Lesson 8: Interpreting Expected Value

Student Outcomes

- Students interpret expected value in context.

Lesson Notes

This lesson develops the interpretation of the expected value as a long-run average of the value of a discrete random variable. As previously observed, the more times an event occurs, the closer the distribution of outcomes gets to the probability distribution and the closer the average value of all the outcomes will get to the expected value. In this lesson, students calculate the expected value of the sum of two dice, roll the dice 10 times to calculate the average value, and then observe that rolling 40 times produces an average value closer to the expected value. Students interpret the expected value of a discrete distribution in the context of the problem. Each student will need two dice.

Classwork

Exploratory Challenge 1/Exercise 1 (2 minutes)

Allow students to read and answer Exercise 1.

Scaffolding:

- For students who struggle, consider using a modified version of this exercise that might include pulling different colored chips or marbles out of a bag or using four-sided dice.
- An extension for advanced students may be given as follows: Construct your own hypothetical discrete random variable with a probability distribution that also has an expected value of 7.

Exploratory Challenge 1/Exercises 1-8

Recall the following problem from the Problem Set in Lesson 7.

Suppose two dice are rolled. The sum of the two numbers showing is a discrete random variable. The following table displays the probability distribution of this random variable.

Sum rolled	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

1. **If you rolled two dice and added the numbers showing a large number of times, what would you expect the average sum to be? Explain why.**

   The expected sum of the two rolled dice is as follows

$$2\left(\frac{1}{36}\right)+3\left(\frac{1}{18}\right)+4\left(\frac{1}{12}\right)+5\left(\frac{1}{9}\right)+6\left(\frac{5}{36}\right)+7\left(\frac{1}{6}\right)+8\left(\frac{5}{36}\right)+9\left(\frac{1}{9}\right)+10\left(\frac{1}{12}\right)+11\left(\frac{1}{18}\right)+12\left(\frac{1}{36}\right)=7$$

   The expected average sum would be 7 because, after a large number of rolls, the distribution of sums would resemble the probability distribution above.

Exploratory Challenge 1/Exercises 2–4 (5 minutes)

Allow students to roll two dice, recording the sum for each of 10 rolls. Students should then use their results to answer Exercises 3 and 4.

2. Roll two dice. Record the sum of the numbers on the two dice in the table below. Repeat this nine more times for a total of 10 rolls.

Sum rolled	2	3	4	5	6	7	8	9	10	11	12
Tally marks											
Relative frequency											

Student responses will vary. Here is one example.

Sum rolled	2	3	4	5	6	7	8	9	10	11	12
Tally marks		$\vert$	$\vert$	$\vert\vert$	$\vert\vert$	$\vert$		$\vert\vert$	$\vert$		
Relative frequency	0	0.1	0.1	0.2	0.2	0.1	0	0.2	0.1	0	0

3. What is the average sum of these 10 rolls?

Answers will vary. For the example above, the average of the sum for these 10 rolls is as follows

$$3(0.1) + 4(0.1) + 5(0.2) + 6(0.2) + 7(0.1) + 9(0.2) + 10(0.1) = 6.4$$

4. How does this average compare to the expected value in Exercise 1? Are you surprised? Why or why not?

Answers will vary, but for most students, the average will not be the same as the expected sum of 7 in Exercise 1. Students may say they are not surprised as there were only 10 rolls of the dice.

Exploratory Challenge 1/Exercises 5–7 (5 minutes)

Allow students to roll two dice, recording the sum of each of the 10 rolls. Combine the results of these 10 rolls to the previous 10 rolls in Exercise 2. Students should then use their combined results of the 20 rolls to answer Exercises 6 and 7.

5. Roll the two dice 10 more times, recording the sums. Combine the sums of these 10 rolls with the sums of the previous 10 rolls for a total of 20 sums.

Sum rolled	2	3	4	5	6	7	8	9	10	11	12
Tally marks											
Relative frequency											

Student responses will vary. Here is one example.

Sum rolled	2	3	4	5	6	7	8	9	10	11	12
Tally marks		$\vert\vert\vert$	$\vert$	$\vert\vert\vert$	$\cancel{\vert\vert\vert\vert}$	$\vert\vert\vert\vert$		$\vert\vert$	$\vert\vert$		
Relative frequency	0	0.15	0.05	0.15	0.25	0.20	0	0.10	0.10	0	0

6. **What is the average sum for these 20 rolls?**

Answers will vary. For the example above, the average sum for these 20 rolls is as follows:

$$3(0.15) + 4(0.05) + 5(0.15) + 6(0.25) + 7(0.20) + 9(0.10) + 10(0.10) = 6.2$$

7. **How does the average sum for these 20 rolls compare to the expected value in Exercise 1?**

Answers will vary, but for most students, the average will still not be the same as the expected sum of 7 in Exercise 1.

Exploratory Challenge 1/Exercise 8 (5 minutes)

Divide the class into pairs. The partners should combine their rolls for a total of 40 rolls. If there are an odd number of students, create one group of three students. The group of three students will combine their rolls for a total of 60 rolls. Students should then find the average.

8. **Combine the sums of your 20 rolls with those of your partner. Find the average of the sum for these 40 rolls.**

Sum rolled	2	3	4	5	6	7	8	9	10	11	12
Frequency											
Relative frequency											

Student responses will vary. Here is one example.

Sum rolled	2	3	4	5	6	7	8	9	10	11	12
Frequency	0	3	3	6	8	5	5	4	4	1	1
Relative frequency	0	0.075	0.075	0.15	0.20	0.125	0.125	0.10	0.10	0.025	0.025

The average sum for these 40 rolls is as follows

$$\begin{gathered}3(0.075) + 4(0.075) + 5(0.15) + 6(0.2) + 7(0.125) \\ + 8(0.125) + 9(0.10) + 10(0.10) + 11(0.025) + 12(0.025) \\ = 6.825\end{gathered}$$

Exploratory Challenge 2/Exercise 9 (5 minutes)

Now assign two pairs to work together (four students). The pairs should combine their rolls for a total of 80 rolls. Students should then find the average sum.

Exploratory Challenge 2/Exercises 9–12

9. **Combine the sums of your 40 rolls above with those of another pair for a total of 80 rolls. Find the average value of the sum for these 80 rolls.**

Student responses will vary. Here is one example.

Sum rolled	2	3	4	5	6	7	8	9	10	11	12
Frequency	0	6	4	13	14	11	9	6	12	3	2
Relative frequency	0	0.075	0.05	0.1625	0.175	0.1375	0.1125	0.075	0.15	0.0375	0.025

The average sum for these 80 rolls is as follow

$$3(0.075) + 4(0.05) + 5(0.1625) + 6(0.175) + 7(0.1375)$$
$$+ 8(0.1125) + 9(0.075) + 10(0.15) + 11(0.0375) + 12(0.025)$$
$$= 7.0375$$

Exploratory Challenge 2/Exercises 10–11 (10 minutes)

Allow time for one person from each group of four students to put the results in a class chart on the board. The use of tally marks may aid students in combining their results. After the class chart is complete, allow students time to calculate the relative frequency for each sum rolled and to calculate the expected sum of the class rolls.

10. **Combine the sums of your 80 rolls with those of the rest of the class. Find the average sum for all the rolls.**

Sum rolled	2	3	4	5	6	7	8	9	10	11	12
Frequency											
Probability											

Student responses will vary. Here is one example for a class of 20 students.

Sum rolled	2	3	4	5	6	7	8	9	10	11	12
Frequency	7	27	33	48	60	62	47	43	39	23	11
Probability	0.0175	0.0675	0.0825	0.12	0.15	0.155	0.1175	0.1075	0.0975	0.0575	0.0275

The average sum of these 400 rolls is as follows

$$2(0.0175) + 3(0.0675) + 4(0.0825) + 5(0.12) + 6(0.15) + 7(0.155)$$
$$+ 8(0.1175) + 9(0.1075) + 10(0.0975) + 11(0.0575) + 12(0.0275)$$
$$= 6.9975$$

Allow time for each group of students to discuss Exercise 11. Then, ask students to share their ideas. Use this opportunity to check for understanding of the lesson.

MP.3

11. Think about your answer to Exercise 1. What do you notice about the averages you have calculated as the number of rolls increase? Explain why this happens.

As the number of rolls increase, the average value approaches the expected value. This happens because as the number of observed values of the random variable increase, the relative frequencies become closer to the actual probabilities in the probability distribution. The expected value of a discrete random variable is a long-run average value for the variable.

Exploratory Challenge 2/Exercise 12 (5 minutes)

Before having students complete Exercise 12, be sure to discuss that the interpretation for the expected value should include the long-run aspect of probability. The interpretation should also be written in the context related to the discrete random variable. Then, allow time for students to work Exercise 12. When students are finished, discuss the answer.

The expected value of a discrete random variable is the long-run mean value of the discrete random variable. Refer back to Exercise 1 where two dice were rolled, and the sum of the two dice was recorded. The interpretation of the expected value of a sum of 7 would be

When two dice are rolled over and over for a long time, the mean sum of the two dice is 7.

Notice that the interpretation includes the context of the problem, which is the random variable sum of two dice, and also includes the concept of *long-run average*.

12. Suppose a cancer charity in a large city wanted to obtain donations to send children with cancer to a circus appearing in the city. Volunteers were asked to call residents from the city's telephone book and to request a donation. Volunteers would try each phone number twice (at different times of day). If there was no answer, then a donation of $\$0$ was recorded. Residents who declined to donate were also recorded as $\$0$. The table below displays the results of the donation drive.

Donation	$\$0$	$\$10$	$\$20$	$\$50$	$\$100$
Probability	0.11	0.35	0.25	0.20	0.09

Find the expected value for the amount donated, *and* write an interpretation of the expected value in context.

The expected amount donated is as follows

$$0(0.11) + 10(0.35) + 20(0.25) + 50(0.20) + 100(0.09) = \$27.50.$$

When a large number of residents are contacted, the mean amount donated is $\$27.50$.

Closing (3 minutes)

- The expected value is the long-run mean of a discrete random variable.
- The interpretation of an expected value must contain the context related to the discrete random variable.
- Ask students to summarize the key ideas of the lesson in writing or by talking to a neighbor. Use this as an opportunity to informally assess student understanding. The lesson summary provides some of the key ideas from the lesson.

> **Lesson Summary**
>
> **The expected value of a discrete random variable is interpreted as the long-run mean of that random variable.**
>
> **The interpretation of the expected value should include the context related to the discrete random variable.**

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 8: Interpreting Expected Value

Exit Ticket

At a large university, students are allowed to register for no more than 7 classes. The number of classes for which a student is registered is a discrete random variable. The expected value of this random variable for students at this university is 4.15.

Write an interpretation of this expected value.

Exit Ticket Sample Solutions

At a large university, students are allowed to register for no more than seven classes. The number of classes for which a student is registered is a discrete random variable. The expected value of this random variable for students at this university is 4.15.

Write an interpretation of this expected value.

If many students at this university are asked how many classes they are registered for, the average number would be 4.15 classes.

Problem Set Sample Solutions

1. **Suppose that a discrete random variable is the number of broken eggs in a randomly selected carton of one dozen eggs. The expected value for the number of broken eggs is 0.48 eggs. Which of the following statements is a correct interpretation of this expected value? Explain why the others are wrong.**
   a. **The probability that an egg will break in one dozen cartons is 0.48, on average.**
   b. **When a large number of one dozen cartons of eggs are examined, the average number of broken eggs in a one dozen carton is 0.48 eggs.**
   c. **The mean number of broken eggs in one dozen cartons is 0.48 eggs.**

   The correct answer is b.

   Answer choice a relates the expected value to a probability, which is incorrect. The expected value is the long-run mean of a random variable.

   Answer choice c does not refer to the long-run aspect of the expected value.

2. **Due to state funding, attendance is mandatory for students registered at a large community college. Students cannot miss more than eight days of class before being withdrawn from a course. The number of days a student is absent is a discrete random variable. The expected value of this random variable for students at this college is 3.5 days. Write an interpretation of this expected value.**

   If many students at this college are asked how many days they have been absent, the average number would be 3.5 days.

3. **The students at a large high school were asked to respond anonymously to the question:**

   How many speeding tickets have you received?

   The table below displays the distribution of the number of speeding tickets received by students at this high school.

Number of tickets	0	1	2	3	4	5
Probability	0.55	0.28	0.09	0.04	0.03	0.01

   Compute the expected number of speeding tickets received. Interpret this mean in context.

   The expected number of speeding tickets received by students at this high school is as follows:

   $0(0.55) + 1(0.28) + 2(0.09) + 3(0.04) + 4(0.03) + 5(0.01) = 0.75$

   In the long-run, students at this high school have received an average of 0.75 speeding tickets.

4. Employees at a large company were asked to respond to the question:

How many times do you bring your lunch to work each week?

The table below displays the distribution of the number of times lunch was brought to work each week by employees at this company.

Number of times lunch brought to work each week	0	1	2	3	4	5
Probability	0.30	0.12	0.12	0.10	0.06	0.30

Compute the expected number of times lunch was brought to work each week. Interpret this mean in context.

$0(0.30) + 1(0.12) + 2(0.12) + 3(0.10) + 4(0.06) + 5(0.30) = 2.4$

In the long-run, employees at this company brought lunch on average lunch 2.4 times each week.

5. Graduates from a large high school were asked the following:

How many total AP courses did you take from Grade 9 through Grade 12?

The table below displays the distribution of the total number of AP courses taken by graduates while attending this high school.

Number of AP courses	0	1	2	3	4	5	6	7	8
Probability	0.575	0.06	0.09	0.12	0.04	0.05	0.035	0.025	0.005

Compute the expected number of total AP courses taken per graduate. Interpret this mean in context.

The expected number of AP courses taken is as follows

$0(0.575) + 1(0.06) + 2(0.09) + 3(0.12) + 4(0.04) + 5(0.05) + 6(0.035) + 7(0.025) + 8(0.005) =$
1.435 *AP courses*

If many high school graduates are asked how many AP courses were taken between Grades 9–12, the average number would be 1.435 AP courses.

6. **At an inspection center in a large city, the tires on the vehicles are checked for damage. The number of damaged tires is a discrete random variable. Create two different distributions for this random variable that have the same expected number of damaged tires. What is the expected number of damaged tires for the two distributions? Interpret the expected value.**

Distribution 1:

Number of damaged tires	0	1	2	3	4
Probability					

Distribution 2:

Number of damaged tires	0	1	2	3	4
Probability					

Answers will vary. Here is one example.

Distribution 1:

Number of damaged tires	0	1	2	3	4
Probability	0.60	0.20	0.10	0.05	0.05

The expected number of damaged tires is as follows

$0(0.60) + 1(0.20) + 2(0.10) + 3(0.05) + 4(0.05) = 0.75$ *tires*

Distribution 2:

Number of damaged tires	0	1	2	3	4
Probability	0.55	0.30	0.05	0.05	0.05

The expected number of damaged tires is as follows

$$0(0.55) + 1(0.30) + 2(0.05) + 3(0.05) + 4(0.05) = 0.75 \textit{ tires}$$

Because this inspection center examines the tires on the vehicles of a large number of customers, the inspectors find an average of 0.75 damaged tires per vehicle.