Lesson 2: Counting Rules—The Fundamental Counting Principle and Permutations

Student Outcomes

- Students use the fundamental counting principle to determine the number of different possible outcomes for a chance experiment consisting of a sequence of steps.
- Students calculate the number of different permutations of a set of n distinct items.
- Students calculate the number of different permutations of k items from a set of n distinct items.

Lesson Notes

A formal definition of permutation as an ordered arrangement is presented. Several contexts are developed in which students apply the multiplication rule to calculate the number of different permutations of a set of n distinct items. The next lesson introduces combinations, which are unordered collections. In the examples and exercises of this lesson, point out that different orderings of the same set of items are considered different outcomes.

Classwork

Example 1 (3 minutes): Fundamental Counting Principle

Similar to examples in the previous lesson, this example and the first exercise ask students to make a list. Suggest an organized list or a tree diagram.

As you introduce this first example, you may wish to illustrate an example or two of a possible three-course dinner. For example: salad, burger, and cheesecake would make up one of the 24 possible fixed dinner choices.

Scaffolding:

- If students struggle with the presented scenario, consider using a parallel example closer to student experience. For example, find the number of outfits that can be generated from three pair of pants, four shirts, and two pairs of shoes.
- Then break the task into parts by posing the following questions:
  - How many different possibilities are there for the pants?
  - How many different possibilities of pants and shirts are there?
  - How many different possibilities of pants, shirts, and shoes are there?

Example 1: Fundamental Counting Principle

A restaurant offers a fixed-price dinner menu for $\$30$. The dinner consists of three courses, and the diner chooses one item for each course.

The menu is shown below:

First Course	Second Course	Third Course
Salad	Burger	Cheesecake
Tomato Soup	Grilled Shrimp	Ice Cream Sundae
French Onion Soup	Mushroom Risotto	
	Ravioli	

Exercises 1–4 (12 minutes)

MP.3

Students should work in small groups (2 or 3 students per group). Allow about 5 minutes for them to complete Exercise 1; then, discuss as a class the strategies that groups used to list all the possibilities. Ask students to explain how they know that they have listed all the possibilities and compare/critique the methods that are shared. Then pose the following question to the class. Allow about 5 minutes for discussion and possible presentation of additional examples as needed.

- In general, how can you find the number of possibilities in situations like this?
  - *I found that the number of possibilities is the same as the product of the choices for each course (or clothing option). So in general, I can multiply the number of choices that each option can occur.*

Convey to students that this generalization is known as the *fundamental counting principle*. Instead of making a list or a tree diagram, another method for finding the total number of possibilities is to use the fundamental counting principle. Suppose that a process involves a sequence of *steps* or *events*. Let n_1 be the number of ways the first step or event can occur and n_2 be the number of ways the second step or event can occur. Continuing in this way, let n_k be the number of ways the k^{th} stage or event can occur. Then the total number of different ways the process can occur is: $n_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_k$.

In the fixed-price dinner example, there are 3 choices for the first course (step 1), 4 choices for the second course (step 2), and 2 choices for the third course (step 3). Using the fundamental counting principle, there are $3 \cdot 4 \cdot 2$, or 24 total choices. For the fixed-dinner example, the students could draw three boxes and label the boxes as shown. Then, identify the number of choices for each box.

First Course (3 choices)	Second Course (4 choices)	Third Course (2 choices)
$\square$	$\square$	$\square$

Scaffolding:

- If students struggle to articulate the principle, consider presenting additional examples and then asking them again to generalize the rule:
- You are ordering lunch. The only choice for an entrée is a hamburger. You need to choose chips or fries. The drink options are bottled water, lemonade, or apple juice. Find the number of possibilities for your lunch.
- You are planning activities for the weekend. On Saturday you can either go to the movies or to the mall. On Sunday you can choose to participate in one of the following sports events: basketball, soccer, flag football, or lacrosse. Find the number of possibilities for weekend activities.

Students can continue working in small groups to complete Exercises 2–4. Confirm answers as a class.

Exercises 1–4

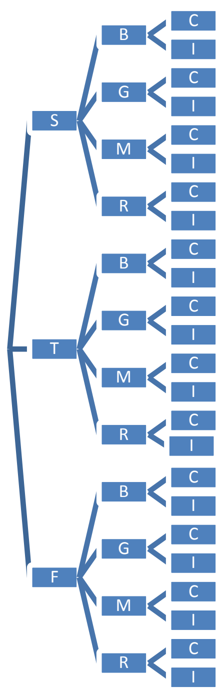
1. **Make a list of all of the different dinner fixed-price meals that are possible. How many different meals are possible?**

Note: The list below shows all the possibilities using abbreviations of the items.

They are listed in order, first course choice, second course choice, and third course choice. For example, SBC would represent salad, burger, and cheesecake.

There are 24 different dinner fixed-price meals possible.

SBC	*TBC*	*FBC*
SBI	*TBI*	*FBI*
SGC	*TGC*	*FGC*
SGI	*TGI*	*FGI*
SMC	*TMC*	*FMC*
SMI	*TMI*	*FMI*
SRC	*TRC*	*FRC*
SRI	*TRI*	*FRI*



2. **For many computer tablets, the owner can set a 4-digit pass code to lock the device.**

   a. **How many digits could you choose from for the first number of the pass code?**

   10

b. How many digits could you choose from for the second number of the pass code? Assume that the numbers can be repeated.

10

c. How many different 4-digit pass codes are possible? Explain how you got your answer.

$10 \cdot 10 \cdot 10 \cdot 10 = 10,000$

There are 10 choices for the first digit, 10 choices for the second digit, 10 choices for the third digit, and 10 choices for the fourth digit. I used the fundamental counting principle and multiplied the number of choices for each digit together to get the number of possible pass codes.

d. How long (in hours) would it take someone to try every possible code if it takes three seconds to enter each possible code?

It would take $30,000$ seconds, which is $8\frac{1}{3}$ hr.

3. The store at your school wants to stock sweatshirts that come in four sizes (small, medium, large, xlarge) and in two colors (red and white). How many different types of sweatshirts will the store have to stock?

$4 \cdot 2 = 8$

4. The call letters for all radio stations in the United States start with either a W (east of the Mississippi river) or a K (west of the Mississippi River) followed by three other letters that can be repeated. How many different call letters are possible?

$2 \cdot 26 \cdot 26 \cdot 26 = 35,152$

Scaffolding:

- The word *stock* has multiple meanings and may confuse English language learners.
- Point out that in this context, *stock* refers to a supply of goods or items kept on hand for sale to customers by a merchant.
- Consider displaying an image of clothing on store shelves to reinforce the meaning of the word.

Example 2 (3 minutes): Permutations

This example introduces the definition of *permutations*. Combinations will be introduced in the next lesson, so the focus of this example is that the order matters for a permutation. The permutation formula is not introduced until Example 3. Give students a moment to read through the example, and ask them to share ideas among their group for how to determine the number of pass codes.

If students struggle, break the task into parts by posing the following questions:

- How many digits can you choose from for the first digit? 10
- How many digits can you choose from for the second digit? (Remember–no repeats) 9
- How many digits can you choose from for the third digit? 8
- How many digits can you choose from for the fourth digit? 7

Encourage students to write out the number choices in the diagram, show the multiplication, and make the connection to the fundamental counting principle.

- How many different 4-digit pass codes are possible if digits cannot be repeated?
  - $10 \cdot 9 \cdot 8 \cdot 7 = 5,040$

- Explain how the fundamental counting principle allows you to make this calculation.
  - *The process of choosing the four digits for the pass code involves a sequence of events. There are* 10 *choices for the first digit,* 9 *choices for the second digit, and so on. So, I can multiply the number of choices that each digit in the pass code can occur.*

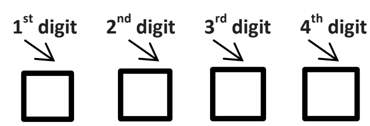
Now introduce the notation for permutations. Explain that finding the number of ordered arrangements of the digits in the pass code is an example of the number of permutations of 10 things taken 4 at a time. This can be written: ${}_{10}P_4 = 10 \cdot 9 \cdot 8 \cdot 7 = 5{,}040$.

Example 2: Permutations

Suppose that the 4-digit pass code a computer tablet owner uses to lock the device *cannot* have any digits that repeat. For example, 1234 is a valid pass code. However, 1123 is not a valid pass code since the digit "1" is repeated.

An arrangement of four digits with no repeats is an example of a permutation. A permutation is an arrangement in a certain order (a sequence).

How many different 4-digit pass codes are possible if digits cannot be repeated?



Exercises 5–9 (8 minutes)

Students should work in small groups (2 or 3 students per group). Allow about 8 minutes for them to complete Exercises 5–9. Encourage students to write the permutation notation and to write out the multiplication. At this point, students should use a calculator to perform the multiplication rather than using the permutation option on the calculator.

Point out to students that when they simplify this formula, the result is the same as using the fundamental counting principle when numbers cannot be repeated. For example: If a password requires three distinct letters, how many different passwords are possible? Students could solve as $26 \cdot 25 \cdot 24$.

When students have finished answering the questions, discuss the answers.

Exercises 5–9

5. **Suppose a password requires three distinct letters. Find the number of permutations for the three letters in the code, if the letters may not be repeated.**

   ${}_{26}P_3 = 26 \cdot 25 \cdot 24 = 15{,}600$

6. **The high school track has 8 lanes. In the 100 meter dash, there is a runner in each lane. Find the number of ways that 3 out of the 8 runners can finish first, second, and third.**

   ${}_{8}P_3 = 8 \cdot 7 \cdot 6 = 336$

7. **There are 12 singers auditioning for the school musical. In how many ways can the director choose first a lead singer and then a stand-in for the lead singer?**

$_{12}P_2 = 12 \cdot 11 = 132$

8. **A home security system has a pad with 9 digits (1 to 9). Find the number of possible 5-digit pass codes:**

a. **if digits can be repeated.**

$9 \cdot 9 \cdot 9 \cdot 9 \cdot 9 = 59{,}049$

b. **if digits cannot be repeated.**

$_9P_5 = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 15{,}120$

MP.8

9. **Based on the patterns observed in Exercises 5–8, describe a general formula that can be used to find the number of permutations of n things taken r at a time, or $n\ Pr$**

Based on the answers to the exercises, a permutation of n things take r at a time can be found using the formula:

$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1).$$

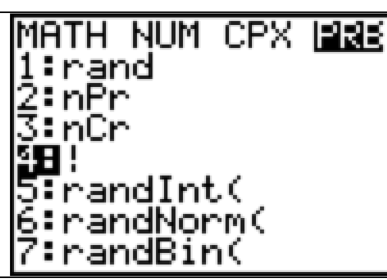
Before moving to the next example, use the following as an opportunity to informally assess student understanding.

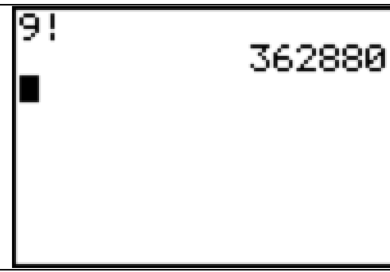
- Explain to your neighbor how to find the number of permutations of n things taken r at a time.

Example 3 (5 minutes): Factorials and Permutations

Factorial notation is introduced in this example. It is also important to tell students that 0! is defined to be equal to 1 (see scaffolding note). After defining factorial notation, show students how to find factorial on their calculator. Use 9! as an example.

Note there are a variety of calculators and software that can be used to find factorials. The steps displayed here refer to the TI-84 graphing calculator and are meant to serve only as a quick reference for teachers:

1. Enter the integer (9) on the Home screen.	
2. Select the MATH menu.	
3. Scroll to PRB.	
4. Select option 4: !	

5. Press ENTER	

Now discuss the permutation formula and show students all the steps prior to using the permutation option on the calculator. For example, ${}_9P_4 = \frac{9!}{(9-4)!} = \frac{9!}{5!} = \frac{9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{5\cdot4\cdot3\cdot2\cdot1} = 9\cdot8\cdot7\cdot6$. Showing these steps reinforces the connection to the fundamental counting principle.

Example 3: Factorials and Permutations

You have purchased a new album with 12 music tracks and loaded it onto your MP3 player. You set the MP3 player to play the 12 tracks in a random order (no repeats). How many different orders could the songs be played in?

This is the permutation of 12 things taken 12 at a time, or
$\mathbf{12\ P12 = 12\cdot\ 11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1 =\ 479,001,600}$.

The notation $12!$ is read "12 factorial" and equals $\mathbf{12\cdot11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}$.

<u>Factorials and Permutations</u>

The factorial of a non-negative integer n is

$$\boldsymbol{n! = n\cdot(n-1)\cdot(n-2)\cdot(n-3)\cdot\ldots\cdot1}.$$

Note: $0!$ is defined to equal 1.

The number of permutations can also be found using factorials. The number of permutations of n things taken r at a time is

$$\mathbf{{}_nP_r} = \frac{\boldsymbol{n!}}{\boldsymbol{(n-r)!}}.$$

Scaffolding:

- Give students an opportunity to explain why $0!$ is equal to 1.
- Consider using the following to demonstrate why $0!$ is equal to 1:
  $5! = 5\cdot(4\cdot3\cdot2\cdot1)$ or
  $5! = 5\cdot4!$
  By rearranging the equation, we get $4! = \frac{5!}{5}$,
  $4! = 4\cdot(3\cdot2\cdot1)$, or
  $4! = 4\cdot3!$
  By rearranging the equation, we get $3! = \frac{4!}{4}$.
  Therefore, $0! = \frac{1!}{1} = 1$.

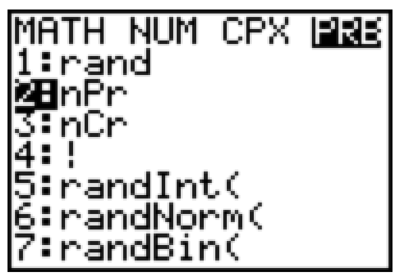
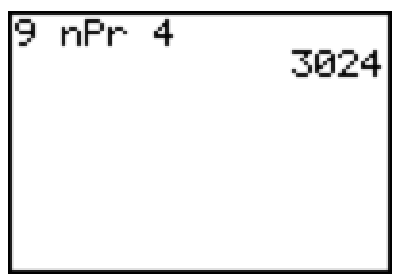
Now have students compare the general formula they described in Exercise 9 with the permutation formula that was just introduced.

- Rewrite the permutation formula by expanding the factorial notation.
  - *Permutation:*

$$_nP_r = \frac{n!}{(n-r)!}$$
$$= \frac{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1) \cdot (n-r) \cdot (n-r-1) \cdot \ldots \cdot 1}{(n-r) \cdot (n-r-1) \cdot \ldots \cdot 1}$$
$$= n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1)$$

- How does this formula compare with the general formula you described in Exercise 9?
  - *The formulas are the same. My formula from Exercise 9 was*
  $$_nP_r = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1).$$
  The permutation formula is just another version of the fundamental counting principle.

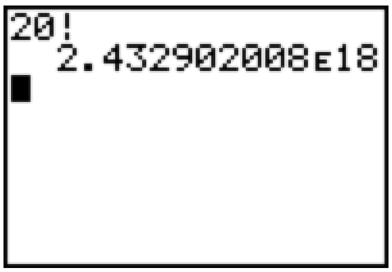
Show students how to find permutations using technology. The following are steps to find 9 P4 using the permutation option on the TI-84 graphing calculator:

1. Enter the integer (9) on the Home screen.	
2. Select the MATH menu.	
3. Scroll to PRB.	
4. Select option 2: nPr.	
5. Enter 4 and press ENTER.	

Exercises 10–15 (7 minutes)

MP.2

Students should work in small groups (2 or 3 students per group). Allow about 7 minutes for them to complete Exercises 10–15. Exercise 13 requires students to reason abstractly and quantitatively as they determine how repetition affects the number of outcomes. Encourage students to write the permutation formula and show the substitution. Some of the answers require the use of scientific notation. You many need to remind students of the notation and how the calculator displays the results. One common student error is that they do not see the E displayed for scientific notation. The screen shot below shows the results for 20!. Some students may record the answer as 2.43 instead of $2.43\ x\ 1018$.



When students have finished answering the questions, discuss the answers.

Exercises 10–15

10. If $9!$ is $362,880$, find $10!$.

$3,628,800$

11. How many different ways can the 16 numbered pool balls be placed in a line on the pool table?

$16!\ or\ {}_{16}P_{16} = 2 \cdot 10^{13}$

12. Ms. Smith keeps eight different cookbooks on a shelf in one of her kitchen cabinets. How many ways can the eight cookbooks be arranged on the shelf?

$8!$ *or* ${}_{8}P_{8} = 40,320$

13. How many distinct 4-letter groupings can be made with the letters from the word *champion* if letters may not be repeated?

$${}_{8}P_{4} = \frac{8!}{(8-4)!} = 1,680$$

14. There are 12 different rides at an amusement park. You buy five tickets that allow you to ride on five different rides. In how many different orders can you ride the five rides? How would your answer change if you could repeat a ride?

Different order: ${}_{12}P_{3} = \frac{12!}{(12-5)!} = 95,040$

Repeat rides: $12^5 = 248,832$

15. In the summer Olympics, 12 divers advance to the finals of the 3-meter springboard diving event. How many different ways can the divers finish 1st, 2nd, or 3rd?

$${}_{12}P_3 = \frac{12!}{(12-3)!} = 1,32$$

Closing (2 minutes)

- How do permutations relate to the fundamental counting principle?
  - *Sample response: Permutations are just another version of the fundamental counting principle.*
- Ask students to summarize the key ideas of the lesson in writing or by talking to a neighbor. Use this as an opportunity to informally assess student understanding. The lesson summary provides some of the key ideas from this lesson.

Lesson Summary

- **Let n_1 be the number of ways the first step or event can occur and n_2 be the number of ways the second step or event can occur. Continuing in this way, let n_k be the number of ways the k^{th} stage or event can occur. Then based on fundamental counting principle, the total number of different ways the process can occur is: $n_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_k$.**
- **The factorial of a non-negative integer n is**
  $$n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \ldots \cdot 1.$$
  Note: $0!$ is defined to equal 1.
- **The number of permutations of n things taken r at a time is**
  $${}_nP_r = \frac{n!}{(n-r)!}.$$

Exit Ticket (5 minutes)

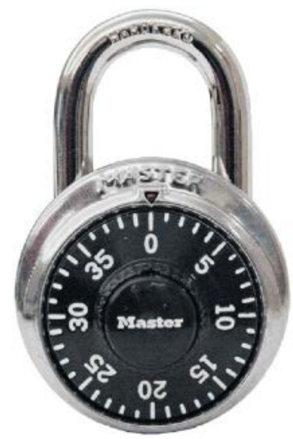
Name ____ Date ____________________

Lesson 2: Counting Rules—The Fundamental Counting Principle and Permutations

Exit Ticket

1. The combination for the lock shown below consists of three numbers.

   a. If the numbers can be repeated, how many different combinations are there? Explain your answer.



   b. If the numbers cannot be repeated, how many different combinations are there? Explain your answer.

2. Jacqui is putting together sets of greeting cards for a school fundraiser. There are four different card options, two different colored envelopes, and four different sticker designs. A greeting card set consists of one type of card, one color for the envelopes, and one sticker design. How many different ways can Jacqui arrange the greeting card sets? Explain how you determined your answer.

Exit Ticket Sample Solutions

1. The combination for the lock shown below consists of three numbers.

   a. If the numbers can be repeated, how many different combinations are there? Explain your answer.

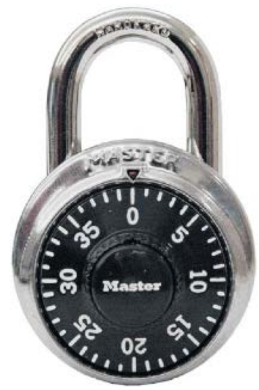
   $40^3 = 64,000$

   Because numbers can be repeated, there are 40 choices for each of the three digits. Therefore, I applied the fundamental counting principle.

   b. If the numbers cannot be repeated, how many different combinations are there? Explain your answer.

   ${}_{40}P_3 = 59,280$

   Since numbers cannot be repeated, this is an example of a permutation.



2. Jacqui is putting together sets of greeting cards for a school fundraiser. There are four different card options, two different colored envelopes, and four different sticker designs. A greeting card set consists of one type of card, one color for the envelopes, and one sticker design. How many different ways can Jacqui arrange the greeting card sets?

   By using the fundamental counting principle, there are $4 \cdot 2 \cdot 4 = 32$ ways to arrange the sets.

Problem Set Sample Solutions

Use this space to describe any specific details about the problem set for teacher reference.

1. For each of the following, show the substitution in the permutation formula and find the answer.

   a. ${}_4P_4$

   $$\frac{4!}{(4-4)!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{0!} = 24$$

   b. ${}_{10}P_2$

   $$\frac{10!}{(10-2)!} = \frac{10!}{8!} = 10 \cdot 9 = 90$$

   c.

   $$\frac{5!}{(5-1)!} = \frac{5!}{4!} = 5$$

2. A serial number for a TV begins with three letters, is followed by six numbers, and ends in one letter. How many different serial numbers are possible? Assume the letters and numbers can be repeated.

   $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 26 = 4.6 \cdot 10^{11}$

3. **In a particular area code, how many phone numbers (###-####) are possible? The first digit cannot be a zero and assume digits can be repeated.**

   $9 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 9,000,000$

4. **There are four NFL teams in the AFC east: Bills, Jets, Dolphins, and Patriots. How many different ways can two of the teams finish first and second?**

   ${}_4P_2 = 12$

5. **How many ways can 3 of 10 students come in first, second, and third place in a spelling contest, if there are no ties?**

   ${}_{10}P_3 = 720$

6. **In how many ways can a president, a treasurer, and a secretary be chosen from among nine candidates if no person can hold more than one position?**

   ${}_9P_3 = 504$

7. **How many different ways can a class of 22 second graders line up to go to lunch?**

   $22! = {}_{22}P_{22} = 1.1 \cdot 10^{21}$

8. **Describe a situation that could be modeled by using ${}_5P_2$.**

   Answers will vary. Suppose that there are five members of a family living at home. The first one home has to take out the garbage, and the second one home has to walk the dog. ${}_5P_2$ can be used to model the number of ways the family members can be assigned the different tasks.

9. **To order books from an online site, the buyer must open an account. The buyer needs a username and a password.**

   a. **If the username needs to be eight letters, how many different usernames are possible:**

      i. **if letters can be repeated?**

         $26^8 = 2 \cdot 10^{11}$

      ii. **if the letters cannot be repeated?**

         ${}_{26}P_8 = 6.3 \cdot 10^{10}$

   b. **If the password must be eight characters, which can be any of the 26 letters, 10 digits, and 12 special keyboard characters, how many passwords are possible:**

      i. **if characters can be repeated?**

         $48^8 = 2.8 \cdot 10^{13}$

      ii. **if characters cannot be repeated?**

         ${}_{48}P_8 = 1.5 \cdot 10^{13}$

c. **How would your answers to part (b) change if the password is case-sensitive? (In other words, *Password* and *password* are considered different because the letter *p* is in uppercase and lowercase.)**

The answers would change because the number of letters that can be used will double to 52, which means the number of characters that can be used is now 74.

So, if characters can be repeated, the answer will be
$72^8 = 7.2 \cdot 10^{14}$.

If characters cannot be repeated, the answer will be ${}_{72}P_8 = 4.8 \cdot 10^{14}$.

10. **Create a scenario to explain why ${}_3P_3 = 3!$.**

Suppose three friends are running in a race. ${}_3P_3$ can be used to model the order in which the three friends finish in 1st, 2nd, and 3rd place. There are three choices for 1st place and two choices for 2nd place, which only leaves one choice for 3rd place. So, there are $3 \cdot 2 \cdot 1 = 3!$, or 6 ways for the friends to finish the race.

11. **Explain why ${}_nP_n = n!$ for all positive integers n.**

Using the permutation formula:

$${}_nP_r = \frac{n!}{(n-r)!}.$$

In this case $r = n$, therefore:

$${}_nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!.$$