Lesson 16: Function Composition

Student Outcome

- Students compose functions and describe the domain and range of the compositions.

Lesson Notes

In previous courses, students have been introduced to functions as a relationship between sets of numbers where each input is assigned a unique output. In this lesson, the students will explore functions and their compositions, including situations where the sets representing the inputs and outputs may not be numerical. They will find the composition of functions in real-world contexts and assess the reasonableness of the compositions.

Classwork

Opening (7 minutes)

Students should be familiar with the term function and its characteristics in a mathematical context. This lesson will broaden the students' understanding of functions to include sets of values that are non-numerical, and it will expand on students' understanding of performing operations on functions to include composing functions. An example addressing competitive free diving will expose the students to a real-world situation where composing functions could be used to solve problems. Explain to the students that free diving is a process by which divers descend a given distance in the water and then swim to the surface, all without any breathing aid, which can be dangerous to competitors because the rapid change in pressure the divers experience as they descend can cause nitrogen bubbles to form in their capillaries, inhibiting blood flow. The students should then analyze the tables provided and respond to the prompt. Several students can share their reflections briefly in a whole-class setting:

	Depth of Free Diver During Descent									
s seconds of descent	0	20	40	60	80	100	120	140	160	180
d depth in meters of diver	0	15	32	44	65	79	90	106	120	133

	Atmospheric Pressure and Ocean Depth									
d depth in meters of diver	0	10	20	30	40	50	60	70	80	90
p pressure in atm on diver	1	2	3	4	5	6	7	8	9	10

Scaffolding:
- Call out times of descent from the first table, and have students call out the corresponding depth.
- Call out depths from the second table, and have student call out the corresponding pressure.

- Based on the information you have been presented about free diving and the data in the tables, how might medical researchers use the concept of functions to predict the atmospheric pressure on a diver during his or her descent?

Example 1 (10 minutes)

This example will provide students with the opportunity to determine whether the tables from the opening scenario have the characteristics of functions and to interpret the domain and range in context. They will also explore the composition of two functions, which will prepare them to compose functions and evaluate the compositions later in the lesson. The example should be completed as part of a teacher-led discussion.

- What do you recall about functions you have studied between numerical sets?
  - *Answers should address that for each input, there is exactly one output; there may be restrictions on the domain and/or range*
- In what sorts of contexts have you analyzed sets of numbers to determine whether the relationship between them represents a function?
  - *Answers will vary but might include the following: mapping diagrams to determine if each input maps to only one output; writing all the inputs and corresponding outputs as ordered pairs and determining if all the x-values are unique; plotting these ordered pairs on a coordinate plane; and determining whether the graph passes the vertical line test, etc.*
- Do the conditions for functions appear to hold for the table relating a free diver's descent time and depth? How about for the table relating depth and pressure? Explain.
  - *Yes for both tables. In the first table, each time entered corresponds to exactly one depth. For the second table, each depth corresponds to exactly one pressure.*
- And which variable is a function of the other for the first table? Explain.
  - *Depth in meters is a function of time in seconds. The depth was recorded given the time spent in descending, so the independent variable or input is the time, and the output is the depth.*
- What about for the second table?
  - *Pressure in atmospheres is a function of depth in meters. The pressure was recorded given the depth of the descending diver, so the independent variable or input is the depth, and the output is the pressure.*
- In the first table, what does the domain and range represent?
  - *Domain is time, in seconds, spent descending, and range is depth, in meters*
- And why can't these values be negative?
  - *It would not make sense for a diver to descend for a negative number of seconds or to a depth that is above the ocean's surface.*
- What other restrictions could we place on the domain and range for this function?
  - *Answers will vary but should address that there is some reasonable upper limit that represents the amount of time a diver can spend descending and a maximum depth the diver can reach.*
- Define and discuss the domain and range of the function represented by the second table.
  - *The domain is the depth of the descending diver in meters, which cannot be negative and has some real-number maximum value that corresponds with the greatest depth a diver can reach. The range is the pressure applied to the diver. This cannot be less than 1 atm, the pressure at sea level, and has a maximum value that corresponds to the maximum depth of the descending diver.*

- How can we use the tables to determine the depth of a diver who has descended 80 seconds, and what is it?
  - *Find the column in the first table where $s = 80$, and find the corresponding value of d, which is 65.*
- How can we represent this relationship using function notation?
  - $f(80) = 65$

MP.2

- So what would $f(20) = 15$ represent in context?
  - *The depth of a descending diver at 20 seconds is 15 meters.*
- How can we use the tables to determine the pressure applied to a diver who has descended 40 meters, and what is it?
  - *Find the column in the second table where $d = 40$, and find the corresponding value of p, which is 5.*
- How can we represent this relationship using function notation, and what does it represent in context?
  - $p(40) = 5$; *a diver has a pressure of 5 atmospheres applied to him or her at a depth of 40 meters.*
- Now how could we use the tables to find the pressure applied to a diver 120 seconds into the descent?
  - *Use the first table to find the depth of the diver at 120 seconds, which is 90 meters, and then use the second table to find the pressure at 90 meters, which is 10 atmospheres.*
- How can we explain this process in terms of functions?
  - *Answers will vary but should address evaluating $f(120)$ and using this as the input for the function $p = g(d)$.*
- And how can the overall process be represented using function notation?
  - $p = g(f(120))$
- Students may struggle with this new concept. Practice with lots of different examples.
- Pause to explain that we have done what mathematicians call composing functions.

Example 1

1. Consider the tables from the opening scenario.

Depth of Free Diver During Descent									
s seconds of descent	20	40	60	80	100	120	140	160	180
d depth in meters of diver	15	32	44	65	79	90	106	120	133

Atmospheric Pressure and Ocean Depth									
d depth in meters of diver	10	20	30	40	50	60	70	80	90
p pressure in atm on diver	2	3	4	5	6	7	8	9	10

Scaffolding:
Have advanced students answer parts (d)-(f) only.

a. Do the tables appear to represent functions? If so, define the function represented in each table using a verbal description.

Both tables appear to represent functions because for each input in the domain, there is exactly one output.

In the first table, the depth of the diver is a function of the time spent descending.

In the second table, the pressure on the diver is a function of the diver's depth.

b. What are the domain and range of the functions?

For the first table, the domain and range are nonnegative real numbers.

For the second table, the domain is nonnegative real numbers, and the range is real numbers greater than or equal to 1.

c. Let's define the function in the first table as $d = f(s)$ and the function in the second table as $p = g(d)$. Use function notation to represent each output, and use the appropriate table to find its value.

i. Depth of the diver at 80 seconds

$d = f(80) = 65$ *meters*

ii. Pressure of the diver at a depth of 60 meters

$p = g(60) = 7$ *atmospheres*

d. Explain how we could determine the pressure applied to a diver after 120 seconds of descending.

We could use the first table to determine the depth that corresponds to a descent time of 120 seconds and then use the second table to find the pressure that corresponds to this depth.

e. Use function notation to represent part (d), and use the tables to evaluate the function.

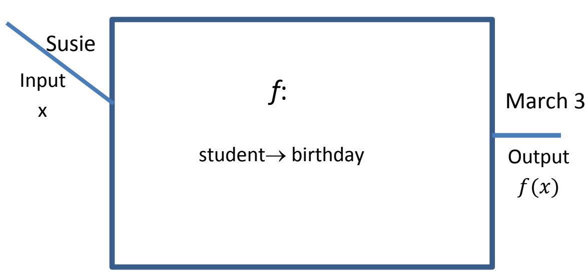
$g\big(f(120)\big) = g(90) = 10$

f. Describe the output from part (e) in context.

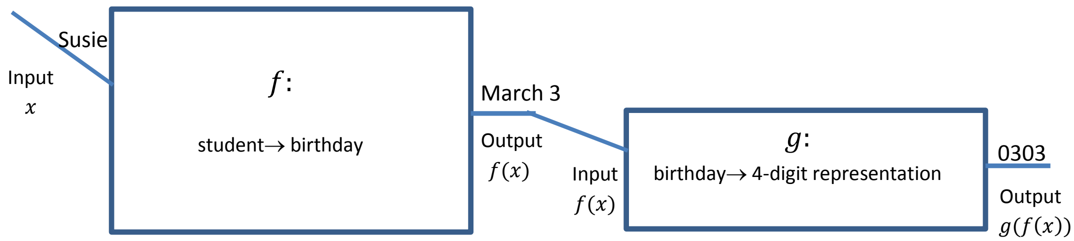
The pressure applied to a diver 120 seconds into a descent is 10 atmospheres.

Discussion: Composing Functions (7 minutes)

- So now we have reviewed the necessary conditions for a relationship between sets to be considered a function and examined a composition of two functions. Let's explore a composition more generally. If we imagine that function f represents a box with an "in" chute and an "out" chute, inputs from the domain would enter through the "in" chute, and the corresponding outputs from the range would exit from the "out" chute like this:



- Look at this diagram. Describe what is occurring.
  - *The input, Susie, is having the function rule f applied to it to produce the output, March 3.*
- And how do we generally notate inputs and outputs for function rule f?
  - *The input is x and the output is $f(x)$.*
- Now if we want to compose two functions, we can imagine placing another box g where the outputs of f are fed into the "in" chute of g, and the function rule for g is applied, producing an output $g\big(f(x)\big)$:



- Explain the process illustrated by the diagram.
  - *Answers might vary but should address that "Susie" was the input into function f, which produced output March 3; March 3, in turn, became the input into function g, which produced the output* 0303.
- So for our function composition, what is the relationship between the inputs and outputs for f and g?
  - *Answers may vary but should address that the output of f represents the input of g.*
- And what is the overall effect of the process on the initial input x?
  - *The function rule f is applied to x, and then g is applied to the result.*
- Let's formalize our description of a function composition:

> The **composite function** of f and g, denoted by $g \circ f$, is the combined function, f followed by g where $f\colon X \to Y$ and $g\colon Y \to Z$, and the range of f lies within the domain of g.

- What other notation can we use to represent the composition $(g \circ f)x$?
  - $g(f(x))$
- How would we interpret $g \circ f$ in our students and birthdays example?
  - *$g \circ f$ represents the four-digit number that corresponds to the birthdate of the enrolled student.*
- And in what order are the functions applied?
  - *From right to left; first the birthday is applied to the student, and then the digits to the birthday.*

Example 2 (5 minutes)

Example 2 below will provide students with the opportunity to examine the composition of functions. The students will compose functions and will interpret them in context, including determining whether compositions are reasonable.

- How do we perform the composition $g \circ f$? Interpret the process in the context of the example.
  - *First, we apply the function rule f to the domain, and then we apply g to the result. For this situation, that means we assign an enrolled student to his or her birthday and then assign the biological father to the birthday, which does not make sense.*

- So what process would be applied to compute $f \circ g$?
  - *We would apply the function rules from right to left. This means that we would assign all enrolled students to his or her biological father and then apply the birthday to the father, which means the output represents the birthday of the enrolled student's biological father.*
- What conclusions can we draw about performing compositions on functions?
  - *Answers will vary but should address that the function rules should be applied from right to left (inside to outside), with the output of each function rule representing the input of the next function applied; not all compositions make sense in context.*
- So if $f(x) = 2x$ and $g(x) = x + 4$, how would you find $f(g(3))$? $(g \circ f)x$?
  - *To find $f(g(3))$, evaluate $g(3)$, and apply f to the result, so $f(g(3)) = f(3+4) = f(7) = 2(7) = 14$; to find $(g \circ f)x$, evaluate $f(x)$,and apply g to the result, so $(g \circ f)x = g(2x) = 2x + 4$.*

Example 2

Consider these functions:

f: Name→ Calendar Date

Assign each enrolled student to his or her birthday.

g: Name→ Name

Assign each person to his or her biological father.

MP.2

Describe the action of each composite function. Determine which composite functions make sense.

a. $g \circ f$

Assign the biological father to the birthday of the enrolled student: function does not make sense.

b. $f \circ f$

Assign the birthday to the birthday of the enrolled student: function does not make sense.

c. $f \circ g$

Assign the birthday to the biological father of the enrolled student: function makes sense.

d. $f \circ g \circ g$

Assign the birthday to the biological father of the biological father of the enrolled student: function makes sense.

Exercises 1–2 (8 minutes)

These exercises should be completed individually. After a few minutes, the students could verify their responses with a partner. Then, selected students could share their responses with the class.

Scaffolding:

- Work through an example using different colors for the expressions representing f and g to help the students visualize how the output of the first applied function represents the input for the second function.

Exercises 1–2

1. **Let $f(x) = x^2$ and $g(x) = x + 5$. Write an expression that represents each composition:**

   a. $(f \circ g)x$

   $(f \circ g)x = f(x+5) = (x+5)^2 = x^2 + 10x + 25$

   b. $g(f(4))$

   $g(f(4)) = g(4^2) = g(16) = (16+5) = 21$

   c. $(f \circ g)(\sqrt{x+5})$

   $(f \circ g)(\sqrt{x+5}) = f(\sqrt{x+5}+5) = (\sqrt{x+5}+5)^2 = x + 10\sqrt{x+5} + 30$

2. **Suppose a sports medicine specialist is investigating the atmospheric pressure placed on competitive free divers during their descent. The following table shows the depth, d, in meters of a free diver s seconds into his descent. The depth of the diver is a function of the number of seconds the free diver has descended, $d = f(s)$.**

s seconds	10	35	55	70	95	115	138	160	175
d depth in meters	8.1	28	45	55	76.0	91.5	110	130	145

The pressure, in atmospheres, felt on a free diver, d, is a function of his or her depth, $p = g(d)$.

d meters	25	35	55	75	95	115	135	155	175
p atm	2.4	3.5	5.5	7.6	9.6	11.5	13.7	15.5	17.6

   a. **How can the researcher use function composition to examine the relationship between the time a diver spends descending and the pressure he or she experiences? Use function notation to explain your response.**

   The function $g(f(s))$ represents the pressure experienced by a diver who has been descending for s seconds. The function f applies a depth in meters to each input s, and the function g assigns a pressure to each depth.

MP.2

b. Explain the meaning of $g(f(0))$ in context.

$g(f(0))$ represents the pressure on a free diver at his or her depth 0 seconds into the descent.

c. Use the charts to approximate these values, if possible. Explain your answers in context.

i. $g(f(70))$

5.5 atmospheres; this is the pressure on the free diver at his depth 70 seconds into his dive ($d = 55$ meters).

ii. $g(f(160))$

13 atmospheres; this is the pressure on the free diver at his depth 160 seconds into his dive ($d = 130$ meters).

Closing (3 minutes)

Have the students paraphrase what they understand about composing functions using bulleted statements. If time permits, have students share their statements, and encourage them to add to their own set of statements based on what is shared.

- To perform the composition $g \circ f$, apply the function rule f to the inputs, and apply g to the result.
- The composition of functions is not a commutative operation. In other words, $g \circ f \neq f \circ g$.
- Not all function compositions produce reasonable outputs in context.

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 16: Function Composition

Exit Ticket

1. Let $f(x) = x^2$ and $g(x) = 2x + 3$. Write an expression that represents each composition:
   a. $(g \circ f)x$

   b. $f\big(f(-2)\big)$

   c. $(f \circ g)\left(\frac{1}{x}\right)$

2. A consumer advocacy company conducted a study to research the pricing of fruits and vegetables. They collected data on the size and price of produce items, including navel oranges. They found that, for a given chain of stores, the price of oranges was a function of the weight of the oranges, $p = f(w)$.

w weight in pounds	0.2	0.25	0.3	0.4	0.5	0.6	0.7
p price in dollars	0.26	0.32	0.39	0.52	0.65	0.78	0.91

The company also determined that the weight of the oranges measured was a function of the radius of the oranges, $w = g(r)$.

r radius in inches	1.5	1.65	1.7	1.9	2	2.1
w in pounds	0.38	0.42	0.43	0.48	0.5	0.53

a. How can the researcher use function composition to examine the relationship between the radius of an orange and its price? Use function notation to explain your response.

b. Use the table to evaluate $f\left(g(2)\right)$, and interpret this value in context.

Exit Ticket Sample Solutions

1. Let $f(x) = x^2$ and $g(x) = 2x + 3$. Write an expression that represents each composition:

   a. $(g \circ f)x$

   $(g \circ f)x = g(x^2) = 2(x^2) + 3 = 2x^2 + 3$

   b. $f(f(-2))$

   $f(f(-2)) = f(4) = 4^2 = 16$

   c. $(f \circ g)\left(\frac{1}{x}\right)$

   $(f \circ g)\left(\frac{1}{x}\right) = f\left(2\left(\frac{1}{x}\right) + 3\right) = \left(\frac{2}{x} + 3\right)^2 = \frac{4}{x^2} + \frac{12}{x} + 9$

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w in pounds	0.38	0.42	0.43	0.48	0.5	0.53

   a. How can the researcher use function composition to examine the relationship between the radius of an orange and its price? Use function notation to explain your response.

   The function $f(g(r))$ represents the price of oranges as a function of the radius of the oranges.

   b. Use the table to evaluate $f(g(2))$, and interpret this value in context.

   $$f(g(2)) = f(0.5) = 0.65$$

   The price of oranges with a radius of 2 inches is $\$0.65$.

Problem Set Sample Solutions

1. Determine whether each rule described represents a function. If the rule represents a function, write the rule using function notation, and describe the domain and range.

   a. Assign to each person his or her age in years.

      Yes. f: People $\rightarrow$ Numbers

      Domain: set of all living people. Range: $\{0, 1, 2, 3, \ldots, 130\}$

   b. Assign to each person his or her height in centimeters.

      Yes. f: People $\rightarrow$ Numbers

      Domain: set of all living people. Range: $\{50, 51, \ldots 280\}$

   c. Assign to each piece of merchandise in a store a bar code.

      Yes. f: Products $\rightarrow$ Bar codes

      Domain: each piece of merchandise in the store. Range: $\{$ *unique bar codes* $\}$

   d. Assign each deli customer a number ticket.

      Yes. f: People $\rightarrow$ number tickets

      Domain: set of people that are waiting in the deli. Range: a ticket number

   e. Assign a woman to her child.

      No. There are many women who will have more than one child and many who will have no children.

   f. Assign to each number its first digit.

      Yes. f: Countering number $\rightarrow$ Counting Numbers

      Domain: set of all counting numbers. Range: $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

   g. Assign each person to his or her biological mother.

      Yes. f: People $\rightarrow$ Women

      Domain: set of all counting numbers. Range: set of women who are mothers.

2. **Let M: people → people**

Assign to each person his or her biological mother.

F: people → people

Assign to each person his or her biological father.

L: people → people

Assign to each person the first letter of his or her name.

A: people → people

Assign to each person his or her age in years.

Which of the following compositions makes sense? For those that do, describe what the composite function is doing.

a. $M \circ F$

It makes sense. The function assigns each person to his or her biological father and then to the father's biological mother, which is the grandmother.

b. $L \circ M$

It makes sense. The function assigns each person to his or her biological mother and then to the first letter of the mother's first name..

c. $M \circ L$

It does not make sense.

d. $A \circ M$

It make sense. The function assigns the biological age to the mother.

e. $A \circ L$

It does not make sense.

f. $F \circ M \circ A$

It does not make sense.

g. $L \circ M \circ F$

It makes sense. The function assigns the biological father to his biological mother and then assigns to her the first letter of her name.

h. $A \circ M \circ M$

It makes sense. The function assigns the biological mother to her biological mother and then assigns to the grandmother her age in years.

3. Let $f(x) = x^2 - x,\ g(x) = 1 - x$.

a. $f \circ g$

$$f(g(x)) = f(1 - x) = 1 - 2x + x^2 - 1 + x = x^2 - x$$

b. $g \circ f$

$$g(f(x)) = g(x^2 - x) = 1 - x^2 + x = -x^2 + x - 1$$

c. $g \circ g$

$$g(g(x)) = g(1 - x) = 1 - 1 + x = x$$

d. $f \circ f$

$$f(f(x)) = f(x^2 - x) = x^4 - 2x^3 + x^2 - x^2 + x = x^4 - 2x^3 + x$$

e. $f(g(2))$

$$f(g(2)) = f(-1) = 2$$

f. $g(f(-1))$

$$g(f(-1)) = g(2) = -1$$

4. Let $f(x) = x^2,\ g(x) = x + 3$.

a. $g(f(5))$

$$g(f(5)) = g(5^2) = g(25) = 25 + 3 = 28$$

b. $f(g(5))$

$$f(g(5)) = f(5 + 3) = f(8) = 8^2 = 64$$

c. $f(g(x))$

$$f(g(x)) = f(x + 3) = (x + 3)^2 = x^2 + 6x + 9$$

d. $g(f(x))$

$$g(f(x)) = g(x^2) = x^2 + 3$$

e. $g\left(f(\sqrt{x + 3})\right)$

$$g\left(f(\sqrt{x + 3})\right) = g(x + 3) = x + 6$$

5. Let $f(x) = x^3$, $g(x) = \sqrt[3]{x}$.

a. $f \circ g$

$$f\big(g(x)\big) = f\big(\sqrt[3]{x}\big) = x$$

b. $g \circ f$

$$g\big(f(x)\big) = g(x^3) = x$$

c. $f\big(g(8)\big)$

$$f\big(g(8)\big) = f(2) = 8$$

d. $g\big(f(2)\big)$

$$g\big(f(2)\big) = g(8) = 2$$

e. $f\big(g(-8)\big)$

$$f\big(g(-8)\big) = f(-2) = -8$$

f. $g\big(f(-2)\big)$

$$g\big(f(-2)\big) = g(-8) = -2$$

6. Let $f(x) = x^2$, $g(x) = \sqrt{x} + 3$.

a. Show that $\big(f(x+3)\big) = |x+3| + 3$.

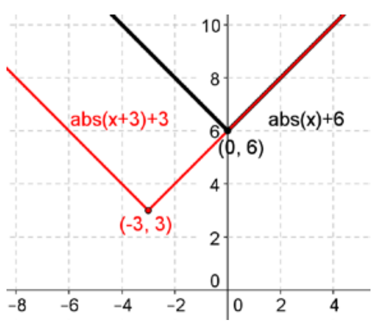
$$g\big(f(x+3)\big) = g((x+3)^2) = \sqrt{(x+3)^2} + 3 = |x+3| + 3$$

b. Does $f(x) = |x+3| + 3 = (x) = |x| + 6$? Graph them on the same coordinate plane.

No, they are not equal.

For $|x+3|+3$, *if* $x+3 \geq 0$, $|x+3|+3 = x+6$; *if* $x+3 < 0$, $|x+3|+3 = -x$

For $|x|+6$, *if* $x \geq 0$, $|x|+6 = x+6$; *if* $x < 0$, $|x|+6 = -x+6$



7. **Given the chart below, find the following:**

	-6	0	2	4
$f(x)$	4	-6	0	2
$g(x)$	2	4	-6	0
$h(x)$	0	2	4	-6
$k(x)$	1	4	0	3

a. $f(g(0))$

2

b. $g(k(2))$

4

c. $k(g(-6))$

0

d. $g(h(4))$

2

e. $g(k(4))$

$g(3)$
is not defined.

f. $f \circ g \circ h(2)$

4

g. $f \circ f \circ f(0)$

2

h. $f \circ g \circ h \circ g(2)$

2

8. Suppose the strep throat virus is spreading in a community. The following table shows the number of people, n, that have the virus d days after the initial outbreak. The number of people who have the virus is a function of the number of days, $n = f(d)$.

d days	0	1	4	8	12	16	20
$n = f(d)$ number of people infected	2	4	14	32	64	50	30

There is only one pharmacy in the community. As the number of people who have the virus increases, the number of boxes of cough drops, b, sold also increases. The number of boxes of cough drops sold on a given day is a function of the number of people who have the virus, $b = g(n)$, on that day.

n number of people infected	0	2	4	9	14	20	28	32	44	48	50	60	64
$b = g(n)$ number of boxes of cough drops sold	1	5	14	16	22	30	42	58	74	86	102	124	136

a. Find $g(f(1))$, and state the meaning of the value in the context of the strep throat epidemic. Include units in your answer.

4. On day one, there were four people infected, and there were fourteen boxes of cough drops sold at the pharmacy.

b. Fill the chart below using the fact that $b = g(f(d))$.

d (days)	0	1	4	8	12	16	20
b number of boxes of cough drops sold	5	14	22	58	136	102	58

c. For each of the following expressions, interpret its meaning in the context of the problem, and if possible, give an approximation of its value.

i. $g(f(4))$

$g(f(4)) = g(14) = 22$

On the fourth day of the outbreak, 22 boxes of cough drops were sold.

ii. $g(f(16))$

$g(f(16)) = g(50) = 102$

On the sixteenth day of the outbreak, 102 boxes of cough drops were sold.

iii. $f(g(9))$

It does not make sense. The function cannot be composed in reverse for this case. The output $g(9)$ represents the number of boxes of cough drops sold when 9 students are infected, which is 16. The composition represents the number of students infected when 16 boxes of cough drops were sold, which cannot be determined because the number of students infected is not a function of the number of boxes of cough drops sold.