

Lesson 13: Games of Chance and Expected Value

Classwork

Example 1: Ducks at the Charity Carnival

One game that is popular at some carnivals and amusement parks involves selecting a floating plastic duck at random from a pond full of ducks. In most cases, the letters S, M, or L appear on the bottom of the duck signifying that the winner receives a small, medium, or large prize, respectively. The duck is then returned to the pond for the next game.

Although the prizes are typically toys, crafts, etc., suppose that the monetary values of the prizes are as follows: Small = \$0.50, Medium = \$1.50, and Large = \$5.00.

The probabilities of winning an item on 1 duck selection are as follows: Small 60%, Medium 30%, and Large 10%.

Suppose a person plays the game 4 times. What is the expected monetary value of the prizes won?

Exercises 1–4

- Let X = the monetary value of the prize that you win playing this game 1 time. Complete the table below and calculate $E(X)$.

| Event | X | Probability of X | $X \cdot$ Probability of X |
|--------|--------|--------------------|------------------------------|
| Small | \$0.50 | 0.6 | |
| Medium | \$1.50 | 0.3 | |
| Large | \$5.00 | 0.1 | |
| Sum: | | | $= E(X)$ |

- Regarding the $E(X)$ value you computed above, could you win exactly that amount on any 1 play of the game?

3. What is the least you could win in 4 attempts? What is the most you could win in 4 attempts?
4. How would you explain the $E(X)$ value in context to someone who had never heard of this measurement? What would you expect for the total monetary value of your prizes from 4 attempts at this game? (To answer this question, check the introduction to this lesson; do not do anything complicated like developing a tree diagram for all of the outcomes from 4 attempts at the game.)

The question above gets at an important feature of expected value. If the probability distribution remains the same for each trial (also called *independent trials*), you can determine the expected value for N trials by computing the expected value of 1 trial and multiplying that $E(X)$ value by N . You do not have to develop a complicated probability distribution covering all the results of all N trials. This rule makes it far easier to compute expected value for situations where several hundred independent trials occur, such as a carnival, lottery, etc.

Example 2: Expected Value for Repeated Trials

In a laboratory experiment, 3 mice will be placed in a simple maze one at a time. In the maze, there is 1 decision point where the mouse can turn either left (L) or right (R). When the 1st mouse arrives at the decision point, the direction the mouse chooses is recorded. The same is done for the 2nd and the 3rd mouse. The researchers conducting the experiment add food in the simple maze such that the long-run relative frequency of each mouse turning left is believed to be 0.7 or 70%.

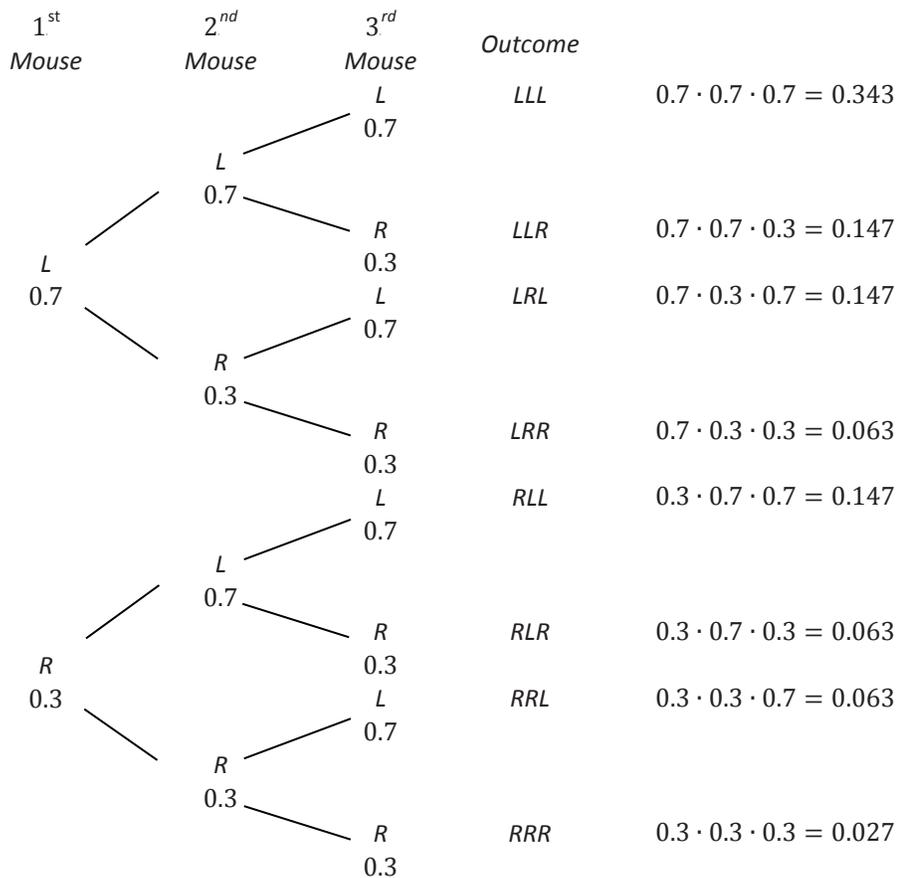
Exercises 5–8

5. Examining the outcomes for just 1 mouse, define the random variable and complete the following table:

| Event | X | Probability of X | $X \cdot$ Probability of X |
|-------|-----|--------------------|------------------------------|
| Left | 1 | 0.7 | |
| Right | 0 | 0.3 | |
| Sum: | | | $= E(X)$ |

6. Using this value and the rule mentioned above, determine the expected number of left turns for 3 mice. (Remember, the value you compute may not be an exact, attainable value for 1 set of 3 mice. Rather, it is the average number of left turns by 3 mice based on many sets of 3 mice.)

The tree diagram below demonstrates the 8 possible outcomes for 3 mice where the first stage of the tree represents the decision made by the 1st mouse and the second stage represents the decision made by the 2nd mouse, and so on.



7. Use the tree diagram to answer the following questions:

a. Complete the following table and compute $E(Y)$, the expected number of left turns for three mice.

| Event | Y | Probability of Y | $Y \cdot$ Probability of Y |
|---------|-----|--------------------|------------------------------|
| 3 Lefts | 3 | 0.343 | |
| 2 Lefts | 2 | | |
| 1 Left | 1 | | |
| 0 Lefts | 0 | 0.027 | |
| Sum: | | | $= E(Y)$ |

b. Verify that the expected number of left turns for three mice, $E(Y)$, is the same as three times the expected number of left turns for 1 mouse, $3 \cdot E(X)$.

8. Imagine that 200 mice are sent through the maze one at a time. The researchers believed that the probability of a mouse turning left due to the food is 0.7. How many left turns would they expect from 200 mice?

Example 3: So How Does the Charity Make Money?

Revisiting the charity carnival of Example 1, recall that when selecting a duck, the average monetary value of the prizes you win per game is \$1.25. How can the charity running the carnival make any money if it is paying out \$1.25 to each player on average for each game?

To address this, in most cases a player must pay to play a game, and that is where the charity (or any other group running such a game) would earn its money.

Imagine that the cost to play the game is \$2.00. What are the expected net earnings for the charity? What are the expected net winnings for a player?

Exercises 9–13

9. Compute a player’s net earnings for each of the three outcomes: small, medium, and large.
10. For 2 of the outcomes, the net earnings result is negative. What does a negative value of net earnings mean in context as far as the player is concerned?
11. Let Y = the *net* amount that you win (or lose) playing the duck game 1 time. Complete the table below and calculate $E(Y)$.

| Event | Y | Probability of Y | $Y \cdot$ Probability of Y |
|--------|-----|--------------------|------------------------------|
| Small | | 0.6 | |
| Medium | | 0.3 | |
| Large | | 0.1 | |
| Sum: | | | $= E(Y)$ |

12. How would you explain the $E(Y)$ value in context to someone who had never heard of this measurement? Write a sentence explaining this value from the perspective of a player; then write a sentence explaining this value from the perspective of the charity running the game.
13. How much money should the charity expect to earn from the game being played 100 times?

Lesson Summary

By computing the expected value, $E(X)$, for the earnings from a game of chance, one can determine the expected average pay off per game.

When this value is positive, the player can expect to come out ahead in the long run. However, in most games of chance, this value is negative and represents how much the group operating the game takes in on average per game. From a player's perspective, a negative expected value means that the player is expected to lose that $E(X)$ amount on average with each trial. Businesses and establishments that intend to make money from players, customers, etc., are counting on situations where the player's expected value is negative.

As long as the probabilities remain the same for each instance of a game or trial, you can compute the expected value of N games as N times the expected value of 1 game.

Problem Set

1. The Maryland Lottery Pick 3 game described in the Exit Ticket has a variety of ways in which a player can bet. Instead of the Front Pair bet of \$0.50 described above with a payout of \$25.00, a player could make a Front Pair bet of \$1.00 on a single ticket for a payout of \$50.00.

Let Y = a player's NET gain or loss from playing 1 game in this manner.

- Compute $E(Y)$.
 - On average, how much does the Maryland Lottery make on each such bet?
 - Assume that for a given time period, 100,000 bets like the one described above were placed. How much money should the Maryland Lottery expect to earn on average from 100,000 bets?
 - Compare your answers to the three questions above with your Exit Ticket answers. How are the answers to these questions and the answers to the Exit Ticket questions related?
2. Another type of carnival or arcade game is a spinning wheel game. Imagine that someone playing a spinning wheel game earns points (pay off) as follows for each spin:
- You gain 2 points 50% of the time.
 - You lose 3 points 25% of the time.
 - You neither gain nor lose any points 25% of the time.

The results of each spin are added to one another, and the object is for a player to accumulate 5 or more points. Negative total point values are possible.

- Develop a model of a spinning wheel that would reflect the probabilities and point values.
- Compute $E(X)$ where X = the number of points earned in a given spin.
- Based on your computation, how many spins on average do you think it might take to reach 5 points?
- Use the spinning wheel you developed in part (a) (or some other randomization device) to take a few spins. See how many spins it takes to reach 5 or more points. Comment on whether this was fewer spins, more spins, or the same number of spins you expected in part (c) above.
- Let Y = the number of spins needed to reach 5 or more points (like the number of spins it took you to reach 5 points in part (d) above), and repeat the simulation process from part (d) many times. Record on a dot plot the various values of Y you obtain. After several simulations, comment on the distribution of Y .