



Lesson 3: Counting Rules—Combinations

Student Outcomes

- Students calculate the number of different combinations of k items selected from a set of n distinct items.

Lesson Notes

This lesson continues counting rules by introducing the number of combinations of n things taken k at a time. A formal definition of *combination* is presented. Several applications will be developed in which students calculate the number of different combinations of k things selected from a set of n distinct items.

Classwork

Example 1 (8 minutes): Combinations

This lesson begins with an example that makes the distinction between situations in which order is important and those in which order is not important. Work through this example with the class. The definition of combinations as a subset of k items selected from a set of n distinct items is formally introduced. Focus on the set of distinct items and that the sequence or order is not important.

Example 1

Seven speed skaters are competing in an Olympic race. The first-place skater earns the gold medal, the second-place skater earns the silver medal, and the third-place skater earns the bronze medal. In how many different ways could the gold, silver, and bronze medals be awarded? The letters A, B, C, D, E, F, and G will be used to represent these seven skaters.

How can we determine the number of different possible outcomes? How many are there?

We can use the permutations formula from the previous lesson. Because each outcome is a way of forming an ordered arrangement of 3 things from a set of 7, the total number of possible outcomes is

$${}_7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = 210.$$

Now consider a slightly different situation. Seven speed skaters are competing in an Olympic race. The top three skaters move on to the next round of races. How many different "top three" groups can be selected?

How is this situation different from the first situation? Would you expect more or fewer possibilities in this situation? Why?

The outcomes in the first situation are medals based on order—first place gets gold, second place gets silver, and third place gets bronze. The outcomes in this situation are not based on order. The top three finishers move on; the others do not. I would expect more possibilities in the first situation because each skater can be first, second, or third and still advance (i.e., there are three different possible finishing positions each skater can attain to earn advancement). In the first situation, there is only one position to earn gold, one for silver, and one for bronze.

Scaffolding:

Below are two questions to ask students who may be struggling with how to begin answering the questions.

- What are two examples of possible outcomes?

Example: BCF and ADC

- Should the outcome CDA be considered as a different outcome than ADC, even though it includes the same three skaters?

Yes, because the order in which skaters finish matters.

Would you consider the outcome where skaters B, C, and A advance to the final to be a different outcome than A, B, and C advancing?

No, this is the same outcome—the same three skaters are advancing to the final competition.

A permutation is an ordered arrangement (a sequence) of k items from a set of n distinct items.

In contrast, a combination is an unordered collection (a set) of k items from a set of n distinct items.

When we wanted to know how many ways there are for seven skaters to finish first, second, and third, order was important. This is an example of a permutation of 3 selected from a set of 7. If we want to know how many possibilities there are for which three skaters will advance to the finals, order is not important. This is an example of a combination of 3 selected from a set of 7.

Scaffolding:

- English language learners may have difficulty with the term *distinct*, so repeated and visual examples of the meaning should be considered.
- To aid in visualizing the meaning of *distinct* (different in a way that you can see, hear, smell, feel, etc.), put three different shapes in a bag, have students reach into the bag to feel each shape, and ask them to identify the distinct shapes and how they know they are different.



- Use seven students from the class to represent each of the seven skaters from the example. Each is distinct. While they are representing a group of skaters, each student is an individual and can be identified by a distinct name such as John or Claire (or your students' names).

After discussing the two situations and the difference between a permutation and a combination, provide an opportunity to check for understanding.

- Turn and talk to your neighbor: What is the difference between a permutation and a combination? With your partner, come up with an example of each.

Encourage students to elaborate if they provide answers such as, "Permutation means order matters, and combination means order doesn't matter." The order of what? Students should understand that in a permutation, the order of possible outcomes of a situation matters, or that outcomes need to happen in a specific order. In a combination, the order of possible outcomes does not matter, or the set of specific outcomes needs to happen in no specific order.

Exercises 1–4 (8 minutes)

Students should work in small groups (2 or 3 students per group). Allow about 8 minutes for them to complete Exercises 1–4. Encourage students to write and/or talk to a partner about how they might answer Exercise 1. To assist students in making their conjectures, it may be beneficial to provide a diagram for them. This could be a circle with four points on it drawn on a whiteboard. When students have finished answering the four questions, discuss the answers. The main point is that if order was important, then segment DC would be different from CD . Since we are only interested in the number of segments, the order of the segment labels is not important. The purpose of this activity is to, again, distinguish between permutations and combinations.

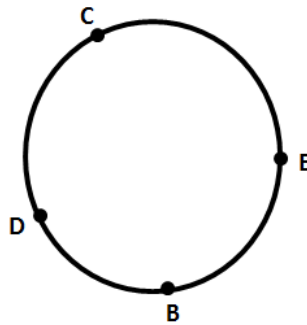
Exercises 1–4

- Given four points on a circle, how many different line segments connecting these points do you think could be drawn? Explain your answer.

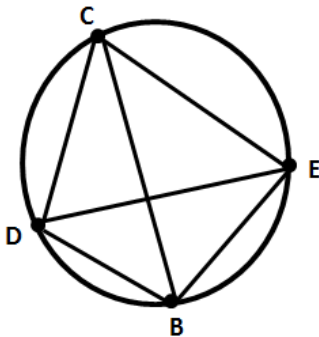
Answers will vary. Expect answers to range from 4 to 42

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2. Draw a circle and place four points on it. Label the points as shown. Draw segments (chords) to connect all the pairs of points. How many segments did you draw? List each of the segments that you drew. How does the number of segments compare to your answer in Exercise 1?



Answer:



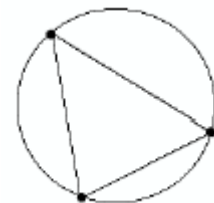
6 segments: $\overline{DC}$, $\overline{DE}$, $\overline{DB}$, $\overline{CB}$, $\overline{CE}$, and $\overline{BE}$

You can think of each segment as being identified by a subset of two of the four points on the circle. Chord $\overline{ED}$ is the same as chord $\overline{DE}$. The order of the segment labels is not important. When you count the number of segments (chords), you are counting combinations of two points chosen from a set of four points.

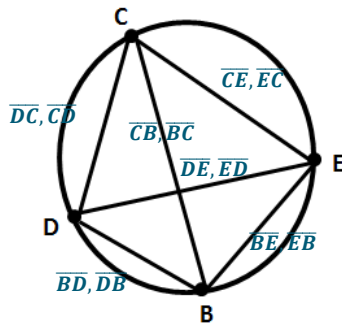
Scaffolding:

For students who struggle, pose the following questions:

- Given two points on a circle, how many different line segments connecting these points do you think could be drawn?
- It may be necessary to begin with two or three points on a circle and build up to four using the diagrams below.



3. Find the number of permutations of two points from a set of four points. How does this answer compare to the number of segments you were able to draw?

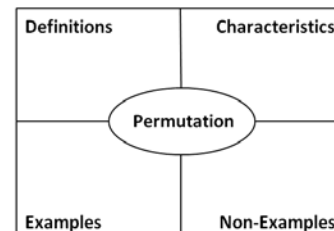


The diagram is the same, but this answer is double the number of combinations. In a permutation, the segment $\overline{DC}$ is counted as different from segment $\overline{CD}$. In this case, there are six combinations, and each segment can be represented two ways. $6 \times 2 = 12$ permutations.

- 4.
- If you add a fifth point to the circle, how many segments (chords) can you draw?
10 segments
 - If you add a sixth point, how many segments (chords) can you draw?
15 segments

Scaffolding:

- Students may associate the term *combination* with a combination of a lock.
- Clarify that in this context, *combination* refers to a selection of items from a group where order does not matter.
- The combination of a lock is a collection of numbers or letters where order *does* matter. (This is an example of a permutation.)
- Consider having students employ Frayer diagrams, such as the one below, to help distinguish between permutation and combination.



Example 2: Combinations Formula (7 minutes)

Briefly summarize the four examples from the lesson. Students should work independently to answer the question.

Example 2

Let's look closely at the four examples we have studied so far.

Choosing gold, silver, and bronze medal skaters	Choosing groups of the top three skaters
Finding the number of segments that can be drawn connecting two points out of four points on a circle	Finding the number of unique segments that can be drawn connecting two points out of four points on a circle

What do you notice about the way these are grouped?

Sample responses: The examples on the left are permutations, and the examples on the right are combinations. In the left examples, the order of the outcomes matter, while in those on the right, the order does not matter.

Now discuss as a class the combinations formula:

The number of combinations of k items selected from a set of n distinct items is

$${}_nC_k = \frac{{}_nP_k}{k!} \text{ or } {}nC_k = \frac{n!}{k!(n-k)!}.$$

The number of permutations of three skaters from the seven is found in Example 1 to be

$${}_7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = 210.$$

This means that there are 210 different ordered arrangements of three skaters from a set of seven skaters.

There are fewer combinations because when order is not important, we do not want to count different orders of the same three skaters as different outcomes.

Consider the set of skaters A, B, and C. There are $3 \cdot 2 \cdot 1 = 3! = 6$ different ordered arrangements of these three skaters, each of which is counted with the permutations formula. So, if we want combinations of 3 from a set of 7, you would need to divide the number of permutations by $3!$. Then

$${}_7C_3 = \frac{{}_7P_3}{3!} = \frac{7!}{3!(7-3)!} = \frac{7!}{3!4!} = 35.$$

In general

$${}_nC_k = \frac{{}_nP_k}{k!} = \frac{n!}{k!(n-k)!}.$$

For example, ${}_9C_4$

$$\begin{aligned} {}_9C_4 &= \frac{9!}{4!(9-4)!} \\ &= \frac{9!}{4! \cdot 5!} \\ &= \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(4 \cdot 3 \cdot 2 \cdot 1)(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)} \\ &= \frac{9 \cdot 8 \cdot 7 \cdot 6}{(4 \cdot 3 \cdot 2 \cdot 1)} \\ &= \frac{3024}{24} \\ &= 126 \end{aligned}$$

Be sure to discuss the connection between the combination formula and the permutation formula. If your students have access to technology (a graphing calculator or computer software), show them how to use technology to calculate the number of permutations and the number of combinations.

- Ask students, with a partner, to explain how to use combinations and permutations to explain the circle segments and point permutations examples. Use this as an opportunity to informally assess student learning.
 - The number of segments (chords) that can be drawn with four points on a circle is the number of combinations of two points selected from a set of four points.

$${}_4C_2 = \frac{{}_4P_2}{2!} = 6 \text{ or } {}_4C_2 = \frac{4!}{2!(4-2)!} = 6$$

Exercises 5–11 (15 minutes)

Students should work in small groups (2 or 3 students per group). Allow about 15 minutes for them to complete Exercises 4–11. Students may use a calculator, but encourage them to write the combination formula and show the substitution prior to using the calculator. In Exercise 7, students need to think carefully about whether to use the formula for permutations or for combinations in solving problems. Also encourage students to think about whether or not order is important in each problem. After students have completed the problems, review the answers.

Exercises 5–11

5. Find the value of each of the following:

- | | |
|-----------------|----|
| a. ${}_9C_2$ | 36 |
| b. ${}_7C_7$ | 1 |
| c. ${}_8C_0$ | 1 |
| d. ${}_{15}C_1$ | 15 |

6. Find the number of segments (chords) that can be drawn for each of the following:

- | | |
|---------------------------|--|
| a. 5 points on a circle | ${}_5C_2 = 10$ |
| b. 6 points on a circle | ${}_6C_2 = 15$ |
| c. 20 points on a circle | ${}_{20}C_2 = 190$ |
| d. n points on a circle | ${}_nC_2 = \frac{{}_nP_2}{2!}$ $= \frac{n!}{2!(n-2)!}$ |

7. For each of the following questions, indicate whether the question posed involves permutations or combinations. Then provide an answer to the question with an explanation for your choice.

- a. A student club has 20 members. How many ways are there for the club to choose a president and a vice-president?
Permutations, 380. The role of president is different than that of vice-president. The order of outcomes matters.
- b. A football team of 50 players will choose two co-captains. How many different ways are there to choose the two co-captains?

Combinations, 1,225. Regardless of order, two players will attain the same outcome of co-captain.

- c. There are seven people who meet for the first time at a meeting. They shake hands with each other and introduce themselves. How many handshakes have been exchanged?
Combinations, 21 People only shake hands with those they have not greeted yet. Each greeting is unique. Person 1 shaking hands with person 2 is the same event as person 2 shaking hands with person 1.
- d. At a particular restaurant, you must choose two different side dishes to accompany your meal. If there are eight side dishes to choose from, how many different possibilities are there?
Combinations, 28 The order of dishes chosen does not matter. Choosing veggies and mac and cheese is the same as choosing mac and cheese and veggies.
- e. How many different four-letter sequences can be made using the letters A, B, C, D, E, and F if letters may not be repeated?
Permutations, 360 Each four-letter sequence is unique. ABCD is different from BACD even though they contain identical letters.

8. How many ways can a committee of 5 students be chosen from a student council of 30 students? Is the order in which the members of the committee are chosen important?

No, ${}_{30}C_5 = 142,506$.

9. Brett has ten distinct t-shirts. He is planning on going on a short weekend trip to visit his brother in college. He has enough room in his bag to pack four t-shirts. How many different ways can he choose four t-shirts for his trip?

${}_{10}C_4 = 210$

10. How many three-topping pizzas can be ordered from the list of toppings below? Did you calculate the number of permutations or the number of combinations to get your answer? Why did you make this choice?

Pizza Toppings

sausage	pepperoni	meatball	onions	olives	spinach
pineapple	ham	green peppers	mushrooms	bacon	hot peppers

${}_{12}C_3 = 220$; I used combinations because the order of toppings chosen does not matter. My pizza is the same if I order pepperoni, olives, and mushrooms or olives, pepperoni, and mushrooms.

Calculated number of combinations—we are choosing 3 items from the 12 distinct toppings, and the order that the 3 toppings are chosen is not important.

11. Write a few sentences explaining how you can distinguish a question about permutations from a question about combinations.

Answers will vary but should address whether order is important or not.

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Closing (2 minutes)

- Ask students to explain the difference between a permutation and combination. If they struggle with the articulating the difference, ask them to provide an example for each instead.
 - *If order does matter, it is a permutation. A phone number.*
 - *If order doesn't matter, it is a combination. Picking five friends from an entire class for a dodge-ball team.*

- Ask students to summarize the key ideas of the lesson in writing or by talking to a neighbor. Use this as an opportunity to informally assess student understanding. The lesson summary provides some of the key ideas from the lesson.

Lesson Summary

A combination is a subset of k items selected from a set of n distinct items.

The number of combinations of k items selected from a set of n distinct items is

$${}_nC_k = \frac{{}_nP_k}{k!} \text{ or } {}nC_k = \frac{n!}{k!(n-k)!}$$

Exit Ticket (5 minutes)

Name _____

Date _____

Lesson 3: Counting Rules—Combinations

Exit Ticket

1. Timika is a counselor at a summer camp for young children. She wants to take 20 campers on a hike and wants to choose a pair of students to lead the way. In how many ways can Timika choose this pair of children?
2. Sean has 56 songs on his MP3 player. He wants to randomly select 6 of the songs to use in a school project. How many different groups of 6 songs could Sean select? Did you calculate the number of permutations or the number of combinations to get your answer? Why did you make this choice?
3. A fast food restaurant has the following options for toppings on their hamburgers: mustard, ketchup, mayo, onions, pickles, lettuce, and tomato. In how many ways could a customer choose four different toppings from these options?
4. Seven colored balls (red, blue, yellow, black, brown, white, and orange) are in a bag. A sample of three balls is selected without replacement. How many different samples are possible?

Exit Ticket Sample Solutions

1. Timika is a counselor at a summer camp for young children. She wants to take 20 campers on a hike and wants to choose a pair of students to lead the way. In how many ways can Timika choose this pair of children?

$${}_{20}C_2 = 190$$

2. Sean has 56 songs on his MP3 player. He wants to randomly select 6 of the songs to use in a school project. How many different groups of 6 songs could Sean select? Did you calculate the number of permutations or the number of combinations to get your answer? Why did you make this choice?

$${}_{56}C_6 = 32,468,436$$

Calculated number of combinations—choosing 6 songs from 56 distinct songs, and order is not important

3. A fast food restaurant has the following options for toppings on their hamburgers: mustard, ketchup, mayo, onions, pickles, lettuce, and tomato. In how many ways could a customer choose 4 different toppings from these options?

$${}_7C_4 = 35$$

4. Seven colored balls (red, blue, yellow, black, brown, white, and orange) are in a bag. A sample of three balls is selected without replacement. How many different samples are possible?

$${}_7C_3 = 35$$

Problem Set Sample Solutions

1. Find the value of each of the following:

a. ${}_9C_8$ 9

b. ${}_9C_1$ 9

c. ${}_9C_9$ 1

2. Explain why ${}_6C_4$ is the same value as ${}_6C_2$.

$${}_6C_4 = \frac{6!}{4! \cdot 2!} \qquad {}_6C_2 = \frac{6!}{2! \cdot 4!}$$

The denominators are the same. This is because the number of ways to choose four from a set of six is the same as the number of ways to select which two to exclude.

3. Pat has 12 books he plans to read during the school year. He decides to take 4 of these books with him while on winter break vacation. He decides to take *Harry Potter and the Sorcerer's Stone* as one of the books. In how many ways can he select the remaining 3 books?

$${}_{11}C_3 = 165$$

4. In a basketball conference of 10 schools, how many conference basketball games are played during the season if the teams all play each other exactly once?

$${}_{10}C_2 = 45$$

5. Which scenario(s) below is represented by ${}_9C_3$? *B and C*
- Number of ways 3 of 9 people can sit in a row of 3 chairs.
 - Number of ways to pick 3 students out of 9 students to attend an art workshop.
 - Number of ways to pick 3 different entrees from a buffet line of 9 different entrees.
6. Explain why ${}_{10}C_3$ would not be used to solve the following problem:
- There are 10 runners in a race. How many different possibilities are there for the runners to finish first, second, and third?
- This is an example of a permutation. The order of how the runners finish is important.*
7. In a lottery, players must match five numbers plus a bonus number. Five white balls are chosen from 59 white balls numbered from 1 to 59, and one red ball (the bonus number) is chosen from 35 red balls numbered 1 to 35. How many different results are possible?
- white ball: ${}_{59}C_5 = 5,006,386$*
- red ball: ${}_{35}C_1 = 35$*
- Number of possible results = $5,006,386 \cdot 35 = 175,223,510$*
8. In many courts, 12 jurors are chosen from a pool of 30 perspective jurors.
- In how many ways can 12 jurors be chosen from the pool of 30 perspective jurors?
- ${}_{30}C_{12} = 86,493,225$*
- Once the 12 jurors are selected, 2 alternates are selected. The order of the alternates is specified. If a selected juror cannot complete the trial, the first alternate is called on to fill that jury spot. In how many ways can the 2 alternates be chosen after the 12 jury members have been chosen?
- ${}_{18}P_2 = 306$*
9. A band director wants to form a committee of 4 parents from a list of 45 band parents.
- How many different groups of 4 parents can the band director select?
- ${}_{45}C_4 = 148,995$*
- How many different ways can the band director select 4 parents to serve in the band parents' association as president, vice-president, treasurer, and secretary?
- ${}_{45}P_4 = 3,575,880$*
- Explain the difference between parts (a) and (b) in terms of how you decided to solve each part.
- Part (a) is an example of finding the number of combinations—how many ways can 4 parents be chosen from 45 distinct parents. The order is not important.*
- Part (b) is an example of finding the number of permutations. The order of how the parents are selected is important.*
10. A cube has faces numbered 1 to 6. If you roll this cube 4 times, how many different outcomes are possible?
- Using the fundamental counting principle: $6^4 = 1,296$*

11. Write a problem involving students that has an answer of ${}_6C_3$.

Answers will vary. One example follows:

There are six seniors on the principal's advisory committee on school improvement. The principal would like to select three of the students to attend a workshop on school improvement. How many ways can the principal select three students out of the six students on the advisory committee?

12. Suppose that a combination lock is opened by entering a three-digit code. Each digit can be any integer between 0 and 9, but digits may not be repeated in the code. How many different codes are possible? Is this question answered by considering permutations or combinations? Explain.

There are 720 possible codes. This question is answered by considering permutations because in a code, the order of the digits is important.

13. Six musicians will play in a recital. Three will perform before intermission, and three will perform after intermission. How many different ways are there to choose which three musicians will play before intermission? Is this question answered by considering permutations or combinations? Explain.

There are 20 possible ways to choose the musicians. This question is answered by considering combinations because order is not important if we just care about which group of three is before the intermission.

14. In a game show, contestants must guess the price of a product. A contestant is given nine cards with the numbers 1 to 9 written on them (each card has a different number). The contestant must then choose three cards and arrange them to produce a price in dollars. How many different prices can be formed using these cards? Is this question answered by considering permutations or combinations? Explain.

There are 504 possible prices. The question is answered by considering permutations because the order of the digits is important. \$123 is different than \$312.

15. a. Using the formula for combinations, show that the number of ways of selecting 2 items from a group of 3 items is the same as the number of ways to select 1 item from a group of 3.

$${}_3C_2 = \frac{3!}{2! \cdot 1!} = \frac{3!}{1! \cdot 2!}$$

- b. Show that ${}_nC_k$ and ${}_nC_{n-k}$ are equal. Explain why this makes sense.

$${}_nC_k = \frac{n!}{k! \cdot (n-k)!} \quad {}nC_{n-k} = \frac{n!}{(n-k)! \cdot k!}$$

The denominators are the same. This is because the number of ways to choose k from a set of n is the same as the number of ways to select which $(n-k)$ to exclude.