which seems to be the most appealing?

Answers vary. Sample response: Stock A seems more appealing because there are lower probabilities of making no profit and losing the entire $\$1,000.00$ investment.

b. Which stock should you decide to invest in and why? Is this what you predicted?

Let X = the value of the investment in Stock A after 1 year.

Event	X	$P(X)$	$X \cdot P(X)$
20% Profit	$\$1,200.00$	0.73	$\$876.00$
No Profit	$\$1,000.00$	0.11	$\$110.00$
Worthless	$-\$1,000.00$	0.16	$-\$160.00$
		Sum:	$\$826.00 = E(X)$

Let Y = the value of the investment in Stock B after 1 year.

Event	Y	$P(Y)$	$Y \cdot P(Y)$
75% Profit	$\$1,750.00$	0.54	$\$945.00$
No Profit	$\$1,000.00$	0.23	$\$230.00$
Worthless	$-\$1,000.00$	0.23	$-\$230.00$
		Sum:	$\$945.00 = E(Y)$

I should invest in Stock B because the total expected value of my investment is $\$945.00$, whereas Stock B is only expected to yield $\$826.00$. Stock B is riskier than Stock A, but the expected value is greater.

4. The clock is winding down in the fourth quarter of the basketball game. The scores are close. It is still anyone's game. As the team's coach, you need to quickly decide which player to put on the court to help ensure your team's success. Luckily, you have the historical data for Player A and Player B in front of you.

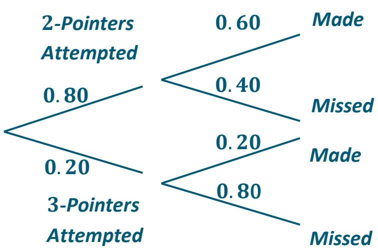
- 80% of Player A's shot attempts have been 2-point field goals, and 60% of them have hit their marks. The remaining shots have been 3-point field goals, and 20% of them have hit their marks.
- 85% of Player B's shot attempts have been 2-point field goals, and 62% of them have hit their marks. The remaining shots have been 3-point field goals, and 22% of them have hit their marks.

a. At first glance, whom would you put in and why?

Answers vary.

b. Based on these statistics, which player might be more likely to help lead your team to victory and why?

Player A Tree Diagram



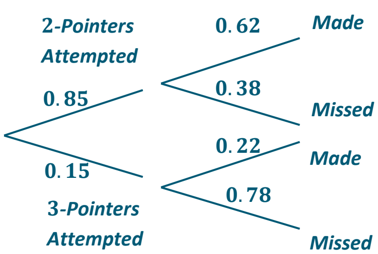
Percent of shots that were 2-pointers and successful $= 0.48$

Percent of shots that were 2-pointers and unsuccessful $= 0.32$

Percent of shots that were 3-pointers and successful $= 0.04$

Percent of shots that were 3-pointers and unsuccessful $= 0.16$

Player B Tree Diagram



Percent of shots that were 2-pointers and successful $= 0.53$

Percent of shots that were 2-pointers and unsuccessful $= 0.32$

Percent of shots that were 3-pointers and successful $= 0.03$

Percent of shots that were 3-pointers and unsuccessful $= 0.12$

Let X = Player A's points per shot.

Event	X	$P(X)$	$X \cdot P(X)$
2-Pointer	2	0.48	0.96
3-Pointer	3	0.04	0.12
Miss	0	0.48	0
		Sum:	$1.08 = E(X)$

Let Y = Player B's points per shot.

Event	Y	$P(Y)$	$Y \cdot P(Y)$
2-Pointer	2	0.53	1.06
3-Pointer	3	0.03	0.09
Miss	0	0.44	0
		Sum:	$1.15 = E(Y)$

From the analysis, it appears as though Player B might be able to score slightly more points than Player A with 1.15 points per shot versus 1.08. However, both players have very close expected points per shot, so either would be a viable candidate.

5. Prior versions of College Board examinations (SAT, AP) awarded the test taker with 1 point for each correct answer and deducted $\frac{1}{4}$ point for each incorrect answer. Current versions have eliminated the point deduction for incorrect responses (test takers are awarded 0 points).

The math section of the SAT contains 44 multiple-choice questions, with choices A–E. Suppose you answer all the questions but end up guessing on 8 questions. How might your math score look different on your score report using each point system? Explain your answer.

Let X = points per question with deduction.

Event	X	$P(X)$	$X \cdot P(X)$
Correct	1	0.2	0.2
Incorrect	-0.25	0.8	-0.2
		Sum:	$0 = E(X)$

Let Y = points per question without deduction.

Event	Y	$P(Y)$	$Y \cdot P(Y)$
Correct	1	0.2	0.2
Incorrect	0	0.8	0
		Sum:	$0.2 = E(Y)$

With the deduction system, a test taker would score an expected 0 points $(0 \cdot 8)$ on the 8 guessed questions, while on the nondeduction system, a test taker would score an expected 1.6 points $(0.2 \cdot 8)$ on the 8 guessed questions.