Lesson 14: Graphing Rational Functions

Classwork

Opening Exercise

State the domain of each of the following functions. Then, determine whether or not the excluded value(s) of x are vertical asymptotes on the graph of the function. Give a reason for your answer.

a. $f(x) = \frac{x^2-3x+2}{x-2}$

b. $f(x) = \frac{x^2+3x+2}{x-2}$

Example 1

Sketch the graph of the rational function $f(x) = \frac{2x^2 - x}{x^2 - 16}$ showing all the key features of the graph. Label the key features on your graph.

Example 2

Graph the function $f(x) = \frac{x^2+5x-6}{x+1}$ showing all key features.

Exercises 1–10

Sketch the graph of each rational function showing all key features. Verify your graph by graphing the function on the graphing calculator.

1. $f(x) = \frac{4x-6}{2x+5}$

2. $f(x) = \frac{(3x-6)(x-4)}{x(x-4)}$

3. $f(x) = \frac{3x-2x^2}{x-2}$

4. $f(x) = \frac{x-2}{3x-2x^2}$

5. $f(x) = \frac{x}{x^2 - 9}$

6. $f(x) = \frac{x^2}{x^2 - 9}$

7. $f(x) = \frac{x^2 - 9}{x}$

8. $f(x) = \frac{x^2 - 9x}{x}$

9. $f(x) = \frac{x^3-8}{x-2}$

10. $f(x) = \frac{x^3-8}{x-1}$

Problem Set

1. List all of the key features of each rational function and its graph, and then sketch the graph showing the key features.
   a. $y = \frac{x}{x-1}$
   b. $y = \frac{x^2-7x+6}{x^2-36}$
   c. $y = \frac{x^3-3x^2-10x}{x^2+8x-65}$
   d. $y = \frac{3x}{x^2-1}$

2. Graph $y = \frac{1}{x^2}$ and $y = \frac{1}{x}$. Compare and contrast the two graphs.

Extension

3. Consider the function $f(x) = \frac{x^3+1}{x}$.
   a. Use the distributive property to rewrite f as the sum of two rational functions g and h.
   b. What is the end behavior of g? What is the end behavior of h?
   c. Graph $y = f(x)$ and $y = x^2$ on the same set of axes. What do you notice?
   d. Summarize what you have discovered in part (b) and (c).

4. Number theory is a branch of mathematics devoted primarily to the study of integers. Some discoveries in number theory involve numbers that are impossibly large such as Skewes' numbers and Graham's number. One Skewes' number is approximately $e^{\left(e^{\left(e^{79}\right)}\right)}$ and Graham's number is so large that to even write it requires 64 lines of writing with a new operation (one that can be thought of as the shortcut for repeated exponentiation). In fact, both of these numbers are so large that the decimal representation of the numbers would be larger than the known universe and dwarf popular large numbers such as googol and googolplex (10^{100} and $10^{\left(10^{100}\right)}$ respectively). These large numbers, although nearly impossible to comprehend, are still not at the "end" of the real numbers, which have no end. Consider the function $f(x) = x^2 - 10^{100}$.
   a. Consider only positive values of x; how long until $f(x) > 0$?
   b. If your answer to part (a) represented seconds, how many billions of years would it take for $f(x) > 0$? (Note: one billion years is approximately 3.15×10^{16} seconds). How close is this to the estimated geological age of the earth (4.54 billion years)?

c. Number theorists frequently only concern themselves with the term of a function that has the most influence as $x \to \infty$. Let $f(x) = x^3 + 10x^2 + 100x + 1000$, and answer the following questions.

   i. Fill out the following table:

x	$f(x)$	x^3	$\frac{x^3}{f(x)}$	$10x^2$	$\frac{10x^2}{f(x)}$	$100x$	$\frac{100x}{f(x)}$	$\frac{1000}{f(x)}$
0								
10								
100								
1000								

   ii. As $x \to \infty$, which term of $f(x)$ dominates the value of the function?

   iii. Find $g(x) = \frac{f(x)}{x}$. Which term dominates $g(x)$ as $x \to \infty$?

d. Consider the formula for a general polynomial, $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ for real numbers a_i, $0 \le i \le n$. Which single term dominates the value of $f(x)$ as $x \to \infty$.