Lesson 21: Logarithmic and Exponential Problem Solving

Classwork

Woolly mammoths, an elephant-like mammal, have been extinct for thousands of years. In the last decade, several well-preserved woolly mammoths have been discovered in the permafrost and icy regions of Siberia. Scientists have determined that some of these mammoths died nearly $40{,}000$ years ago using a technique called radiocarbon (Carbon-14) dating.

This technique was introduced in 1949 by the American chemist Willard Libby and is one of the most important tools archaeologists use for dating artifacts that are less than $50{,}000$ years old. Carbon-14 is a radioactive isotope present in all organic matter. Carbon-14 is absorbed in small amounts by all living things. The ratio of the amount of normal carbon (Carbon-12) to the amount Carbon-14 in all living organisms remains nearly constant until the organism dies. Then, the Carbon-14 begins to decay because it is radioactive.

Exploratory Challenge/Exercises 1–14

By examining the amount of Carbon-14 that remains in an organism after death, one can determine its age. The half-life of Carbon-14 is $5{,}730$ years, meaning that the amount of Carbon-14 present is reduced by a factor of $\frac{1}{2}$ every $5{,}730$ years.

1. Complete the table.

Years Since Death	0	$5{,}730$							
C-14 Atoms Remaining Per 1.0×10^8 C-12 Atoms	$10{,}000$								

Let C be the function that represents the number of C-14 atoms remaining per 1.0×10^8 C-12 atoms t years after death.

2. What is $C(11460)$? What does it mean in this situation?

3. Estimate the number of C-14 atoms per 1.0×10^8 C-12 atoms you would expect to remain in an organism that died 10,000 years ago.

4. What is $C^{-1}(625)$? What does it represent in this situation?

5. Suppose the ratio of C-14 to C-12 atoms in a recently discovered woolly mammoth was found to be 0.000001. Estimate how long ago this animal died.

6. Explain why the $C^{-1}(100)$ represents the answer to Exercise 5.

7. What type of function best models the data in the table you created in Exercise 1? Explain your reasoning.

8. Write a formula for C in terms of t. Explain the meaning of any parameters in your formula.

9. Graph the set of points $(t, C(t))$ from the table and the function C to verify that your formula is correct.

10. Graph the set of points $(C(t), t)$ from the table. Draw a smooth curve connecting those points. What type of function would best model this data? Explain your reasoning.

11. Write a formula that will give the years since death as a function of the amount of C-14 remaining per 1.0×10^8 C-12 atoms.

12. Use the formulas you have created to accurately calculate the following:
    a. The amount of C-14 atoms per 1.0×10^8 C-12 atoms remaining in a sample after $10{,}000$ years.

    b. The years since death of a sample that contains 100 C-14 atoms per 1.0×10^8 C-12 atoms.

c. $C(25{,}000)$

d. $C^{-1}(1000)$

13. A baby woolly mammoth that was discovered in 2007 died approximately $39{,}000$ years ago. How many C-14 atoms per 1.0×10^8 C-12 atoms would have been present in the tissues of this animal when it was discovered?

14. A recently discovered woolly mammoth sample was found to have a red liquid believed to be blood inside when it was cut out of the ice. Suppose the amount of C-14 in a sample of the creature's blood contained $3{,}000$ atoms of C-14 per 1.0×10^8 atoms of C-12. How old was this woolly mammoth?

Exercises 15–18

Scientists can infer the age of fossils that are older than $50,000$ years by using similar dating techniques with other radioactive isotopes. Scientists use radioactive isotopes with half-lives even longer than Carbon-14 to date the surrounding rock in which the fossil is embedded.

A general formula for the amount A of a radioactive isotope that remains after t years is

$$A = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$$

where A_0 is the amount of radioactive substance present initially, and h is the half-life of the radioactive substance.

15. Solve this equation for t to find a formula that will infer the age of a fossil by dating the age of the surrounding rocks.

16. Let $(x) = A_0\left(\frac{1}{2}\right)^{\frac{x}{h}}$. What is $A^{-1}(x)$?

17. Verify that A and A^{-1} are inverses by showing that $A\left(A^{-1}(x)\right) = x$ and $A^{-1}\left(A(x)\right) = x$.

18. Explain why, when determining the age of organic materials, archaeologists and anthropologists would prefer to use the logarithmic function to relate the amount of a radioactive isotope present in a sample and the time since its death?

Problem Set

1. A particular bank offers 6% interest per year compounded monthly. Timothy wishes to deposit $\$1{,}000$.
   a. What is the interest rate per month?
   b. Write a formula for the amount A Timothy will have after n months.
   c. Write a formula for the number of months it will take Timothy to have A dollars.
   d. Doubling-Time is the amount of time it takes for an investment to double. What is the doubling-time of Timothy's investment?
   e. In general, what is the doubling-time of an investment with an interest rate of $\frac{r}{12}$ per month?

2. A study done from 1950 through 2000 estimated that the world population increased on average by 1.77% each year. In 1950, the world population was 2519 million.
   a. Write a formula for the world population t years after 1950. Use p to represent world population.
   b. Write a formula for the number of years it will take to reach a population of p.
   c. Use your equation in part (b) to find when the model predicts that the world population will be 10 billion.

3. Consider the case of a bank offering r (given as a decimal) interest per year compounded monthly, if you deposit $\$P$.
   a. What is the interest rate per month?
   b. Write a formula for the amount A you will have after n months.
   c. Write a formula for the number of months it will take to have A dollars.
   d. What is the doubling-time of an investment earning 7% interest per year, compounded monthly? Round up to the next month.

4. A half-life is the amount of time it takes for a radioactive substance to decay by half. In general, we can use the equation $A = P\left(\frac{1}{2}\right)^t$ for the amount of the substance remaining after t half-lives.
   a. What does P represent in this context?
   b. If a half-life is 20 hours, rewrite the equation to give the amount after h hours.
   c. Use the natural logarithm to express the original equation as having base e.
   d. The formula you wrote in part (c) is frequently referred to as the "Pert" formula, that is, Pe^{rt}. Analyze the value you have in place for r in part (c). What do you notice? In general, what do you think r represents?
   e. Jess claims that any exponential function can be written with base e; is she correct? Explain why.

5. If caffeine reduces by about 10% per hour, how many hours h does it take for the amount of caffeine in a body to reduce by half (round up to the next hour)?

6. Iodine-123 has a half-life of about 13 hours, emits gamma-radiation, and is readily absorbed by the thyroid. Because of these facts, it is regularly used in nuclear imaging.
   a. Write a formula that gives you the percent p of iodine-123 left after t half-lives.
   b. What is the decay rate per hour of iodine-123? Approximate to the nearest millionth.
   c. Use your result to part (b). How many hours h would it take for you to have less than 1% of an initial dose of iodine-123 in your system? Round your answer to the nearest tenth of an hour.

7. An object heated to a temperature of $50°\ C$ is placed in a room with a constant temperature of $10°\ C$ to cool down. The object's temperature T after t minutes can be given by the function $T(t) = 10 + 40e^{-0.023105t}$.
   a. How long will it take for the object to cool down to $30°\ C$?
   b. Will it take longer for the object to cool from $50°\ C$ to $30°\ C$ or from $30°\ C$ to $10.1°\ C$?
   c. Will the object ever be $10°\ C$ if kept in this room?
   d. What is the domain of T^{-1}? What does this represent?

8. The percent of usage of the word "judgment" in books can be modeled with an exponential decay curve. Let P be the percent as a function of x, and let x be the number of years after 1900, then $P(x) = 0.0220465 \cdot e^{-0.0079941x}$.
   a. According to the model, in what year was the usage 0.1% of books?
   b. When will the usage of the word "judgment" drop below 0.001% of books? This model was made with data from 1950 to 2005. Do you believe your answer will be accurate? Explain.
   c. Find P^{-1}. What does the domain represent? What does the range represent?