Lesson 7: Expected Value of a Discrete Random Variable

Student Outcomes

- Students calculate the expected value of a discrete random variable.

Lesson Notes

This lesson, which is divided into two parts, develops the concept of the expected value of a discrete random variable as the mean of the distribution of the discrete random variable. In Exploratory Challenge 1, the method for computing the expected value of a discrete random variable is developed. In Exploratory Challenge 2, this lesson relates the method for computing the expected value of a discrete random variable to previous work with vectors, i.e., the computation of the dot product of two vectors. This lesson is designed for students to work in pairs. Each student will need a die. Either the teacher or each pair of students will need a timer.

Classwork

Exploratory Challenge 1/Exercises 1–5 (5 minutes)

Read the instructions for the new game, Six Up. Be sure that students understand the rules for the game. Allow them to discuss and answer Exercises 1–5. When students are ready, discuss the answers.

Exploratory Challenge1/Exercises 1–5

A new game, Six Up, involves two players. Each player rolls his or her die, counting the number of times a "six" is rolled. The players roll their die for up to one minute. The first person to roll 15 sixes wins. If no player rolls 15 sixes in the one-minute time limit, then the player who rolls the greatest number of sixes wins that round. The player who wins the most rounds wins the game.

Suppose that your class will play this game. Your teacher poses the following question:

How many sixes would you expect to roll in one round?

MP.3

1. **How would you answer this question?**

   Answers will vary. Anticipate a wide range of answers to this question.

2. **What discrete random variable should you investigate to answer this question?**

   The discrete random variable is the number of sixes rolled in a round.

3. **What are the possible values for this discrete random variable?**

   The values are $0, 1, 2, 3, 4, \ldots, 15$.

Scaffolding:

Teachers may choose to use a simpler version of this question. For example:

If you flip a coin 10 times, how many times would you expect the coin to land on heads? What about if you flipped it 20 times?

4. **Do you think these possible values are all equally likely to be observed?**

   No, some outcomes are more likely than others. For example, it would be unlikely to see only 0 or 1 six rolled in a round.

5. **What might you do to estimate the probability of observing each of the different possible values?**

   We could play many rounds of the game to see how often each number occurs. This would allow us to estimate the probabilities.

Exploratory Challenge 1/Exercise 6–8 (12 minutes)

Be sure that students understand the rules of the game. Students should roll the die as quickly as possible to try to be the first to roll 15 sixes (or to roll the greatest number of sixes). You may time each minute, giving signals to begin and end the round. (The pairs of students may be allowed to time their own minutes, if preferred.)

Exploratory Challenge 1/Exercises 6–8

You and your partner will play the Six Up game. Roll the die until the end of one minute or until you or your partner rolls 15 sixes. Count the number of sixes you rolled. Remember to stop rolling if either you or your partner rolls 15 sixes before the end of one minute.

6. **Play five rounds. After each round, record the number of sixes that you rolled.**

   Student answers will vary. Values range from 0 to 15. One possible answer is shown.

   The number of sixes rolled in each round is

Round 1	Round 2	Round 3	Round 4	Round 5
5	9	14	11	15

7. **On the board, put a tally mark for the number of sixes rolled in each round.**

   Student answers will vary. Values range from 0 to 15. One possible answer is shown.

   The frequencies for the number of sixes rolled by the class are shown below:

Number of sixes rolled	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Frequency	0	0	0	0	3	3	3	3	6	3	5	10	9	4	2	9

8. **Using the data summarized in the frequency chart on the board, find the mean number of sixes rolled in a round.**

Answers will vary. For the example given in Exercise 6:

The total number of observations is 60.

The mean is

$$\frac{0(0)+1(0)+2(0)+3(0)+4(3)+5(3)+6(3)+7(3)+8(6)+9(3)+10(5)+11(10)+12(9)+13(4)+14(2)+15(9)}{60}$$

$= 10.4$ *sixes.*

Exploratory Challenge 1/Exercises 9–13 (8 minutes)

Students should answer Exercises 9–13. They may work with their partner. Be sure that students understand that the relative frequency is a proportion.

Exploratory Challenge 1/Exercises 9–13

9. **Calculate the relative frequency (*proportion)* for each value of the discrete random variable (i.e., the number of sixes rolled) by dividing the frequency for each possible value of the number of sixes rolled by the total number of rounds (the total number of tally marks). (The relative frequencies can be interpreted as estimates of the probabilities of observing the different possible values of the discrete random variable.)**

Student answers will vary. For the above example

The proportions are displayed in the table below. The observed number for each x*-value is divided by* 60*, the total number of observations of the variable.*

Number of sixes rolled	0	1	2	3	4	5	6	7
Relative frequency	$\frac{0}{60}=0$	$\frac{0}{60}=0$	$\frac{0}{60}=0$	$\frac{0}{60}=0$	$\frac{3}{60}=0.05$	$\frac{3}{60}=0.05$	$\frac{3}{60}=0.05$	$\frac{3}{60}=0.05$

Number of sixes rolled	8	9	10	11	12	13	14	15
Relative frequency	$\frac{6}{60}=0.1$	$\frac{3}{60}=0.05$	$\frac{5}{60}=0.083$	$\frac{10}{60}=0.167$	$\frac{9}{60}=0.15$	$\frac{4}{60}=0.067$	$\frac{2}{60}=0.033$	$\frac{9}{60}=0.15$

10. Multiply each possible value for the number of sixes rolled by the corresponding probability (relative frequency).

Answers will vary. The products for the example given above are the following:

$0(0) = 0$ $1(0) = 0$ $2(0) = 0$ $3(0) = 0$ $4(0.05) = 0.2$ $5(0.05) = 0.25$

$6(0.05) = 0.3$ $7(0.05) = 0.35$ $8(0.1) = 0.8$ $9(0.05) = 0.45$ $10(0.083) = 0.83$

$11(0.167) = 1.837$ $12(0.15) = 1.8$ $13(0.067) = 0.871$ $14(0.033) = 0.462$ $15(0.15) = 2.25$

11. Find the sum of the calculated values in Exercise 10. This number is called the *expected value* of the discrete random variable.

Student answers will vary. For the above example:

The expected value is

$0+0+0+0+0.2+0.25+0.3+0.35+0.8+0.45+0.83+1.837+1.8+0.871+0.462+2.25$

$= 10.4$ *sixes.*

12. What do you notice about the sum in Exercise 11 and the mean that you calculated in Exercise 8?

The two values are the same. The procedures in Exercises 9, 10, and, 11 combine to create a procedure for calculating the expected value of a discrete random variable.

13. The *expected value* of a random variable, x, is also called the *mean* of the distribution of that random variable. Why do you think it is called the mean?

Because it is like calculating a long-run average value for the random variable.

Before moving on to the next exercise, check for understanding by asking students to answer the following:

- What is expected value and what did it represent in this example?
  - *It is the mean of the distribution of the random variable "the number of sixes rolled." In this case, the expected value was* 10.4 *sixes.*

Exploratory Challenge 1/Exercise 14 (3 minutes)

Explain the formula presented in the text.

Be sure to discuss symbols in the formula:

$$\text{expected value} = \sum xp$$

- The mathematical symbol, Σ, denotes the word *summation*.
- This equation indicates to find the sum of the products of each individual x-value times its corresponding probability.

Read the question and have students answer it.

Scaffolding:

- The word *sigma* may need rehearsal.
- The symbol denotes the word *summation*.
- Students may need to be shown how to use the formula with the probability distribution. Consider using the following as an example:

X	0	1	2
Probability	0.3	0.3	0.4

Exploratory Challenge 1/Exercise 14

The expected value for a discrete random variable is computed using the following equation:

$$\text{expected value} = \sum(\text{each value of the random variable } (x)) \times (\text{the corresponding probability } (p))$$

or

$$\text{expected value} = \sum xp$$

where x is a possible value of the random variable, and p is the corresponding probability.

The following table provides the probability distribution for the number of heads occurring when two coins are flipped.

Number of heads	0	1	2
Probability	0.25	0.5	0.25

14. If two coins are flipped many times, how many heads would you expect to occur, on average?

$0(0.25) + 1(0.5) + 2(0.25) = 1$ *head*

You would expect 1 head to occur when two coins are flipped.

Scaffolding

- Students might need to think of two different coins, for example a penny and a nickel.
- This will help students understand that there are two ways to get one head. The penny can land on heads and the nickel lands tails, *or* the penny lands on tails and the nickel lands heads.
- Have students who are above grade level calculate the expected value of a more complicated random variable:

Number of Cars	0	1	2	3	4
Probability	0.07	0.44	0.24	0.13	0.12

Exploratory Challenge 1/Exercises 15–16 (5 minutes)

These exercises focus on the interpretation of expected value. Make sure that students understand that expected value is interpreted as a long-run average. Then have students work in pairs to complete Exercises 15 and 16. You may want to have some students share their answers to Exercise 16 with the class.

Exploratory Challenge 1/Exercises 15–16

15. The estimated expected value for the number of sixes rolled in one round of the Six Up game was 10.4. Write a sentence interpreting this value.

If a player were to play many rounds of the Six Up game, the average number of sixes rolled by that player would be about 10.4 sixes per round.

16. Suppose that you plan to change the rules of the Six Up game by increasing the one-minute time limit for a round. You would like to set the time so that most rounds will end by a player reaching 15 sixes. Considering the estimated expected number of sixes rolled in a one minute round, what would you recommend for the new time limit. Explain your choice.

Answers will vary. Look for a justification based on the expected value. For example, a student might say 1.5 minutes, reasoning that if the average number of sixes rolled in one minute is about 10, increasing the time by 10% to 1.5 minutes should be enough time because the average would then be around 15. Or a student might say two minutes should be enough time because if the average number of sixes in one minute is around 19, in two minutes, it would be very likely that one of the players would get to 15 sixes.

Exploratory Challenge 2/Exercises 17–19 (5 minutes)

Read the description of the two vectors. You might ask students to recall how to find a dot product. Have students answer the questions and then discuss answers as a class.

Exploratory Challenge 2/Exercises 17–19

Suppose that we convert to two vectors the above table displaying the discrete distribution for the number of heads occurring when two coins are flipped.

Let vector A be the number of heads occurring.

Let vector B be the corresponding probabilities.

$$A = \langle 0, 1, 2 \rangle \qquad B = \langle 0.25, 0.5, 0.25 \rangle$$

17. **Find the dot product of these two vectors.**

    $0(0.25) + 1(0.5) + 2(0.25) = 1$

18. **Explain how the dot product computed in Exercise 17 compares to the expected value computed in Exercise 14.**

    The dot product and the expected value are equal.

MP.2

19. **How do these two processes, finding the expected value of a discrete random variable and finding the dot product of two vectors, compare?**

    The two processes are the same.

Closing (2 minutes)

Students should understand that:

- The long-run relative frequency or proportion of each possible value of a discrete random variable can be interpreted as an estimate of the probability of the values of observing that value.
- The expected value of a random variable is the mean of the distribution of that random variable.
- Ask students to summarize the key ideas of the lesson in writing or by talking to a neighbor. Use this as an opportunity to informally assess student understanding. The lesson summary provides some of the key ideas from the lesson.

> **Lesson Summary**
>
> **The *expected value* of a random variable is the *mean* of the distribution of that random variable.**
>
> **The expected value of a discrete random variable is the *sum* of the *products* of each possible value (x) and the corresponding probability.**
>
> **The process of computing the expected value of a discrete random variable is similar to the process of computing the dot product of two vectors.**

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 7: Expected Value of a Discrete Random Variable

Exit Ticket

At a carnival, one game costs $1 to play. The contestant gets one shot in an attempt to bust a balloon. Each balloon contains a slip of paper with one of the following messages.

- Sorry, you do not win, but you get your dollar back. (The contestant has not lost the $1 cost.)
- Congratulations, you win $2. (The contestant has won $1.)
- Congratulations, you win $5. (The contestant has won $4.)
- Congratulations, you win $10. (The contestant has won $9.)

If the contestant does not bust a balloon, then the $1 cost is forfeited. The table below displays the probability distribution of the discrete random variable, or ***net*** winnings for this game.

Net winnings	-1	0	1	4	9
Probability	0.25	$?$	0.3	0.08	0.02

1. What is the sum of the probabilities in a discrete probability distribution? Why?

2. What is the probability that a contestant will bust a balloon and receive the message, "Sorry, you do not win, but you get your dollar back"?

3. What is the net amount that a contestant should expect to win per game if the game were to be played many times?

Exit Ticket Sample Solutions

At a carnival, one game costs $\$1$ to play. The contestant gets one shot in an attempt to bust a balloon. Each balloon contains a slip of paper with one of the following messages.

- **Sorry, you do not win, but you get your dollar back. (The contestant has not lost the $\$1$ cost.)**
- **Congratulations, you win $\$2$. (The contestant has won $\$1$.)**
- **Congratulations, you win $\$5$. (The contestant has won $\$4$.)**
- **Congratulations, you win $\$10$. (The contestant has won $\$9$.)**

If the contestant does not bust a balloon, then the $\$1$ cost is forfeited. The table below displays the probability distribution of the discrete random variable, or *net* winnings for this game.

Net winnings	-1	0	1	4	9
Probability	0.25	$?$	0.3	0.08	0.02

1. **What is the sum of the probabilities in a discrete probability distribution? Why?**

   The sum of the probabilities in a discrete probability distribution is one. In a discrete distribution, every possible x-value of the random variable is listed. Thus, the sum of the corresponding probabilities must equal one.

2. **What is the probability that a contestant will bust a balloon and receive the message, "Sorry, you do not win, but you get your dollar back"?**

   If you receive the message, "Sorry, you do not win, but you get your dollar back," then your net winnings is $\$0$. The probability of winning $\$0$ is

   $$1-(0.25+0.3+0.08+0.02)=0.35$$

3. **What is the net amount that a contestant should expect to win per game if the game were to be played many times?**

   The net amount that a contestant should expect to win is the expected value of the probability distribution.

   $$-1(0.25)+0(0.35)+1(0.3)+4(0.08)+9(0.02)=\$0.55$$

Problem Set Sample Solutions

1. The number of defects observed in the paint of a newly manufactured car is a discrete random variable. The probability distribution of this random variable is shown in the table below.

Number of defects	0	1	2	3	4	5
Probability	0.02	0.15	0.40	0.35	0.05	0.03

If large numbers of cars were inspected, what would you expect to see for the average number of defects per car?

The expected number is

$$0(0.02) + 1(0.15) + 2(0.40) + 3(0.35) + 4(0.05) + 5(0.03) = 2.35 \textit{ defects.}$$

Note: Since the expected number of defects is the mean of the probability distribution, we do not round to a whole number.

2.

a. Interpret the expected value calculated in Problem 1. Be sure to give your interpretation in context.

If many cars were inspected and the number of defects was observed for each car, the average number of defects per car would be about 2.35*.*

b. Explain why it is not reasonable to say that every car will have the expected number of defects.

The expected value is an average. It represents a typical value for a random variable, but individual values may be greater than or less than the expected value.

3. Students at a large high school were asked how many books they read over the summer. The number of books read is a discrete random variable. The probability distribution of this random variable is shown in the table below.

Number of books read	0	1	2	3
Probability	0.12	0.33	0.48	0.07

If a large number of students were asked how many books they read over the summer, what would you expect to see for the average number of books read?

The expected number is

$$0(0.12) + 1(0.33) + 2(0.48) + 3(0.07) = 1.5 \textit{ books.}$$

4. Suppose two dice are rolled. The sum of the two numbers showing is a discrete random variable. The following table displays the probability distribution of this random variable.

Sum rolled	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

If you rolled two dice a large number of times, what would you expect the average of the sum of the two numbers showing to be?

The expected sum of the two rolled dice is

$$2\left(\frac{1}{36}\right)+3\left(\frac{1}{18}\right)+4\left(\frac{1}{12}\right)+5\left(\frac{1}{9}\right)+6\left(\frac{5}{36}\right)+7\left(\frac{1}{6}\right)+8\left(\frac{5}{36}\right)+9\left(\frac{1}{9}\right)+10\left(\frac{1}{12}\right)+11\left(\frac{1}{18}\right)+12\left(\frac{1}{36}\right)=7.$$

5. Explain why it is not possible for a random variable whose only possible values are 3, 4, and 5 to have an expected value greater than 6.

Expected value is an average. The average of a large number of observations of this variable cannot be greater than 6 if the variable can only have values that are 6 or less.

6. Consider a discrete random variable with possible values 1, 2, 3, and 4. Create a probability distribution for this variable so that its expected value would be greater than 3 by entering probabilities into the table below. Then calculate the expected value to verify that it is greater than 3.

Value of variable	1	2	3	4
Probability				

Answers will vary. One possible answer is as follows:

Value of variable	1	2	3	4
Probability	0.1	0.1	0.1	0.7

with expected value $= 1(0.1) + 2(0.1) + 3(0.1) + 4(0.7) = 3.4$