Lesson 13: Games of Chance and Expected Value

Student Outcomes

- Students analyze simple games of chance.
- Students calculate expected payoff for simple games of chance.
- Students interpret expected payoff in context.

Lesson Notes

When students are presented with a complete probability distribution for given random variable (X), they can calculate the *expected value*, $E(X)$, of the random variable. $E(X)$ is obtained from the following formula:

$$E(X) = \sum (\text{Value of } x) \cdot (\text{Probability of that value}) = \sum (X \cdot \text{Probability of } X)$$

This value, $E(X)$, can be interpreted as the long-run average of X over many repeated trials, such as over many rolls of the dice, plays at a lottery game, attempts at a carnival game, etc.

In addition to asking students to compute $E(X)$, this lesson has students examine the context of this value within the framework of payouts for simple games of chance. Specifically, students should recognize that a negative expected value (e.g., $-\$0.25$) states that on average, the player will lose this amount for every attempt at the game. While a player might not lose exactly $\$0.25$ on each game, if the player were to play the game over and over again, in the long run, we would expect the player's total losses to equal $\$0.25$ multiplied by the number of games played. Keep in mind that the actual total winnings (or losses) after a finite series of games will vary from player to player.

The material covered in Lessons 13 and 14 provides an opportunity for students to explore long-run behavior through simulation. Some students may benefit from developing or creating simulations with either manipulatives or technology. For example, on the ducks question below, students could use 10 colored disks (e.g., 6 red, 3 white, and 1 blue) in a bag and simulate the game by selecting a disk (and then returning it). Students could also make a spinning wheel as shown in Problem 2 of the Problem Set. Random number generators in popular and/or public software could also be used.

Classwork

Example 1 (2 minutes): Ducks at the Charity Carnival

MP.3

Have the class read the description of the game in Example 1. Ask students to write a brief response on paper around the following question to compare to Exercise 2 and be ready to share their response with a partner when they begin work on Example 3:

- You have to pay a fee to play a carnival game. What fee would you be willing to pay? Explain.
  - Sample Response: I would pay $\$1.00$ to play the game because I might not win, but the carnival needs the money to cover their costs. A dollar seems appropriate to pay for a couple of minutes of fun. More than $\$1.00$ might be too much, and not many people would play.

Make sure students understand the game before they begin to work on the exercises.

Example 1: Ducks at the Charity Carnival

One game that is popular at some carnivals and amusement parks involves selecting a floating plastic duck at random from a pond full of ducks. In most cases, the letters S, M, or L appear on the bottom of the duck signifying that the winner receives a small, medium, or large prize, respectively. The duck is then returned to the pond for the next game.

Although the prizes are typically toys, crafts, etc., suppose that the monetary values of the prizes are as follows: Small = $\$0.50$, Medium = $\$1.50$, and Large = $\$5.00$.

The probabilities of winning an item on 1 duck selection are as follows: Small 60%, Medium 30%, and Large 10%.

Suppose a person plays the game 4 times. What is the expected monetary value of the prizes won?

Scaffolding:

- For English language learners, consider displaying the following sentence frame on the board at the front of the classroom to aid in writing a response: I would be willing to pay ____ to play a game if the prize was worth ____.
- For advanced learners, provide an opportunity for further exploration around expected value. For instance:
- You want the carnival to make more money in the long run from these games. How would you change the expected value to help make that happen?

Exercises 1–4 (10 minutes)

Students should work in pairs to complete Exercises 1–4.

Exercises 1–4

1. Let X = the monetary value of the prize that you win playing this game 1 time. Complete the table below and calculate $E(X)$.

Event	X	Probability of X	$X \cdot$ Probability of X	
Small	$\$0.50$	0.6		
Medium	$\$1.50$	0.3		
Large	$\$5.00$	0.1		
		Sum:		$= E(X)$

Answer:

Event	X	Probability of X	$X \cdot$ Probability of X	
Small	$\$0.50$	0.6	$\$0.30$	
Medium	$\$1.50$	0.3	$\$0.45$	
Large	$\$5.00$	0.1	$\$0.50$	
		Sum:	$\$1.25$	$= E(X)$

2. Regarding the $E(X)$ value you computed above, can you win exactly that amount on any 1 play of the game?

No, you cannot win exactly $\$1.25$ on any 1 play. On average, you are expected to win $\$1.25$ per play if you were to play many times.

3. What is the least you could win in 4 attempts? What is the most you could win in 4 attempts?

The least amount that could be won in 4 attempts is $\$2.00$ ($4 \cdot \$0.50$), and the most is $\$20.00$ ($4 \cdot \$5.00$).

MP.2

4. How would you explain the $E(X)$ value in context to someone who had never heard of this measurement? What would you *expect* for the total monetary value of your prizes from 4 attempts at this game? (To answer this question, check the introduction to this lesson; do not do anything complicated like developing a tree diagram for all of the outcomes from 4 attempts at the game.)

On average, a person is expected to win $\$5.00$ ($4 \cdot E(X)$) for every 4 attempts. The winnings could be as little as $\$2.00$ (4 small) or as large as $\$20.00$ (4 larges). If 4 attempts were made many times, the winnings would average out to be about $\$5.00$.

The question above gets at an important feature of expected value. If the probability distribution remains the same for each trial (also called *independent trials*), you can determine the expected value for N trials by computing the expected value of 1 trial and multiplying that $E(X)$ value by N. You do not have to develop a complicated probability distribution covering all the results of all N trials. This rule makes it far easier to compute expected value for situations where several hundred independent trials occur, such as a carnival, lottery, etc.

Example 2 (2 minutes): Expected Value for Repeated Trials

Read the details of the experiment as a class.

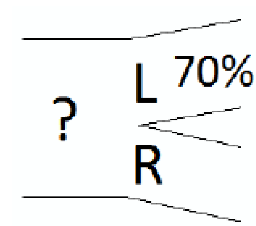
Example 2: Expected Value for Repeated Trials

In a laboratory experiment, 3 mice will be placed in a simple maze one at a time. In the maze, there is 1 decision point where the mouse can turn either left (L) or right (R). When the 1^{st} mouse arrives at the decision point, the direction the mouse chooses is recorded. The same is done for the 2^{nd} and the 3^{rd} mouse. The researchers conducting the experiment add food in the simple maze such that the long-run relative frequency of each mouse turning left is believed to be 0.7 or 70%.

Scaffolding:

For struggling students:

- Have them read the problem aloud and then turn to a partner and summarize, or
- Provide a visual such as the one below to assist in understanding the scenario.



Exercises 5–8 (8 minutes)

Have students work in pairs or small groups to answer Exercises 5–8. When students are finished, discuss the answers to make sure that all students were able to calculate the expected values correctly. Check in with students as they work to ensure quality responses. Each student's level of understanding can be gauged during this time, and misconceptions and errors can be addressed and corrected. Make sure all of the students are prepared to share their responses.

Exercises 5–8

5. **Examining the outcomes for just 1 mouse, define the random variable and complete the following table:**

Event	X	Probability of X	$X \cdot$ Probability of X	
Left	1	0.7		
Right	0	0.3		
		Sum:		$= E(X)$

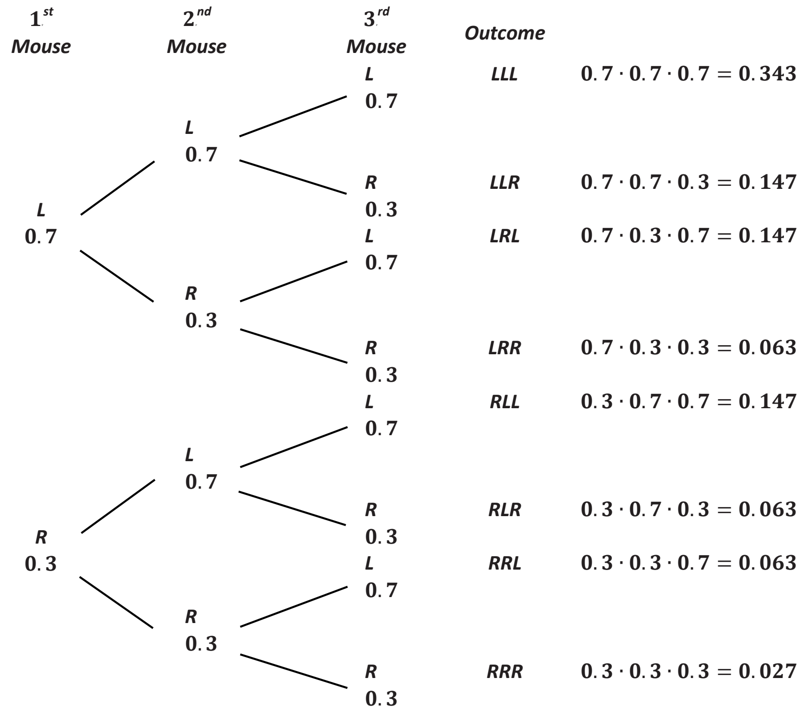
X = the number of left turns made by 1 mouse

Event	X	Probability of X	$X \cdot$ Probability of X	
Left	1	0.7	0.7	
Right	0	0.3	0	
		Sum:	0.7	$= E(X)$

6. **Using this value and the rule mentioned above, determine the expected number of left turns for 3 mice. (Remember, the value you compute may not be an exact, attainable value for 1 set of 3 mice. Rather, it is the *average* number of left turns by 3 mice based on many sets of 3 mice.)**

   The expected value is 2.1 left turns. As stated in the student exercise, the value may not be an exact, attainable value for 1 set of 3 mice. Rather, it is the average number of left turns by 3 mice based on numerous attempts.

The tree diagram below demonstrates the 8 possible outcomes for 3 mice where the first stage of the tree represents the decision made by the 1^{st} mouse and the second stage represents the decision made by the 2^{nd} mouse, and so on.



Scaffolding:

- Tree diagrams were used in Grades 7 and 11. Students new to the curriculum may or may not be familiar with the construction or use of such diagrams. Briefly demonstrate how to construct the tree diagram presented in the text for students who are not familiar with the concept.
- For advanced learners, a more complex situation may be appropriate, such as a maze with three directional choices—right, left, and center.

7. Use the tree diagram to answer the following questions:

a. Complete the following table and compute $E(Y)$, the expected number of left turns for 3 mice.

Event	Y	Probability of Y	$Y \cdot$ Probability of Y
3 Lefts	3	0.343	
2 Lefts	2		
1 Left	1		
0 Lefts	0	0.027	
		Sum:	$= E(Y)$

Event	Y	Probability of Y	$Y \cdot$ Probability of Y
3 Lefts	3	0.343	1.029
2 Lefts	2	$3 \cdot 0.147 = 0.441$	0.882
1 Left	1	$3 \cdot 0.063 = 0.189$	0.189
0 Lefts	0	0.027	0
		Sum:	2.1 $= E(Y)$

b. Verify that the expected number of left turns for 3 mice, $E(Y)$, is the same as 3 times the expected number of left turns for 1 mouse, $3 \cdot E(X)$.

Yes. $E(Y) = 2.1 = 3 \cdot E(X) = 3 \cdot 0.7$

8. Imagine that 200 mice are sent through the maze one at a time. The researchers believed that the probability of a mouse turning left due to the food is 0.7. How many left turns would they expect from 200 mice?

$200 \cdot 0.7 = 140$ *left turns*

Example 3 (3 minutes): So How Does the Charity Make Money?

Revisit the charity game from Example 1. Students have expressed in writing what cost might justify them playing the game. Have students share their responses with their partners.

Now students have to consider why the charity is holding a carnival and how it will benefit from the game. Have students discuss the following questions with their partners:

- What is the purpose of having a charity carnival?
  - *To raise or earn money for a charitable cause.*
- Consider how much you would pay to play a game. How much should a prize be worth for the charity to earn money?
  - *Answers will vary. To earn money for the charity, the prizes cannot be worth more money than the game brings in. In addition, in the end, all the money brought in needs to be worth more than the time and expenses spent to hold the carnival. I would think in general that prizes should be worth* $\$1.00$ *or less in order for the charity to raise a significant amount of money.*

Read the example as a class and remind them of the carnival game from Example 1.

Example 3: So How Does the Charity Make Money?

Revisiting the charity carnival of Example 1, recall that when selecting a duck, the average monetary value of the prizes you win per game is $\$1.25$. How can the charity running the carnival make any money if it is paying out $\$1.25$ to each player on average for each game?

To address this, in most cases a player must *pay* to play a game, and that is where the charity (or any other group running such a game) would earn its money.

Imagine that the cost to play the game is $\$2.00$. What are the expected net earnings for the charity? What are the expected net winnings for a player?

Scaffolding:

- English language learners may not understand the use of the word *net* in this context.
- Point out that *net earnings* refers to the amount of money the player won in the game, less the amount of money paid to play the game.
- For example, if the player won the $\$5.00$ prize, he would actually have a net earnings of only $\$3.00$ (equal to the $\$5.00$ he wins, less the $\$2.00$ he paid to play).

Exercises 9–13 (12 minutes)

Have students work on Exercises 9–13. Students should continue to work with partners. Informally assess student mastery by circulating and listening to students explain to one another and by looking at their work.

Exercises 9–13

9. Compute a player's net earnings for each of the 3 outcomes: small, medium, and large.

Small $= \$0.50 - \$2.00 = -\$1.50$

Medium $= \$1.50 - \$2.00 = -\$0.50$

Large $= \$5.00 - \$2.00 = +\$3.00$

10. For two of the outcomes, the net earnings result is negative. What does a negative value of net earnings mean in context as far as the player is concerned?

A negative expected value from the player's perspective means a loss.

Scaffolding:

For advanced learners, remove Exercises 9–11 and ask them to answer the following questions:

- What are the expected net winnings for a player?
- How does this relate to the earnings for the charity?

11. Let Y = the *net* amount that you win (or lose) playing the duck game 1 time. Complete the table below and calculate $E(Y)$.

Event	Y	Probability of Y	$Y \cdot$ Probability of Y
Small		0.6	
Medium		0.3	
Large		0.1	
		Sum:	$= E(Y)$

Event	Y	Probability of Y	$Y \cdot$ Probability of Y
Small	$-\$1.50$	0.6	$-\$0.90$
Medium	$-\$0.50$	0.3	$-\$0.15$
Large	$\$3.00$	0.1	$\$0.30$
		Sum:	$-\$0.75$ $= E(Y)$

MP.2

12. How would you explain the $E(Y)$ value in context to someone who had never heard of this measurement? Write a sentence explaining this value from the perspective of a player; then write a sentence explaining this value from the perspective of the charity running the game.

To a player: The game is designed such that if you play it over and over again, the people running the game expect that you will lose $\$0.75$ *to them on average per attempt.*

To the people running the game: The game is designed such that if people play this game over and over again, we can expect to make about $\$0.75$ *on average per attempt.*

13. How much money should the charity expect to earn from the game being played 100 times?

$100 \cdot \$0.75 = \$75.00$ *is expected from* 100 *attempts.*

Closing (3 minutes)

- How does a state lottery make money for its state? Thinking back to the duck game, imagine a simple lottery game that pays out $\$100.00$ in 1% of cases (like a scratch-off game). How much would the state expect to make on every 100 players if 99 of those players paid $\$2.00$ to play and won nothing while 1 of the 100 players paid $\$2.00$ and earned a net gain of $\$98.00$? Note: Make a table if you think it will help students.
  - *The state earned* $99 * \$2.00 = \$198.00$ *from the* 99 *losers and lost* $\$98.00$ *for the* 1 *winner. That means the state comes out ahead* $\$100.00$ *from these* 100 *players, or* $\$1.00$ *per ticket sold.*
  - You could also say that the state brought in $\$200.00$ ($\$2.00$ per player) and paid out $\$100.00$ to the 1 winner, thus coming out ahead $\$100.00$.
- Ask students to summarize the main ideas of the lesson in writing or with a neighbor. Use this as an opportunity to informally assess comprehension of the lesson. The Lesson Summary below offers some important ideas that should be included.

Lesson Summary

By computing the expected value, $E(X)$, for the earnings from a game of chance, one can determine the expected average payoff per game.

When this value is positive, the player can expect to come out ahead in the long run. However, in most games of chance, this value is negative and represents how much the group operating the game takes in on average per game. From a player's perspective, a negative expected value means that the player is expected to lose that $E(X)$ amount on average with each trial. Businesses and establishments that intend to make money from players, customers, etc., are counting on situations where the player's expected value is negative.

As long as the probabilities remain the same for each instance of a game or trial, you can compute the expected value of N games as N times the expected value of 1 game.

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 13: Games of Chance and Expected Value

Exit Ticket

As posted on the Maryland Lottery's website for its Pick 3 game, the chance of winning with a Front Pair bet is 0.01.

A Front Pair bet is successful if the front pair of numbers you select match the Pick 3 number's first 2 digits. For example, a bet of $12X$ would be a winner if the Pick 3 number is $120, 121, 122$, etc. In other words, 10 of the $1{,}000$ possible Pick 3 numbers (1%) would be winners, and thus, the probability of winning is 0.01 or 1%.

A successful bet of $\$0.50$ pays out $\$25.00$ for a net gain to the player of $\$24.50$.

a. Define the random variable X and compute $E(X)$.

b. On average, how much does the Maryland Lottery make on each such bet?

c. Assume that for a given time period, $100{,}000$ bets like the one described above were placed. How much money should the Maryland Lottery Agency expect to earn on average from $100{,}000$ bets?

Note: According to the Maryland Lottery Gaming and Control Agency's Annual Report for Fiscal Year 2012, the Pick 3 game accounted for $\$254.60$ million in net sales. (http://mlgca.com/annual-report/ accessed November 17, 2013)

Exit Ticket Sample Solutions

As posted on the Maryland Lottery's website for its Pick 3 game, the chance of winning with a Front Pair bet is 0.01. (http://mdlottery.com/games/pick-3/payouts/ accessed on November 17, 2013)

A Front Pair bet is successful if the front pair of numbers you select match the Pick 3 number's first 2 digits. For example, a bet of $12X$ would be a winner if the Pick 3 number is $120, 121, 122$, etc. In other words, 10 of the $1,000$ possible Pick 3 numbers (1%) would be winners, and thus, the probability of winning is 0.01 or 1%.

A successful bet of $\$0.50$ pays out $\$25.00$ for a net gain to the player of $\$24.50$.

a. Define the random variable X and compute $E(X)$.

Let X = a player's NET gain or loss from playing 1 game in this manner.

$$E(X) = 0.99 \cdot -\$0.50 + 0.01 \cdot \$24.50 = -\$0.495 + \$0.245 = -\$0.25$$

b. On average, how much does the Maryland Lottery make on each such bet?

The Maryland Lottery makes on average $\$0.25$ for each such bet.

c. Assume that for a given time period, $100,000$ bets like the one described above were placed. How much money should the Maryland Lottery expect to earn on average from $100,000$ bets?

$$100,000 \cdot \$0.25 = \$25,000.00$$

Note: According to the Maryland Lottery Gaming and Control Agency's Annual Report for Fiscal Year 2012, the Pick 3 game accounted for $\$254.60$ million in net sales. (http://mlgca.com/annual-report/ accessed November 17, 2013)

Problem Set Sample Solutions

1. The Maryland Lottery Pick 3 game described in the Exit Ticket has a variety of ways in which a player can bet. Instead of the Front Pair bet of $\$0.50$ described above with a payout of $\$25.00$, a player could make a Front Pair bet of $\$1.00$ on a single ticket for a payout of $\$50.00$.

   Let Y = a player's NET gain or loss from playing 1 game in this manner.

   a. Compute $E(Y)$.

   $$E(Y) = 0.99 \cdot -\$1.00 + 0.01 \cdot \$49.00 = -\$0.99 + \$0.49 = -\$0.50$$

   b. On average, how much does the Maryland Lottery make on each such bet?

   The Maryland Lottery makes on average $\$0.50$ for each such bet.

   c. Assume that for a given time period, $100,000$ bets like the one described above were placed. How much money should the Maryland Lottery expect to earn on average from $100,000$ bets?

   $$100,000 \cdot \$0.50 = \$50,000.00$$

   d. Compare your answers to the three questions above with your Exit Ticket answers. How are the answers to these questions and the answers to the Exit Ticket questions related?

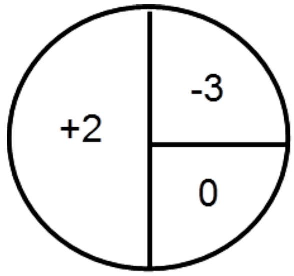
   Each of these answers is double the corresponding answers above.

2. Another type of carnival or arcade game is a spinning wheel game. Imagine that someone playing a spinning wheel game earns points (payoff) as follows for each spin:

- **You gain 2 points 50% of the time.**
- **You lose 3 points 25% of the time.**
- **You neither gain nor lose any points 25% of the time.**

The results of each spin are added to one another, and the object is for a player to accumulate 5 or more points. Negative total point values are possible.

a. Develop a model of a spinning wheel that would reflect the probabilities and point values.



b. Compute $E(X)$ where X = the number of points earned in a given spin.

$E(X) = 2 \cdot 0.50 + (-3) \cdot 0.25 + 0 \cdot 0.25 = 0.25$ *points*

c. Based on your computation, how many spins on average do you think it might take to reach 5 points?

$\frac{5}{0.25} = 20$. *It would take on average 20 spins.*

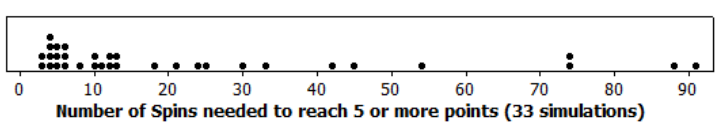
d. Use the spinning wheel you developed in part (a) (or some other randomization device) to take a few spins. See how many spins it takes to reach 5 or more points. Comment on whether this was fewer spins, more spins, or the same number of spins you expected in part (c) above.

Answers will vary. In most cases, a player can reach 5 or more points with very few spins, often far fewer than the average of 20.

e. Let Y = the number of spins needed to reach 5 or more points (like the number of spins it took you to reach 5 points in part (d) above), and repeat the simulation process from part (d) many times. Record on a dot plot the various values of Y you obtain. After several simulations, comment on the distribution of Y.

Answers will vary with students' work.

For parts (d) and (e), 1 simulation distribution (computer generated) of Y = the number of spins needed to reach 5 or more points was as follows:



The distribution is very skewed right, and sometimes it takes many spins to get to 5 or more points. The average number of spins needed based on these 33 simulations was 23.12 spins, close to the $E(Y) = 20$. In most cases, a player can reach 5 or more points with very few spins, often far fewer than the average of 20. (Note: This may be a good chance to remind students that it is important to see the center, shape, and spread of a distribution before drawing conclusions about a variable.)