Lesson 11: Estimating Probability Distributions Empirically

Student Outcomes

- Students use empirical data to estimate probabilities associated with a discrete random variable.
- Students interpret probabilities in context.

Lesson Notes

This lesson engages students in estimating probability distributions of discrete random variables using data they have collected in class. If you have access to polling software (Google Forms is a free option and SurveyMonkey has a basic, free account option. There are many choices available for various platforms–computer, phone/tablet, clickers, etc.), you can collect the data in real time at the start of class. If not, have students complete a paper survey and then provide them with summarized data. If possible, try to provide the results graphically for Question 1 and in tabular form for Question 2, as the exercises ask for the alternate representation. Students should have technology available to do the calculations so they can focus on understanding what the results represent. Note that in some cases, the sum of the probabilities in an estimated probability distribution might not exactly equal 1 because of rounding.

Depending on the time available, you may choose to have some students analyze the data from Question 1 and the rest the data from Question 2, or you may want everyone to analyze the data from both questions, in which case you might like to give two polls, one at the beginning of Exploratory Challenge 1 and the second when you start Exploratory Challenge 2. If you have a small class, or would like a larger set of responses, you might choose to give the poll to other classes. You could also have the class design a way to randomly sample students in a certain grade or set of classes to revisit ideas about taking random samples.

If you divide the class and have each student work on only one of the two examples, you might have time to begin Lesson 12, which might take more than one class period.

A note about rounding decimals: Typically in statistics, rounding is not as big of an issue as it is in, for instance, a calculus or a chemistry course. Most of the time, statisticians are not looking for an absolutely correct numerical figure; they are trying to use numbers to explain trends and make generalizations. In general, rounding to two or three decimal places is sufficient. A teacher may want to establish a rule with a class, but it is also acceptable to have a bit of subjectivity.

Classwork

Exploratory Challenge 1/Exercise 1 (5 minutes)

Have students collect the data for Questions 1 and 2. This can be done as a *speed date*, in which the whole class pairs off, one student in each pair provides data to the other, and then after twenty seconds, students switch partners. This is done until all students have met one another and everyone has a complete set of data. Check to make sure all data is in whole numbers and that everyone has collected data from everyone else.

Exploratory Challenge 1/Exercise 1

In this lesson, you will use empirical data to estimate probabilities associated with a discrete random variable and interpret probabilities in context.

1. Collect the responses to the following questions from your class.

Question 1: **Estimate to the nearest whole number the number of hours per week you spend playing games on computers or game consoles.**

Question 2: **If you rank each of the following subjects in terms of your favorite (number 1), where would you put mathematics: $1, 2, 3, 4, 5$, or 6?**
English, foreign languages, mathematics, music, science, and social studies

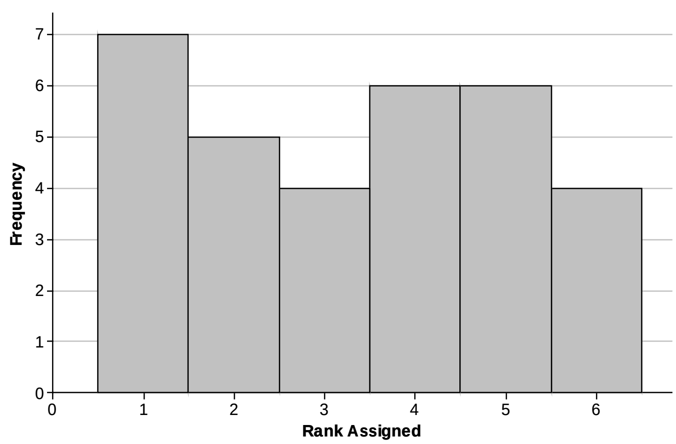
Responses may be displayed in raw form, tables, or graphs depending on the method you used to collect data from the class. The responses below are based on a class of 32 students.

Sample responses for Question 1 are given below.

Table: Number of hours playing games on computers or consoles

Hours	0	1	4	9	10	11	14	17	28	35
Frequency	3	2	5	4	6	5	3	2	1	1

Sample responses for Question 2 are given below.



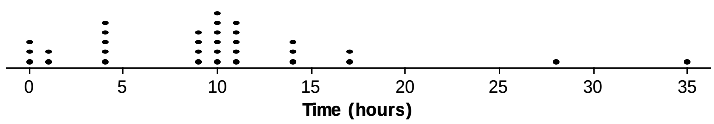
Exploratory Challenge 1/Exercises 2–5 (15 minutes): Computer Games

Students are asked to make a dot plot of the responses to Question 1. Be sure they have used a number line with the complete scale rather than only those numbers that were given as responses. (i.e., Even if no one responded 10 hours, but some did respond 11 hours, the horizontal axis should include both 10 and 11.)

Exploratory Challenge 1/Exercises 2–5: Computer Games

2. **Create a dot plot of the data from Question 1 in the poll: the number of hours per week students in class spend playing computer or video games.**

Responses will vary. The plot below is based on the sample data given in Exercise 1.



3. **Consider the chance experiment of selecting a student at random from the students at your school. You are interested in the number of hours per week a student spends playing games on computers or game consoles.**

   a. **Identify possible values for the random variable number of hours spent playing games on computers or game consoles.**

   Values include whole numbers from 1 to some number less than $24x7 = 168$. (Records show that some teens actually do play games for two or three days straight, but this is unusual.)

   b. **Which do you think will be more likely: a randomly chosen student at your school will play games for less than 9 hours per week or for more than 15 hours per week? Explain your thinking.**

   Responses will vary.

   Depending on the poll results students see displayed, if they think that their class is representative of students at the school, they may choose either option. Using the sample data above, students might suggest that chances seem more likely a randomly chosen student would play computer games less than 9 hours per week.

MP.4

   c. **Assume that your class is representative of students at your school. Create an estimated probability distribution for the random variable number of hours per week a randomly selected student at your school spends playing games on computers or game consoles.**

   The table below is a sample answer based on data given in Exercise 1.

Hours	0	1	4	9	10	11	14	17	28	35
Probability	0.09	0.06	0.16	0.13	0.19	0.16	0.09	0.06	0.03	0.03

   d. **Use the estimated probability distribution to check your answer to part (b).**

   Responses will vary. A response based on sample data indicates that 0.25 of the class play less than 9 hour per week, while 0.12 play more than 15 hours of games on computers or game consoles.

Scaffolding:

For advanced learners, consider replacing Exercises 2–4 with the following:

- How many hours do you think a typical student spends per week playing computer or video games?

Ask students to explain in writing how they arrived at their answer. Students should use an estimated probability distribution and expected value to support their answers.

For struggling students, consider reading the problems out loud, providing visual aids to help explain the problems, providing sample data for Exercise 3, or providing an exemplar response for the statements in Exercise 5.

4. Use the data your class collected to answer the following questions.

   a. What is the expected value for the number of hours students at your school play video games on a computer or game console?

   Responses will vary. Based on the sample data, the expected value would be 9.7 hours.

   b. Interpret the expected value you calculated in part (a).

   Responses will vary. If you were to randomly select a student and then randomly select a student again and do this many times finding the number of hours spent playing games, the average number of hours would be about 9.7 hours per week.

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5. Again, assuming that the data from your class is representative of students at your school, comment on each of the following statements.

   a. It would not be surprising to have 20 students in a random sample of 200 students from the school who do not play computer or console games.

   Responses will vary. Using the sample data, the estimated probability that students do not play any computer games is 0.09, which is about 18 of the 200 students. 20 is close to 18, so I would not be surprised.

   b. It would be surprising to have 60 students in a random sample of 200 students from the school spend more than 10 hours per week playing computer or console games.

   Responses will vary. According to the sample data, the probability of playing more than 10 hours a week is 0.37, which would be about 74 students. 60 students is not too close to 74, but I do not think it is far enough away to be that unlikely; so, I do not think it would be surprising.

   c. It would be surprising if more than half of the students in a random sample of 200 students from the school played less than 9 hours of games per week.

   Responses will vary. Using the sample data, the estimated probability that students play less than 9 hours of computer games per week is 0.31, which is about 62 students. Half of 200 is 100, so I would be surprised if 100 of the students played less than 9 hours of games per week.

Exploratory Challenge 2/Exercises 6–10 (15 minutes): Favorite Subject

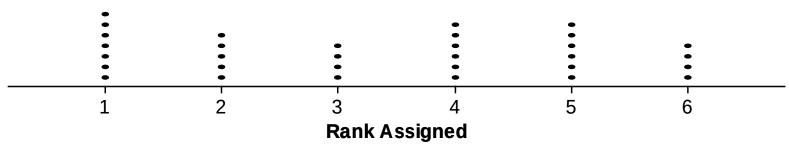
As mentioned before, you might begin this exercise set by giving students Question 2 as a second poll.

If students respond to Exercise 6, part (b), relying on their beliefs, stress the need to use data to support their answers; making decisions based on evidence is the point of studying statistics.

Exploratory Challenge 2/Exercises 6–10: Favorite Subject

6. **Create a dot plot of your responses to Question 2 in the poll.**

   Responses will vary. Based on the sample data, the plot for Question 2 would be the following.



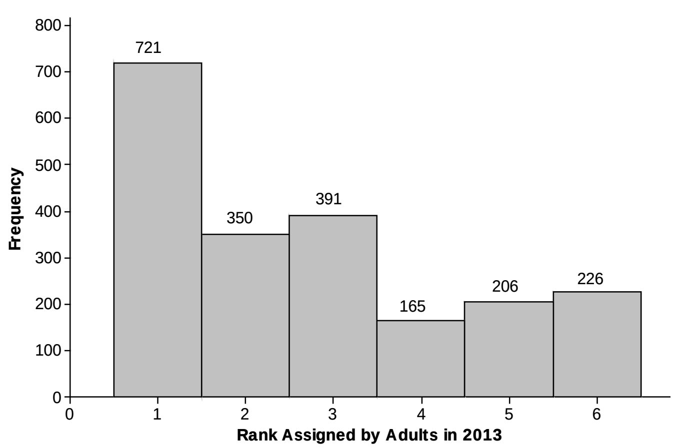
   a. **Describe the distribution of rank assigned.**

   Responses will vary. For the sample data, the distribution of the ranks is fairly uniform, with not that much difference in the frequency with which each of the ranks was chosen.

   b. **Do you think it is more likely that a randomly selected student in your class would rank mathematics high (1 or 2) or that he or she would rank it low (5 or 6)? Explain your reasoning.**

   Responses will vary. Using the counts in the sample data, students might observe that 12 people ranked math as 1 or 2, while only 10 people ranked math as a 5 or 6. So, two more people ranked it high than did low.

7. **The graph displays the results of a 2013 poll taken by a polling company of a large random sample of $2,059$ adults 18 and older responding to Question 2 about ranking mathematics.**



   a. **Describe the distribution of the rank assigned to mathematics for this poll.**

   Responses will vary.

   The distribution is skewed right with more people ranking mathematics with a 1 or 2 than with a 5 or 6.

Scaffolding:

For advanced learners, consider replacing Exercises 1–5 with the following:

- How do you think a typical student would rank mathematics?
- Do you think a typical adult would rank mathematics higher or lower than a typical high school student?

Ask students to explain in writing how they arrived at their answer. Students should use an estimated probability distribution and expected value to support their answers.

For struggling students, consider providing sentence frames for English language learners, or altering Exercise 1, part (a), to be a multiple choice question:

- Would you describe the distribution as uniform, approximately normal, skewed left, or skewed right?

In addition, perhaps adding sentence starters or key points could be given for Exercise 2 to ensure thorough responses.

b. **Do you think the proportion of students who would rank mathematics 1 is greater than the proportion of adults who would rank mathematics 1? Explain your reasoning.**

Responses will vary. If students think that their class is representative of students in general, they may calculate the proportions or percentage and say more adults ranked it 1 than the students in their class. They may also ignore the data and respond with their belief.

8. **Consider the chance experiment of randomly selecting an adult and asking them what rank they would assign to mathematics. The variable of interest is the rank assigned to mathematics.**

a. **What are possible values of the random variable?**

The values are $1, 2, 3, 4, 5$, and 6.

b. **Using the data from the large random sample of adults, create an estimated probability distribution for the rank assigned to mathematics by adults in 2013.**

Table: Rank assigned to mathematics (adults in 2013)

Rank	1	2	3	4	5	6
Probability	0.35	0.17	0.19	0.08	0.10	0.11

MP.4

c. **Assuming that the students in your class are representative of students in general, use the data from your class to create an estimated probability distribution for the rank assigned to mathematics by students.**

Responses will vary. Sample response based on data from Question 2:

Table: Rank assigned to mathematics (based on sample class data for Question 2)

Rank	1	2	3	4	5	6
Probability	0.22	0.16	0.13	0.19	0.19	0.13

Note: Values may not add up to exactly 1 due to rounding.

9. **Use the two estimated probability distributions from Exercise 8 to answer the following questions.**

a. **Do the results support your answer to Exercise 7, part (b)? Why or why not?**

Responses will vary. The probability distributions indicate that adults rank mathematics as 1 with a probability of 0.35, while students do so with a probability of 0.22.

b. **Compare the probability distributions for the rank assigned to mathematics for adults and students.**

Responses will vary. Using the sample data, some students might note that the probabilities for ranking mathematics 6 and last for adults and for students were not that far apart. Others might note that for adults, the probabilities are quite large for a rank of 1, then decrease with a little fluctuation. The probabilities for students are all about the same for all of the ranks.

c. **Do adults or students have a greater probability of ranking mathematics in the middle (either a 3 or 4)?**

Responses will vary. Using the sample data, the estimated probability that a randomly chosen adult will rank mathematics as a 3 or 4 is 0.27 and that a randomly chosen student will rank it as a 3 or 4 is 0.32. These probabilities are not that far apart, so it does not seem that the two groups are that different.

10. **Use the probability distributions from Exercise 2 to answer the following questions.**
    a. **Find the expected value for the estimated probability distribution of rank assigned by adults in 2013.**

       The expected value is 2.74.

    b. **Interpret the expected value calculated in part (a).**

       Responses will vary. Given that the sample of $2,059$ people was random, if you asked lots of adults how they would rank mathematics, in the long run, the average ranking would be 2.74.

    c. **How does the expected value for the rank students assign to mathematics compare to the expected value for the rank assigned by adults?**

       Responses will vary. Using the sample data, the expected value for students is 3.42, which indicates that students ranked mathematics lower than adults.

MP.2

Closing (2 minutes)

- Explain how the average value of a probability distribution helps us to answer statistical questions.
  - *Responses will vary. The average value over the long run should approach the expected value and so should be somewhat close to the expected value. If it is not, I would probably be surprised.*
- Ask students to summarize the main ideas of the lesson in writing or with a neighbor. Use this as an opportunity to informally assess comprehension of the lesson. The Lesson Summary below offers some important ideas that should be included.

Lesson Summary

In this lesson you learned that

- **You can estimate probability distributions for discrete random variables using data collected from polls or other sources.**
- **Probabilities from a probability distribution for a discrete random variable can be interpreted in terms of long-run behavior of the random variable.**
- **An expected value can be calculated from a probability distribution and interpreted as a long-run average.**

Exit Ticket (8 minutes)

Name ______________________________ Date ______________

Lesson 11: Estimating Probability Distributions Empirically

Exit Ticket

The table shows the number of hours, to the nearest half-hour per day, that teens spend texting, according to a random sample of 870 teenagers aged 13–18 in a large urban city.

Table: Number of hours teenagers spend texting per day

Hours	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
Frequency	170	82	220	153	92	58	40	15	12	18	10

1. What random variable is of interest here? What are the possible values for the random variable?

2. Create an estimated probability distribution for the time teens spend texting.

3. What is the estimated probability that teens spend less than an hour per day texting?

4. Would you be surprised if the average texting time for a smaller random sample of teens in the same city was three hours? Why or why not?

Exit Ticket Sample Solutions

The table shows the number of hours, to the nearest half-hour per day, that teens spend texting, according to a random sample of 870 teenagers aged 13–18 in a large urban city.

Table: Number of hours teenagers spend texting per day

Hours	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
Frequency	170	82	220	153	92	58	40	15	12	18	10

1. What random variable is of interest here? What are the possible values for the random variable?

   The variable of interest is the number of hours to the nearest half-hour teens spend texting per day. Possible values are $0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5,$ *and* 5.

2. Create an estimated probability distribution for the time teens spend texting.

Hours	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
*Frequency**	0.20	0.09	0.25	0.18	0.11	0.07	0.05	0.02	0.01	0.02	0.01

   **Total does not equal* 1.00 *due to rounding.*

3. What is the estimated probability that teens spend less than an hour per day texting?

   0.29

4. Would you be surprised if the average texting time for a smaller random sample of teens in the same city was three hours? Why or why not?

   Responses will vary. Sample response:

   I would be surprised because the expected value is 1.36 *hours, and three hours is much larger than the expected value.*

Problem Set Sample Solutions

1. The results of a 1989 poll in which each person in a random sample of adults ranked mathematics as a favorite subject are in the table below. The poll was given in the same city as the poll in Exercise 6.

   Table: Rank assigned to mathematics by adults in 1989

Rank	1	2	3	4	5	6
Frequency	56	43	12	19	39	61

   a. Create an estimated probability distribution for the random variable that is the rank assigned to mathematics.

   Table: Rank assigned to mathematics by adults in 1989

Rank	1	2	3	4	5	6
Probability	0.24	0.19	0.05	0.08	0.17	0.27

b. **An article about the poll reported, "Americans have a bit of a love-hate relationship with mathematics." Do the results support this statement? Why or why not?**

Responses will vary.

Some students might note that the probability of ranks 1 and 2 is 0.43, and the probability of ranks 5 and 6 is 0.44. So, the probability that a randomly chosen adult from this city liked mathematics was almost the same as the probability that he or she did not like mathematics. Others might point out the probability that a randomly chosen person ranked math first was 0.24 and last was 0.27, which are only 0.03 apart, so the probability that they love math was about the same as the probability that they hate math.

c. **How is the estimated probability distribution of the rank assigned to mathematics by adults in 1989 different from the estimated probability distribution for adults in 2013?**

Table: Rank assigned to mathematics by adults in 2013

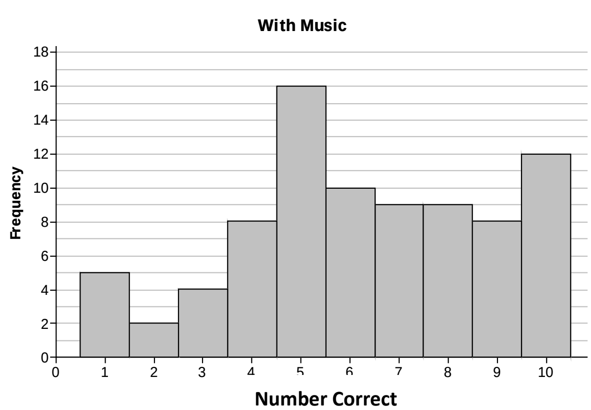
Rank	1	2	3	4	5	6
Frequency	0.35	0.17	0.19	0.08	0.1	0.11

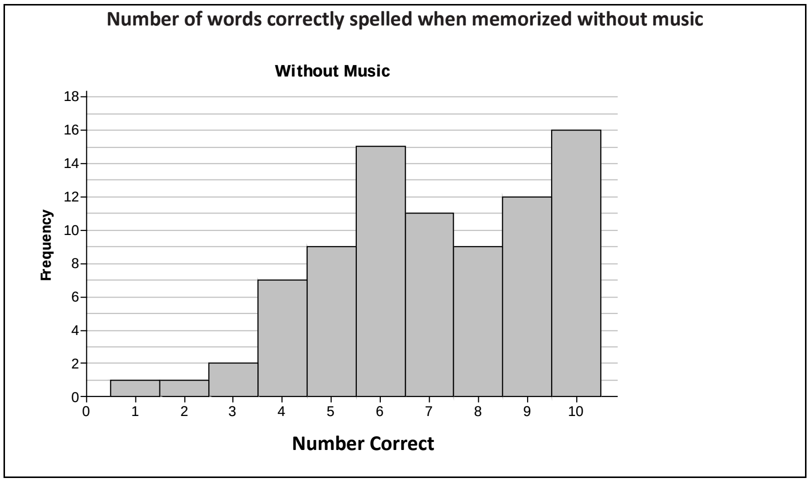
Responses will vary.

Students might say that the probability of ranking mathematics first was greater in 2013 or that the probabilities of ranking mathematics first and third in 2013 were greater than in 1989. The probability of ranking mathematics sixth in 2013 was less than half of the probability of ranking it sixth in 1989. The expected value in 2013 was a rank of 2.74, and in 1989 the expected value was lower with a rank of 3.56.

2. **A researcher investigated whether listening to music made a difference in people's ability to memorize the spelling of words. A random sample of 83 people memorized the spelling of 10 words with music playing, and then they were tested to see how many of the words they could spell. These people then memorized 10 different words without music playing and were tested again. The results are given in the two displays below.**

Number of words correctly spelled when memorized with music





a. What do you observe from comparing the two distributions?

Responses will vary.

Some may observe that the people in the sample seemed to memorize more words without music than with music.

b. Identify the variable of interest. What are possible values it could take on?

The random variable is the number of words spelled correctly. Possible values are integers 0 to 10.

c. Assume that the group of people that participated in this study are representative of adults in general. Create both a table and a graph of the estimated probability distributions for number of words spelled correctly when memorized with music and number of words spelled correctly when memorized without music. What are the advantages and disadvantages of using a table? A graph?

Responses will vary.

With music:

Words	1	2	3	4	5	6	7	8	9	10
Frequency	5	2	4	8	16	10	9	9	8	12
Probability	0.06	0.02	0.05	0.10	0.19	0.12	0.11	0.11	0.10	0.14

Without music:

Words	1	2	3	4	5	6	7	8	9	10
Frequency	1	1	2	7	9	15	11	9	12	16
Probability	0.01	0.01	0.02	0.08	0.11	0.18	0.13	0.11	0.14	0.19

Students might suggest the advantages of using a table are that you can see the actual probabilities, and the advantages of using a graph are that you can see the pattern or trend. It is almost easier to see that more words were spelled correctly when they were memorized without music from the graph because you can see the distribution seems to be more skewed left than the one for the number of words memorized with music.

A disadvantage of using a table might be that the differences in the probabilities are not always easy to sort out quickly; a disadvantage of using the graph is that you almost have to estimate the probabilities because you cannot see the actual values. (Note that some interactive technology will show the values if the cursor is dragged over the bar or point.)

d. **Compare the probability that a randomly chosen person who memorized words with music will be able to correctly spell at least eight of the words to the probability for a randomly chosen person who memorized words without music.**

Responses will vary.

Students should compare the two probabilities: $P(W \geq 8 \text{ with music}) = 0.35$*, while* $P(W \geq 8 \text{ without music}) = 0.44$*. There is a* 0.9 *difference in the probability they will be able to correctly spell at least eight words for the two conditions (music and no music).*

e. **Make a conjecture about which of the two estimated probability distributions will have the largest expected value. Check your conjecture by finding the expected values. Explain what each expected value means in terms of memorizing with and without music.**

The expected value for the number of words spelled correctly when memorizing with music is 6.27 *words; without music it is* 6.99 *words. Assuming that the group of people that participated in this study are representative of adults in general, if you gave people lots and lots of lists of* 10 *words to memorize with and without music, over the long run, with music they would be able to spell* 6.27 *words and without music* 6.99 *words. The difference does not seem to be very large.*

3. **A random variable takes on the values $0, 2, 5$, and 10. The table below shows a frequency distribution based on observing values of the random variable and the estimated probability distribution for the random variable based on the observed values. Fill in the missing cells in the table.**

Table: Distribution of observed values of a random variable

Variable	0	2	5	10
Frequency	18	12	2	$?$
Probability	0.3	0.2	$??$	0.47

The missing cell in the probability row has to be 0.03 *because the sum of the probabilities has to be* 1*. The missing cell in the frequency row has to be 28 because* $\frac{18}{32+x} = 0.3$*,* $\frac{12}{32+x} = 0.2$*, and* $\frac{2}{32+x} = 0.03$*, and solving for* x *in any of the equations yields* $x = 28$*.*