Lesson 13: Horizontal and Vertical Asymptotes of Graphs of Rational Functions

Classwork

Opening Exercise

Determine the end behavior of each rational function below. Graph each function on the graphing calculator and explain how the graph supports your analysis of the end behavior.

1. $f(x) = \frac{x^2-3}{x^3}$

2. $f(x) = \frac{x^2-3}{x^2+1}$

3. $f(x) = \frac{x^3}{x^2-3}$

Example 1

Consider the rational function $f(x) = \frac{2x-1}{x-4}$.

a. State the domain of f.

b. Determine the end behavior of f.

c. State the horizontal asymptote of the graph of $y = f(x)$.

d. Graph the function on the graphing calculator and make a sketch on your paper.

Exercises 1–9

State the domain and end behavior of each rational function. Identify all horizontal and vertical asymptotes on the graph of each rational function. Then, verify your answer by graphing the function on the graphing calculator.

1. $f(x) = \frac{-x+6}{2x+3}$

2. $f(x) = \frac{3x-6}{x}$

3. $f(x) = \frac{3}{x^2-25}$

4. $f(x) = \frac{x^2-2}{x^2+2x-3}$

5. $f(x) = \frac{x^2-5x-4}{x+1}$

6. $f(x) = \frac{5x}{x^2+9}$

Write an equation for a rational function whose graph has the given characteristic. Graph your function on the graphing calculator to verify.

7. A horizontal asymptote of $y = 2$ and a vertical asymptote of $x = -2$.

8. A vertical asymptote of $x = 6$ and no horizontal asymptote.

9. A horizontal asymptote of $y = 6$ and no vertical asymptote.

Lesson Summary

- Let a be a real number. The line given by $x = a$ is a *vertical asymptote* of the graph of $y = f(x)$ if at least one of the following statements is true.
  - As $x \to a$, $f(x) \to \infty$.
  - As $x \to a$, $f(x) \to -\infty$.
- Let L be a real number. The line given by $y = L$ is a *horizontal asymptote* of the graph of $y = f(x)$ if at least one of the following statements is true.
  - As $x \to \infty$, $f(x) \to L$.
  - As $x \to -\infty$, $f(x) \to L$.

Problem Set

1. State the domain of each rational function. Identify all horizontal and vertical asymptotes on the graph of each rational function.
   a. $y = \frac{3}{x^3-1}$
   b. $y = \frac{2x+2}{x-1}$
   c. $y = \frac{5x^2-7x+12}{x^3}$
   d. $y = \frac{3x^6-2x^3+1}{16-9x^6}$
   e. $f(x) = \frac{6-4x}{x+5}$
   f. $f(x) = \frac{4}{x^2-4}$

2. Sketch the graph of each function in Exercise 1 with asymptotes and excluded values from the domain drawn on the graph.

3. Factor out the highest power of x in each of the following, and cancel common factors if you can. Assume x is nonzero.
   a. $y = \frac{x^3+3x-4}{3x^3-4x^2+2x-5}$
   b. $y = \frac{x^3-x^2-6x}{x^3+5x^2+6x}$
   c. $y = \frac{2x^4-3x+1}{5x^3-8x-1}$
   d. $y = -\frac{9x^5-8x^4+3x+72}{7x^5+8x^4+8x^3+9x^2+10x}$
   e. $y = \frac{3x}{4x^2+1}$

4. Describe the end behavior of each function in Exercise 3.

5. Using the equations that you wrote in Exercise 3, make some generalizations about how to quickly determine the end behavior of a rational function.

6. Describe how you may be able to use the end behavior of the graphs of rational functions, along with the excluded values from the domain and the equations of any asymptotes, to graph a rational function without technology.