Lesson 21: Logarithmic and Exponential Problem Solving

Student Outcomes

- Students understand the inverse relationship between logarithms and exponents and apply their understanding to solve real-world problems.

Lesson Notes

This lesson uses the context of radiocarbon dating to understand and apply the inverse relationship between logarithms and exponents when solving problems. A quick internet search reveals several news stories about recent discoveries of woolly mammoth remains. Radiocarbon dating is one of several methods used by archaeologists and anthropologists to date their findings. Students have studied real-world situations that are modeled by exponential functions since Algebra I. In Algebra II, they were able to develop more precise solutions to modeling and application problems involving exponential functions because they learned to use logarithms to solve equations analytically **(F-BF.B.4, F.LE.A.4)**. Students also learned about situations that could be modeled by logarithmic functions. In this lesson, students use the inverse relationship between logarithms and exponents as a basis for making sense of and solving real-world problems. Throughout the lesson, students create models, compute using models, and interpret the results (MP.4).

Class Work

Opening (5 minutes)

Have students read the Opening to themselves and jot down one question that they have about the reading. Have them share and discuss this question with a partner. If time permits, you can have students research the answers to any questions that you or others in the class cannot answer. The following web-based resources can help you and your students learn more about radiocarbon dating and wooly mammoth discoveries.

http://en.wikipedia.org/wiki/Radiocarbon_dating

http://science.howstuffworks.com/environmental/earth/geology/carbon-14.htm

http://en.wikipedia.org/wiki/Woolly_mammoth

http://ngm.nationalgeographic.com/2009/05/mammoths/mueller-text

After a brief whole-class discussion, move students on to the Exploratory Challenge.

Woolly mammoths, an elephant-like mammal, have been extinct for thousands of years. In the last decade, several well-preserved woolly mammoths have been discovered in the permafrost and icy regions of Siberia. Scientists have determined that some of these mammoths died nearly $40,000$ years ago using a technique called radiocarbon (Carbon-14) dating.

This technique was introduced in 1949 by the American chemist Willard Libby and is one of the most important tools archaeologists use for dating artifacts that are less than $50,000$ years old. Carbon-14 is a radioactive isotope present in all organic matter. Carbon-14 is absorbed in small amounts by all living things. The ratio of the amount of normal carbon (Carbon-12) to the amount Carbon-14 in all living organisms remains nearly constant until the organism dies. Then, the Carbon-14 begins to decay because it is radioactive.

Exploratory Challenge/Exercises 1–14 (20 minutes)

Organize students into small groups, and give them about 10 minutes to work these exercises. The goal is for them to create an exponential function to model the data. Monitor groups to make sure they are completing the table entries correctly. There is quite a bit of number sense and quantitative reasoning required in these exercises.

Students may wish to construct the graphs on graph paper. Pay attention to how they scale the graphs. Graphing calculators or other graphing technology can also be used to construct the scatter plots and graphs of the functions.

Exploratory Challenge/Exercises 1–14

By examining the amount of Carbon-14 that remains in an organism after death, one can determine its age. The half-life of Carbon-14 is $5,730$ years, meaning that the amount of Carbon-14 present is reduced by a factor of $\frac{1}{2}$ every $5,730$ years.

1. **Complete the table.**

Years Since Death	0	$5,730$	$11,460$	$17,190$	$22,920$	$28,650$	$34,380$	$40,110$	$45,840$
C-14 Atoms Remaining Per 1.0×10^8 C-12 Atoms	$10,000$	$5,000$	$2,500$	$1,250$	625	312.5	156.25	78.125	38.0625

Let C be the function that represents the number of C-14 atoms remaining per 1.0×10^8 C-12 atoms t years after death.

MP.3

2. **What is $C(11460)$? What does it mean in this situation?**

   $C(11460) = 2500$. Each time the years since death increase by $5,730$, you have to reduce the C-14 atoms per 1.0×10^8 C-12 atoms by a factor of $\frac{1}{2}$. After the organism has been dead for $11,460$ years, the organism contains $2,500$ C-14 atoms per 1.0×10^8 C-12 atoms.

3. **Estimate the number of C-14 atoms per 1.0×10^8 C-12 atoms you would expect to remain in an organism that died $10,000$ years ago.**

   It will be slightly more than $2,500$ atoms so approximately $3,100$.

Scaffolding:

Let students use a calculator to do the heavy computational lifting. Students could also model the table information using a spreadsheet. The regression features could be used to generate an exponential model for this situation as well.

4. **What is $C^{-1}(625)$? What does it represent in this situation?**

   $C^{-1}(625) = 22,920$. This represents the number of years since death when there are 625 C-14 atoms per 1.0×10^8 C-12 atoms remaining in the sample.

5. **Suppose the ratio of C-14 to C-12 atoms in a recently discovered woolly mammoth was found to be 0.000001. Estimate how long ago this animal died.**

   We need to write this as a ratio with denominator equal to 1.0×10^8.

$$0.000001 = 1.0 \times 10^{-6} = \frac{1.0 \times 10^2}{1.0 \times 10^8}$$

   So if the number of C-12 atoms is 1.0×10^8, then the number of C-14 atoms would be 100. This animal would have died between $34,000$ and $40,000$ years ago.

6. **Explain why the $C^{-1}(100)$ represents the answer to Exercise 5.**

   $C^{-1}(100)$ means the value of x when $C(x) = 100$. We wanted the time when there would be 100 C-14 atoms for every 1.0×10^8 C-12 atoms.

7. **What type of function best models the data in the table you created in Exercise 1? Explain your reasoning.**

Since the data is being multiplied by a constant factor each time the years increase by the same amount, this data would be modeled best by an exponential function.

Exercises 8 and 9 provide an opportunity for you to check for student understanding of essential prerequisite skills related to writing and graphing exponential functions. In both Algebra I and Algebra II, students have modeled exponential functions when given a table of values. The challenge for students will be to see how well they deal with the half-life parameter. Students may choose to use a calculator to do an exponential regression equation as well. Then in Exercise 10, you can assess how well students understand that logarithmic and exponential functions are inverses. They create the graph of the inverse of C by exchanging the t and $C(t)$ coordinates and plotting the points. The result of their work in Lesson 20 is that they should now understand that the inverse of any exponential function of the form $y = a \cdot b^{cx}$ will be a logarithmic function with base b. Use the questions below to provide additional support as groups are working.

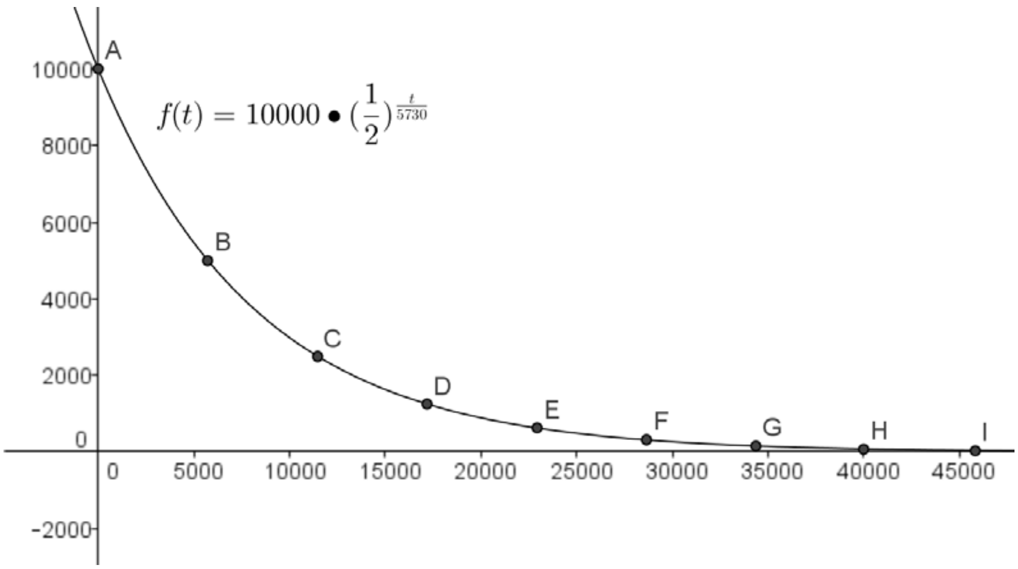
- What type of function makes sense to use to model C?
  - *An exponential function because whenever the time increased by a consistent amount, the value of C was multiplied by $\frac{1}{2}$.*
- How can you justify your choice of function using the graph in Exercise 9?
  - *I can see that every $5{,}730$ years, the amount of C is reduced by a factor of $\frac{1}{2}$. The graph is decreasing but will never reach 0, so it appears to look like a model for exponential decay.*
- What is the base of the function? What is the y-intercept?
  - *The base is $\frac{1}{2}$. The y-intercept would be the amount when $t = 0$.*
- How does the half-life parameter affect how you will write the formula for C in Exercise 8?
  - *Since the years are not increasing by 1 but by $5{,}730$, we need to divide the time variable by $5{,}730$ in our exponential function.*
- How do you know that the graph of the function in Exercise 10 is the inverse of the graph of the function in Exercise 9?
  - *Because we exchanged the domain and range values from the table to create the graph of the inverse.*
- What type of function is the inverse of an exponential function?
  - *In the last lesson, we learned that the inverse of an exponential function is a logarithmic function.*

8. **Write a formula for C in terms of t. Explain the meaning of any parameters in your formula.**

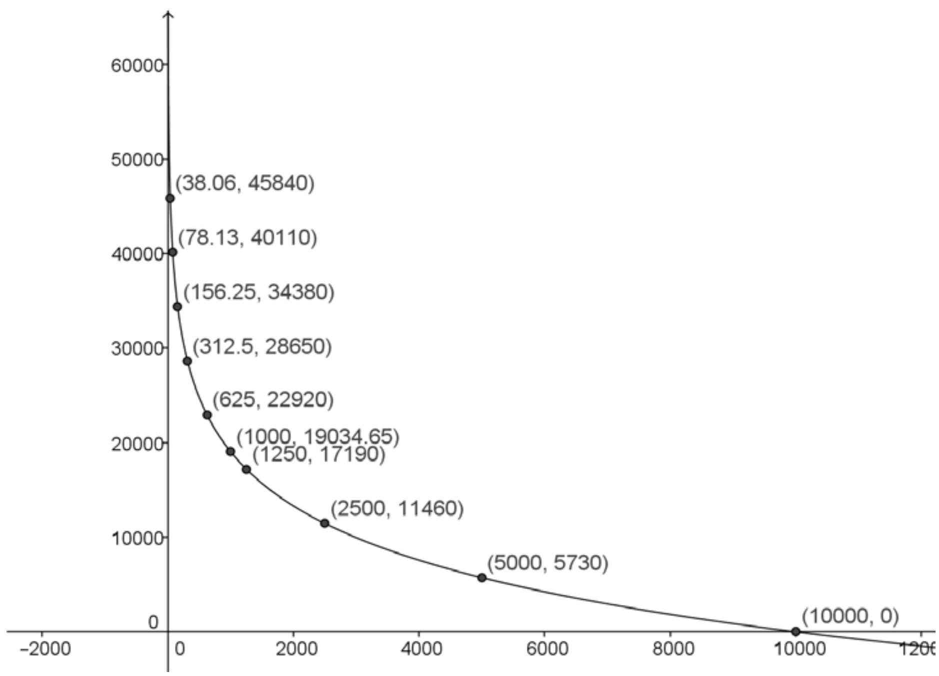
$$C(t) = 10000\left(\frac{1}{2}\right)^{\frac{t}{5730}}$$

The $10{,}000$ represents the number of C-14 atoms per 1.0×10^8 C-12 atoms present at the death of the organism. The $5{,}730$ and the $\frac{1}{2}$ indicate that the amount is halved every $5{,}730$ years.

9. Graph the set of points $(t, C(t))$ from the table and the function C to verify that your formula is correct.



10. Graph the set of points $(C(t), t)$ from the table. Draw a smooth curve connecting those points. What type of function would best model this data? Explain your reasoning.



This data would best be modeled by a logarithmic function since the data points represent points on the graph of the inverse of an exponential function.

MP.4

11. Write a formula that will give the years since death as a function of the amount of C-14 remaining per 1.0×10^8 C-12 atoms.

Take the equation $C(t) = 10000\left(\frac{1}{2}\right)$ and use the variables x and y in place of t and $C(t)$. Find the inverse by exchanging the x and y variables and solving for y.

$$y = 10000\left(\frac{1}{2}\right)^{\frac{x}{5730}}$$

$$x = 10000\left(\frac{1}{2}\right)^{\frac{y}{5730}}$$

$$\frac{x}{10000} = \left(\frac{1}{2}\right)^{\frac{y}{5730}}$$

$$\log_2\left(\frac{x}{10000}\right) = \log_2\left(\frac{1}{2}\right)^{\frac{y}{5730}}$$

$$\log_2\left(\frac{x}{10000}\right) = \log_2\left(2^{-1}\right)^{\frac{y}{5730}}$$

$$\log_2\left(\frac{x}{10000}\right) = \frac{y}{5730}\log_2\left(2^{-1}\right)$$

$$\log_2\left(\frac{x}{10000}\right) = -\frac{y}{5730}$$

$$y = -5730\log_2\left(\frac{x}{10000}\right)$$

$$f(x) = -5730\log_2\left(\frac{x}{10000}\right)$$

In this formula, x is the number of C-14 atoms for every 1.0×10^8 C-12 atoms, and $f(x)$ is the time since death.

12. Use the formulas you have created to accurately calculate the following:

a. The amount of C-14 atoms per 1.0×10^8 C-12 atoms remaining in a sample after $10,000$ years.

$$y = 10000\left(\frac{1}{2}\right)^{\frac{10000}{5730}}$$

$$C(10,000) \approx 2981$$

There will be approximately $2,981$ C-14 atoms per 1.0×10^8 C-12 atoms in a sample that died $10,000$ years ago.

b. The years since death of a sample that contains 100 C-14 atoms per 1.0×10^8 C-12 atoms.

$$f(x) = -5730\log_2\left(\frac{100}{10000}\right)$$

$$f(100) \approx 38069$$

An organism containing 100 C-14 atoms per 1.0×10^8 C-12 atoms died $38,069$ years ago.

c. $C(25,000)$

$$y = 10000\left(\frac{1}{2}\right)^{\frac{25000}{5730}}$$

$$C(25000) \approx 486$$

After an organism has been dead for $25,000$ years, there will be approximately 486 C-14 atoms per 1.0×10^8 C-12 atoms.

d. $C^{-1}(1000)$

$$f(x) = -5730 \log_2\left(\frac{1000}{10000}\right)$$

To find this amount, evaluate $f(1000) \approx 19035$.

When there are $1,000$ C-14 atoms per 1.0×10^8 C-12 atoms, the organism will have been dead for $19,035$ years.

13. A baby woolly mammoth that was discovered in 2007 died approximately $39,000$ years ago. How many C-14 atoms per 1.0×10^8 C-12 atoms would have been present in the tissues of this animal when it was discovered?

Evaluate $C(39000) \approx 89$ atoms of C-14 per 10×10^8atoms of C-12.

14. A recently discovered woolly mammoth sample was found to have a red liquid believed to be blood inside when it was cut out of the ice. Suppose the amount of C-14 in a sample of the creature's blood contained $3,000$ atoms of C-14 per 1.0×10^8 atoms of C-12. How old was this woolly mammoth?

Evaluate $C^{-1}(3000) \approx 9953$. The woolly mammoth died approximately $10,000$ years ago.

Have students present different portions of this Exploratory Challenge, and then lead a short discussion to make sure all students have been able to understand the inverse relationship between logarithmic and exponential functions.

Discuss different approaches to finding the logarithmic function that represented the inverse of C. One of the challenges for students may be dealing with the notation. When applying inverse relationships in real-world situations, you must be sure to explain the meaning of the variables.

Exercises 15-18 (10 minutes)

In Exercises 15-18, students generalize the work they did in earlier exercises. Because the half-life of C-14 is 5,730 years, the carbon-dating technique only produces valid results for samples up to 50,000 years old. For older samples, scientists can use other radioactive isotopes to date the rock surrounding a fossil and infer the fossil's age from the age of the rock.

Exercises 15–18

Scientists can infer the age of fossils that are older than $50,000$ years by using similar dating techniques with other radioactive isotopes. Scientists use radioactive isotopes with half-lives even longer than Carbon-14 to date the surrounding rock in which the fossil is embedded.

A general formula for the amount A of a radioactive isotope that remains after t years is $A = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$

where A_0 is the amount of radioactive substance present initially, and h is the half-life of the radioactive substance.

15. **Solve this equation for t to find a formula that will infer the age of a fossil by dating the age of the surrounding rocks.**

$$A = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$$
$$\frac{A}{A_0} = \left(\frac{1}{2}\right)^{\frac{t}{h}}$$
$$\frac{A}{A_0} = 2^{-\frac{t}{h}}$$
$$\log_2\left(\frac{A}{A_0}\right) = -\frac{t}{h}$$
$$t = -h\log_2\left(\frac{A}{A_0}\right)$$

16. **Let $(x) = A_0\left(\frac{1}{2}\right)^{\frac{x}{h}}$. What is $A^{-1}(x)$?**

$$A^{-1}(x) = -h\,log_2\left(\frac{x}{A_0}\right)$$

17. **Verify that A and A^{-1} are inverses by showing that $A\left(A^{-1}(x)\right) = x$ and $A^{-1}\left(A(x)\right) = x$.**

$$\begin{aligned} A\left(A^{-1}(x)\right) &= A_0\left(\frac{1}{2}\right)^{\frac{-h\log_2\left(\frac{x}{A_0}\right)}{h}} \\ &= A_0\left(2^{-1}\right)^{-\log_2\left(\frac{x}{A_0}\right)} \\ &= A_0(2)^{\log_2\left(\frac{x}{A_0}\right)} \\ &= A_0\left(\frac{x}{A_0}\right) \\ &= x \end{aligned}$$

And

$$\begin{aligned} A^{-1}\left(A(x)\right) &= -h\log_2\left(\frac{A_0\left(\frac{1}{2}\right)^{\frac{x}{h}}}{A_0}\right) \\ &= -h\log_2\left(\frac{1}{2}\right)^{\frac{x}{h}} \\ &= -h\cdot\frac{x}{h}\cdot\log_2\left(\frac{1}{2}\right) \\ &= -x\cdot -1 \\ &= x \end{aligned}$$

Scaffolding:

- For students that need a more concrete approach, select particular values for A_0 and h and have students solve the equation for t.
- Do this a few times and then have students try to generalize the solution steps.
- You could also use $A = A_0(2)^{-\frac{t}{h}}$ instead of $\frac{1}{2}$ so that students quickly see the connection to base 2 logarithms.

18. **Explain why, when determining the age of organic materials, archaeologists and anthropologists would prefer to use the logarithmic function to relate the amount of a radioactive isotope present in a sample and the time since its death?**

    Defining the years since death as a function of the amount of radioactive isotope makes more sense since archaeologists and anthropologists are trying to determine the number of years since death of an organism from a sample. They will know the amount of the radioactive isotope and can use that as an input into the formula to generate the number of years since its death.

Closing (5 minutes)

Ask students to respond to the questions below with a partner or individually in writing.

- Which properties of logarithms and exponents are most helpful when verifying that these types of functions are inverses?
    - *The definition of logarithm, which states that if $x = b^y$, then $y = \log_b(x)$, and the identities, $\log_b(b^x) = x$ and $b^{\log_b(x)} = x$.*
- What is the inverse function of $f(x) = 3^x$? What is the inverse of $g(x) = \log(x)$?
    - $f^{-1}(x) = \log_3(x)$, *and* $g^{-1}(x) = 10^x$.
- Suppose a function f has a domain that represents time in years and a range that represents the number of bacteria. What would the domain and range of f^{-1} represent?
    - *The domain would be the number of bacteria, and the range would be the time in years.*

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 21: Logarithmic and Exponential Problem Solving

Exit Ticket

Darrin drank a latte with 205 mg of caffeine. Each hour, the caffeine in Darrin's body diminishes by about 8%.

a. Write a formula to model the amount of caffeine remaining in Darrin's system after each hour.

b. Write a formula to model the number of hours since Darrin drank his latte based on the amount of caffeine in Darrin's system.

c. Use your equation in part (b) to find how long it will take for the caffeine in Darrin's system to drop below 50 mg.

Exit Ticket Sample Solutions

Darrin drank a latte with 205 mg of caffeine. Each hour, the caffeine in Darrin's body diminishes by about 8%.

a. **Write a formula to model the amount of caffeine remaining in Darrin's system after each hour.**

$$c(t) = 205 \cdot (1 - 8\%)^t$$
$$c(t) = 205 \cdot (0.92)^t$$

b. **Write a formula to model the number of hours since Darrin drank his latte based on the amount of caffeine in Darrin's system.**

$$c = 205(0.92)^t$$
$$\frac{c}{205} = 0.92^t$$
$$\ln\left(\frac{c}{205}\right) = \ln(0.92)^t$$
$$\ln\left(\frac{c}{205}\right) = t \cdot \ln(0.92)$$
$$t = \frac{\ln\left(\frac{c}{205}\right)}{\ln(0.92)}$$

Thus,

$$t(c) = \frac{\ln\left(\frac{c}{205}\right)}{\ln(0.92)}.$$

Alternatively,

$$c = 205(0.92)^t$$
$$\frac{c}{205} = 0.92^t$$
$$\log_{0.92}\left(\frac{c}{205}\right) = t.$$

And by the change of base property,

$$t(c) = \frac{ln\left(\frac{c}{205}\right)}{ln(0.92)}.$$

c. **Use your equation in part (b) to find how long it will take for the caffeine in Darrin's system to drop below 50 mg.**

$$t = \frac{\ln\left(\frac{50}{205}\right)}{\ln(0.92)}$$

It will take approximately 16.922 hours for the caffeine to drop to 50 mg; therefore, it would take approximately 17 hours for the caffeine to drop below 50 mg.

Problem Set Sample Solutions

1. **A particular bank offers 6% interest per year compounded monthly. Timothy wishes to deposit $\$1,000$.**

   a. **What is the interest rate per month?**

$$\frac{0.06}{12} = 0.005$$

b. Write a formula for the amount A Timothy will have after n months.

$$A = 1000(1.005)^n$$

c. Write a formula for the number of months it will take Timothy to have A dollars.

$$n = \frac{\ln\left(\frac{A}{1000}\right)}{\ln(1.005)}$$

d. Doubling-Time is the amount of time it takes for an investment to double. What is the doubling-time of Timothy's investment?

$$n = \frac{\ln(2)}{\ln(1.005)}$$
$$\approx 138.98$$

It will take 139 months for Timothy's investment to double.

e. In general, what is the doubling-time of an investment with an interest rate of $\frac{r}{12}$ per month?

$$n = \frac{\ln(2)}{\ln\left(1 + \frac{r}{12}\right)}$$

2. A study done from 1950 through 2000 estimated that the world population increased on average by 1.77% each year. In 1950, the world population was 2519 million.

a. Write a formula for the world population t years after 1950. Use p to represent world population.

$$p = 2519(1.0177)^t$$

b. Write a formula for the number of years it will take to reach a population of p.

$$t = \frac{\ln\left(\frac{p}{2519}\right)}{\ln(1.0177)}$$

c. Use your equation in part (b) to find when the model predicts that the world population will be 10 billion.

$$t = \frac{\ln\left(\frac{10000}{2519}\right)}{\ln(1.0177)}$$
$$\approx 78.581$$

According to the model, it will take about $78\frac{1}{2}$ years from 1950 for the world population to reach 10 billion; this would be in 2028.

3. Consider the case of a bank offering r (given as a decimal) interest per year compounded monthly, if you deposit $\$P$.

a. What is the interest rate per month?

$$\frac{r}{12}$$

b. Write a formula for the amount A you will have after n months.

$$A = P\left(1 + \frac{r}{12}\right)^n$$

c. Write a formula for the number of months it will take to have A dollars.

$$\ln\left(\frac{A}{P}\right) = n \cdot \ln\left(1 + \frac{r}{12}\right)$$
$$n = \frac{\ln\left(\frac{A}{P}\right)}{\ln\left(1 + \frac{r}{12}\right)}$$

d. What is the doubling-time of an investment earning 7% interest per year, compounded monthly? Round up to the next month.

$$2 = \left(1 + \frac{0.07}{12}\right)^n$$
$$\ln(2) = n \cdot \ln\left(1 + \frac{0.07}{12}\right)$$
$$n = \frac{\ln(2)}{\ln\left(1 + \frac{0.07}{12}\right)} \approx 119.17$$

It would take 120 months or 10 years in order to double an investment earning 7% interest per year, compounded monthly.

4. A half-life is the amount of time it takes for a radioactive substance to decay by half. In general, we can use the equation $A = P\left(\frac{1}{2}\right)^t$ for the amount of the substance remaining after t half-lives.

a. What does P represent in this context?

The initial amount of the radioactive substance.

b. If a half-life is 20 hours, rewrite the equation to give the amount after h hours.

$$t = \frac{h}{20}$$
$$A = P\left(\frac{1}{2}\right)^{\frac{h}{20}}$$

c. Use the natural logarithm to express the original equation as having base e.

$$A = Pe^{\ln\left(\frac{1}{2}\right)t}$$

d. The formula you wrote in part (c) is frequently referred to as the "Pert" formula, that is, Pe^{rt}. Analyze the value you have in place for r in part (c). What do you notice? In general, what do you think r represents?

$$r = \ln\left(\frac{1}{2}\right)$$

It seems like the r value will always represent the natural logarithm of the growth rate per t. If t is a number of half-lives, then r is the natural logarithm of $\frac{1}{2}$.

e. **Jess claims that any exponential function can be written with base e; is she correct? Explain why.**

She is correct. No matter what the original rate is, say b, we can always rewrite the rate as $e^{\ln(b)}$, so e is always a possible base for an exponential function. Similarly, we could always rewrite logarithms in terms of the natural logarithm.

5. **If caffeine reduces by about 10% per hour, how many hours h does it take for the amount of caffeine in a body to reduce by half (round up to the next hour)?**

$$\frac{1}{2} = 1 \cdot (0.9)^h$$
$$h = \frac{\ln\left(\frac{1}{2}\right)}{\ln(0.9)}$$
$$\approx 6.5788$$

It will take about 7 hours for the caffeine to reduce by half.

6. **Iodine-123 has a half-life of about 13 hours, emits gamma-radiation, and is readily absorbed by the thyroid. Because of these facts, it is regularly used in nuclear imaging.**

a. **Write a formula that gives you the percent p of iodine-123 left after t half-lives.**

$$A = P\left(\frac{1}{2}\right)^t$$
$$\frac{A}{P} = \left(\frac{1}{2}\right)^t$$
$$p = \left(\frac{1}{2}\right)^t$$

b. **What is the decay rate per hour of iodine-123? Approximate to the nearest millionth.**

$$t = \frac{h}{13}$$
$$p = \left(\frac{1}{2}\right)^{\frac{h}{13}}$$
$$p = \left(\left(\frac{1}{2}\right)^{1/13}\right)^h$$
$$p \approx (0.948078)^h$$

Iodine-123 decays by about 0.051922 per hour, or 5.1922%.

c. **Use your result to part (b). How many hours h would it take for you to have less than 1% of an initial dose of iodine-123 in your system? Round your answer to the nearest tenth of an hour.**

$$0.01 = (0.948078)^h$$
$$h = \frac{\ln(0.01)}{\ln(0.948078)}$$
$$\approx 86.4$$

It would take approximately 86.4 hours for you to have less than 1% of an initial does of iodine-123 in your system.

7. An object heated to a temperature of $50°\ C$ is placed in a room with a constant temperature of $10°\ C$ to cool down. The object's temperature T after t minutes can be given by the function $T(t) = 10 + 40e^{-0.023105t}$.

   a. How long will it take for the object to cool down to $30°\ C$?

$$30 = 10 + 40e^{-0.023105t}$$
$$\frac{1}{2} = e^{-0.023105t}$$
$$t = \frac{\ln\left(\frac{1}{2}\right)}{-0.023105} \approx 29.9999$$

   About 30 minutes.

   b. Will it take longer for the object to cool from $50°\ C$ to $30°\ C$ or from $30°\ C$ to $10.1°\ C$?

   Since it is an exponential decay function, it will take longer for the object to cool from $30°\ C$ to $10.1°\ C$ than it will take for the object to cool from $50°\ C$ to $30°\ C$. The function levels off as it approaches $10°\ C$, so it takes progressively longer. It would take an additional 70 minutes to cool down to $10.1°\ C$ after getting to $30°\ C$.

   c. Will the object ever be $10°\ C$ if kept in this room?

   It will effectively be $10°\ C$ eventually, but mathematically it will never get there. After 400 minutes, the temperature will be about $10.0001°\ C$.

   d. What is the domain of T^{-1}? What does this represent?

   The domain of the inverse will represent the possible temperatures that the object could be, so $(10, 50]$.

8. The percent of usage of the word "judgment" in books can be modeled with an exponential decay curve. Let P be the percent as a function of x, and let x be the number of years after 1900, then $P(x) = 0.0220465 \cdot e^{-0.0079941x}$.

   a. According to the model, in what year was the usage 0.1% of books?

$$P^{-1}(x) = \frac{\ln\left(\frac{x}{0.0220465}\right)}{-0.0079951}$$

   According to the inverse of the model, we get a value of -189, which corresponds to the year 1711.

   b. When will the usage of the word "judgment" drop below 0.001% of books? This model was made with data from 1950 to 2005. Do you believe your answer will be accurate? Explain.

   We get a value of 387, which corresponds to the year $2,287$. It is unlikely that the model would hold up well in either part (a) or part (b) because these years are so far in the past and future.

   c. Find P^{-1}. What does the domain represent? What does the range represent?

$$x = 0.0220465 \cdot e^{-0.0079941y}$$
$$-0.0079941y = \ln\left(\frac{x}{0.0220465}\right)$$
$$y = \frac{\ln\left(\frac{x}{0.0220465}\right)}{-0.0079941}$$

   The domain is the percent of books containing the word "judgment" while the range is the number of years after $1,900$.