



Lesson 12: End Behavior of Rational Functions

Student Outcomes

- Students describe the end behavior of rational functions.

Lesson Notes

This lesson offers students opportunities to use tables to analyze the end behavior of rational functions and the behavior of rational functions as they approach restricted input values. This prepares students for subsequent lessons in which they graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior (**F-IF.C.7d**).

Classwork

Opening Exercise (3 minutes)

Our work in Algebra II showed us how to analyze the end behavior of polynomials. We'll begin with a set of exercises that provides an opportunity to recall those skills, and then we'll analyze the end behavior of rational functions.

Opening Exercise

Analyze the end behavior of each function below. Then choose one of the functions and explain how you determined the end behavior.

a. $f(x) = x^4$

As $x \rightarrow \infty$, $f(x) \rightarrow \infty$. As $x \rightarrow -\infty$, $f(x) \rightarrow \infty$.

We have $2^4 = 16$, and $3^4 = 81$. In general, as we use larger and larger inputs, this function produces larger and larger outputs, exceeding all bounds.

$f(x)$ is an even function, so the same remarks apply to negative inputs.

b. $g(x) = -x^4$

As $x \rightarrow \infty$, $f(x) \rightarrow -\infty$. As $x \rightarrow -\infty$, $f(x) \rightarrow -\infty$.

We have $-2^4 = -16$, and $-3^4 = -81$. In general, as we use larger and larger inputs, this function produces larger and larger negative outputs, exceeding all bounds.

$f(x)$ is an even function, so the same remarks apply to negative inputs.

c. $h(x) = x^3$

As $x \rightarrow \infty$, $f(x) \rightarrow \infty$. As $x \rightarrow -\infty$, $f(x) \rightarrow -\infty$.

We have $2^3 = 8$, and $3^3 = 27$. In general, as we use larger and larger inputs, this function produces larger and larger outputs, exceeding all bounds.

We also have $(-2)^3 = -8$, and $(-3)^3 = -27$. In general, as we use lesser and lesser inputs, this function produces larger and larger negative outputs, exceeding all bounds.

Scaffolding:

- As necessary, remind students that analyzing the end behavior of a function entails finding what value $f(x)$ approaches as $x \rightarrow \infty$ or as $x \rightarrow -\infty$.
- Also consider showing students the graphs of $f(x) = x^2$ and $g(x) = x^3$, and then asking them to describe the end behavior verbally.

d. $k(x) = -x^3$

As $x \rightarrow \infty$, $f(x) \rightarrow -\infty$. As $x \rightarrow -\infty$, $f(x) \rightarrow \infty$.

We have $-2^3 = -8$, and $-3^3 = -27$. In general, as we use larger and larger inputs, this function produces larger and larger negative outputs, exceeding all bounds.

We also have $-(-2)^3 = 8$, and $-(-3)^3 = 27$. In general, as we use lesser and lesser inputs, this function produces larger and larger outputs, exceeding all bounds.

Discussion: Power Functions (3 minutes)

The following discussion builds on the material developed in the opening exercises. Students analyze simple **power functions** $g(x) = a \cdot x^n$. This analysis is essential to understanding the end behavior of polynomials, as well as that of rational functions.

- Let's take a close look at one of these functions to make sure the reasoning is clear. For $h(x) = x^3$, do you think the outputs grow without bound? For example, do the outputs ever exceed one trillion? Think about this question for a moment, and then share your response with a partner. Try to be as specific as possible.
 - One trillion is 10^{12} . If we take $x = 10^4$ as an input, we get $f(10^4) = (10^4)^3 = 10^{12}$. So any input larger than 10^4 will produce an output that is larger than one trillion.
- Good. Now let's focus on the sign of the output. Take $f(x) = x^{13}$ and $g(x) = x^{14}$. When $x = -1000$, is the output positive or negative? Explain your thinking to a partner, being as specific as possible about how you reached your conclusion.
 - Let's consider $f(-1000) = (-1000)^{13}$. The expression $(-1000)^{13}$ represents the product of 13 negative numbers. Each pair of negative numbers has a positive product, but since there is an odd number of factors, the final product must be negative.
 - Now let's consider $g(-1000) = (-1000)^{14}$. This time there are 7 pairs of negative numbers, each of which has a positive product. Therefore, the final product must be positive.

Discussion: Reciprocals of Power Functions (8 minutes)

- Okay, so we see that analyzing the end behavior of a power function is straightforward. Now let's turn our attention to some very simple rational functions.
- What do you know about the function $f(x) = \frac{1}{x}$? Give several examples of input-output pairs associated with this function, then share them with a partner.
 - This function takes a number and returns its reciprocal. For example, $f(2) = \frac{1}{2}$, $f(10) = \frac{1}{10}$, $f(-3) = -\frac{1}{3}$, $f(1) = 1$, and $f\left(\frac{1}{5}\right) = 5$. At $x = 0$, $f(x)$ is undefined.
- Describe the end behavior for $f(x) = \frac{1}{x}$.
 - As $x \rightarrow \infty$, $f(x) \rightarrow 0$, and as $x \rightarrow -\infty$, $f(x) \rightarrow 0$.

- Good. Can you confirm this using a table? Organize several input-output pairs in a table to confirm the behavior of the graph of $f(x) = \frac{1}{x}$ as x approaches 0 from the positive side and as $x \rightarrow \infty$.

x	0.001	0.01	0.1	1	10	100	1000
$f(x)$	$\frac{1}{0.001} = 1000$	$\frac{1}{0.01} = 100$	$\frac{1}{0.1} = 10$	$\frac{1}{1} = 1$	$\frac{1}{10} = 0.1$	$\frac{1}{100} = 0.01$	$\frac{1}{1000} = 0.001$

- What does the table indicate about $f(x) = \frac{1}{x}$ as $x \rightarrow \infty$ and as $x \rightarrow 0$ from the positive side?
 - It approaches 0 as $x \rightarrow \infty$ and it approaches ∞ as $x \rightarrow 0$ from the positive side.
- Now let's examine the behavior of the function when the input is a negative number.

x	-0.001	-0.01	-0.1	-1	-10	-100	-1000
$f(x)$	$\frac{1}{-0.001} = -1000$	$\frac{1}{-0.01} = -100$	$\frac{1}{-0.1} = -10$	$\frac{1}{-1} = -1$	$\frac{1}{-10} = -0.1$	$\frac{1}{-100} = -0.01$	$\frac{1}{-1000} = -0.001$

- What does the table indicate about $f(x) = \frac{1}{x}$ as $x \rightarrow -\infty$ and as $x \rightarrow 0$ from the negative side?
 - It approaches 0 as $x \rightarrow -\infty$ and it approaches $-\infty$ as $x \rightarrow 0$ from the negative side.
- What pattern do you notice between the tables?
 - It appears that $f(-x) = -f(x)$.
- What type of function displays the type of pattern shown in the tables?
 - Odd functions.
- How can you prove that $f(x)$ is odd?
 - $f(x)$ is odd if $f(-x) = -f(x)$. We have $f(-x) = \frac{1}{-x}$ and $-f(x) = -\frac{1}{x}$. These expressions are indeed equivalent, so $f(x)$ is odd.
- Good. Now let's try analyzing $g(x) = \frac{1}{x^2}$. What can we say about the end behavior of this function? How does the function behave as x approaches 0 from the right and left sides?

Scaffolding:

- Recall that even functions such as $g(x) = \frac{1}{x^2}$ exhibit symmetry with respect to the y -axis.
- Odd functions such as $h(x) = \frac{1}{x^3}$ exhibit symmetry with respect to the origin.

x	-100	-10	-1	$-\frac{1}{10}$	$-\frac{1}{100}$	$\frac{1}{100}$	$\frac{1}{10}$	1	10	100
$g(x)$	$\frac{1}{10,000}$	$\frac{1}{100}$	1	100	10,000	10,000	100	1	$\frac{1}{100}$	$\frac{1}{10,000}$

- What does the table indicate about $g(x) = \frac{1}{x^2}$ as $x \rightarrow \infty$ and as $x \rightarrow 0$ from the positive side?
 - *It approaches 0 as $x \rightarrow \infty$, and it approaches ∞ as $x \rightarrow 0$ from the positive side.*
- What does the table indicate about $g(x) = \frac{1}{x^2}$ as $x \rightarrow -\infty$ and as $x \rightarrow 0$ from the negative side?
 - *It approaches 0 as $x \rightarrow -\infty$, and it approaches ∞ as $x \rightarrow 0$ from the negative side.*
- What pattern do you notice between the tables?
 - *It appears that there is symmetry.*
- What type of function displays the type of symmetry shown in the tables?
 - *Even functions.*
- How can you prove that $g(x) = \frac{1}{x^2}$ is even?
 - *A function is even if $g(-x) = g(x)$. We have $g(-x) = \frac{1}{(-x)^2}$, and $g(x) = \frac{1}{x^2}$. These expressions are equivalent, so $g(x)$ is even.*
- Now let's analyze the end behavior of $f(x) = \frac{1}{x^3}$ and $g(x) = \frac{1}{x^4}$.
 - *As $x \rightarrow \infty$, $f(x) \rightarrow 0$, and as $x \rightarrow -\infty$, $f(x) \rightarrow 0$.*
 - *As $x \rightarrow \infty$, $g(x) \rightarrow 0$, and as $x \rightarrow -\infty$, $g(x) \rightarrow 0$.*
- What about the behavior of $f(x)$ and $g(x)$ as the functions approach 0 from the right and left sides?
 - *$f(x) \rightarrow \infty$ as $x \rightarrow 0$ from the right side.*
 - *$f(x) \rightarrow -\infty$ as $x \rightarrow 0$ from the left side.*
 - *$g(x) \rightarrow \infty$ as $x \rightarrow 0$ from the right side.*
 - *$g(x) \rightarrow \infty$ as $x \rightarrow 0$ from the left side.*
- Can you see how to generalize your results?
 - *If n is any whole number, then as $x \rightarrow \infty$, $\frac{1}{x^n} \rightarrow 0$, and as $x \rightarrow -\infty$, $\frac{1}{x^n} \rightarrow 0$.*
 - *If n is any odd whole number, then as $x \rightarrow 0$ from the positive side, $\frac{1}{x^n} \rightarrow \infty$, and as $x \rightarrow 0$ from the negative side, $\frac{1}{x^n} \rightarrow -\infty$.*
 - *If n is any even whole number, then as $x \rightarrow 0$ from the positive side, $\frac{1}{x^n} \rightarrow \infty$, and as $x \rightarrow 0$ from the negative side, $\frac{1}{x^n} \rightarrow \infty$.*

Scaffolding:

- Recall that even functions such as $g(x) = \frac{1}{x^2}$ exhibit symmetry with respect to the y -axis.
- Odd functions such as $h(x) = \frac{1}{x^3}$ exhibit symmetry with respect to the origin.

MP.7
&
MP.8**Discussion: General Polynomials (5 minutes)**

- So far we've examined only the simplest rational functions $f(x) = \frac{1}{x^n}$. What can we say about the end behavior of a more complex rational function such as $f(x) = \frac{5x^3 - 2x^2 + 4x - 16}{2x^3 + 10x^2 - x + 4}$? To answer this question, we'll need to review some principles regarding the end behavior of polynomials.
- Let $g(x) = 5x^3 - 2x^2 + 4x - 16$ and $h(x) = 2x^3 + 10x^2 - x + 4$. Explain why the functions have similar end behavior.
 - *Answers will vary but should address that the functions are polynomials of the same degree and both have positive leading coefficients..*

MP.3

- We saw in the opening exercises that functions of the form $y = x^n$ are simple to analyze, but recall that things get a little more complicated when there are several terms involved. For instance, consider $f(x) = x^3 - 5x^2 - 3x - 1$. See what you can learn by computing $f(4)$ and $f(6)$ term by term. You may use a calculator for this purpose.
 - $f(4) = 64 - 80 - 12 - 1 = -29$
 - $f(6) = 216 - 180 - 18 - 1 = 17$
- In the case where $x = 4$, the second term is larger than the first, and so you ended up subtracting more than you started with, making the answer negative. But in the case where $x = 6$, the first term was large enough that the result was still positive even after performing three subtractions.
- More than anything else, it's the size of the exponent that matters in this analysis. You may recall that the end behavior of a polynomial is determined by the term with the largest exponent. For instance, in the example above, $f(x)$ has the same end behavior as the simple power function $g(x) = x^3$. Do you recall how to demonstrate this formally? Try factoring out x^3 from each term of $f(x)$ to see why its end behavior is similar to $g(x)$.
 - $f(x) = x^3 - 5x^2 - 3x - 1 = x^3 \cdot \left(1 - \frac{5}{x} - \frac{3}{x^2} - \frac{1}{x^3}\right)$
- Use this new form for $f(x)$ to compare $f(10^4)$ and $g(10^4)$.
 - $g(10^4) = (10^4)^3 = 10^{12}$
 - $f(10^4) = 10^{12} \cdot \left(1 - \frac{5}{10^4} - \frac{3}{10^8} - \frac{1}{10^{12}}\right)$
- Notice that for this large input, the output of f is about 99.9% of the value of g . If you use an even larger input, the relative value of f will grow even closer to that of g . Without actually doing the calculations, can you see intuitively why this is the case? Explain your thinking to a neighbor.
 - *Except for the first term, the terms in parentheses are extremely small, so the multiplier has a value that is very close to 1. This explains why the relative values of the two functions are very close when the input is a large number.*
- Let's examine each component in the expression $x^3 \cdot \left(1 - \frac{5}{x} - \frac{3}{x^2} - \frac{1}{x^3}\right)$. What happens when $x \rightarrow \infty$?
 - As $x \rightarrow \infty$, $x^3 \rightarrow \infty$, $1 \rightarrow 1$, $\frac{5}{x} \rightarrow 0$, $\frac{3}{x^2} \rightarrow 0$, and $\frac{1}{x^3} \rightarrow 0$.
 - Thus, $1 - \frac{5}{x} - \frac{3}{x^2} - \frac{1}{x^3} \rightarrow 1 - 0 - 0 - 0 = 1$.
- Good. This confirms our observation that $f(x)$ and $g(x)$ have similar end behavior. We can say that the function $g(x) = x^3$ is an end-behavior model for $f(x) = x^3 - 5x^2 - 3x - 1$.
- As it happens, knowing how to analyze the end behavior of a polynomial is really all that is needed to understand the end behavior of a rational function. Let's look at some examples.

Example 1 (6 minutes)

In this example, students use what they know about polynomials to analyze a rational function.

- Based on our reasoning with $f(x)$ and $g(x)$ above, what would be an end-behavior model for $f(x) = x^4 - 2x^2 + 6x - 3$?
 - $g(x) = x^4$

MP.3

- Consider the function $f(x) = \frac{5x^3 - 2x^2 + 4x - 16}{2x^3 + 10x^2 - x + 4}$. What can we say about the end behavior of this function? Well, let's break this problem up into two smaller problems: What if we consider the numerator and the denominator separately? Find an end-behavior model for the numerator and a model for the denominator, then take turns sharing your results with a partner.
 - The function $5x^3$ is an end-behavior model for the numerator.*
 - The function $2x^3$ is an end-behavior model for the denominator.*
- Now let's combine these observations: When we form the rational expression $f(x)$, does it seem reasonable that its end behavior is well approximated by the quotient $\frac{5x^3}{2x^3}$? And since $\frac{5x^3}{2x^3} = \frac{5}{2}$, it looks as though the outputs of f must get close to $\frac{5}{2}$ when x is very large. Use a calculator to verify this claim. Try, for example, using 10^{10} as an input to $f(x)$.
 - $f(10^{10}) \approx 2.499999999$
- You have to admit that is pretty darn close to $\frac{5}{2}$.
- The general principle here is to focus on the term with the highest power, ignoring all of the other terms. Looking at the quotient of the two terms with the largest power gives you an idea of what the end behavior of a rational function is like.
- Let's take one last look at this example using factoring. Factor out the highest power of x from the numerator and denominator of $f(x) = \frac{5x^3 - 2x^2 + 4x - 16}{2x^3 + 10x^2 - x + 4}$. This should help you to see how all of the pieces in this discussion fit together.

$$\square \text{ We have } f(x) = \frac{5x^3 - 2x^2 + 4x - 16}{2x^3 + 10x^2 - x + 4} = \frac{x^3 \left(5 - \frac{2}{x} + \frac{4}{x^2} - \frac{16}{x^3} \right)}{x^3 \left(2 + \frac{10}{x} - \frac{1}{x^2} + \frac{4}{x^3} \right)}.$$

- The quotient of $\frac{x^3}{x^3}$ is 1. The fractions in the expression are all approaching 0 as $x \rightarrow \infty$, so the*

expression $\frac{5 - \frac{2}{x} + \frac{4}{x^2} - \frac{16}{x^3}}{2 + \frac{10}{x} - \frac{1}{x^2} + \frac{4}{x^3}}$ is approaching $\frac{5}{2}$. This confirms our view that the function is approaching $\frac{5}{2}$ as

$x \rightarrow \infty$.

Example 2 (5 minutes)

- Now that we've seen that we can use our knowledge of polynomials to develop end-behavior models for rational functions, let's get some additional practice with this technique.
- Consider the function $f(x) = \frac{2x^3 + 3x - 1}{x^2 + x + 1}$. Find an end-behavior model for this function.
 - The function $2x^3$ is an end-behavior model for the numerator.*
 - The function x^2 is an end-behavior model for the denominator.*
 - Thus, the expression $\frac{2x^3}{x^2} = 2x$ is an end-behavior model for $f(x)$.*
- What conclusions can you draw about the end behavior of f based on your analysis?
 - We know that as $x \rightarrow \infty$, $2x \rightarrow \infty$, so we conclude that $f(x) \rightarrow \infty$ as well.*
 - Similarly, we know that as $x \rightarrow -\infty$, $2x \rightarrow -\infty$, so we conclude that $f(x) \rightarrow -\infty$ as well.*

- Use factoring to confirm that your analysis of the end behavior of f is correct.
 - $f(x) = \frac{2x^3+3x-1}{x^2+x+1} = \frac{x^3 \cdot (2 + \frac{3}{x^2} - \frac{1}{x^3})}{x^2 \cdot (1 + \frac{1}{x} + \frac{1}{x^2})} = x \cdot \frac{2 + \frac{3}{x^2} - \frac{1}{x^3}}{1 + \frac{1}{x} + \frac{1}{x^2}}$
 - Each fraction is approaching 0 as $x \rightarrow \infty$, so the expression on the right is approaching $\frac{2}{1}$. This confirms that $y = 2x$ is an end-behavior model for $y = f(x)$.
- So far, so good. Now let's use technology to further our understanding of this function. When you enter the function into your calculator, do the numerical results confirm your analysis?

x	100	500	1000	5000
$y = f(x)$	198	398	1998	9998

x	-100	-500	-1000	-5000
$y = f(x)$	-202	-1002	-2002	-10,002

- The numerical data confirms our views about the end behavior of $f(x)$.

Example 3 (4 minutes)

- Let's look at one final example together. What happens if we switch the numerator and the denominator in the previous example? We get the expression $f(x) = \frac{x^2+x+1}{2x^3+3x-1}$. What do you suppose the end behavior of this function is like? Make a conjecture and share it with a neighbor.
 - In the original function, the outputs got very large. Since this function is the reciprocal of the previous example, perhaps the outputs will be very small.
- Find an end-behavior model for this function.
 - The function x^2 is an end-behavior model for the numerator.
 - The function $2x^3$ is an end-behavior model for the denominator.
 - Thus, the expression $\frac{x^2}{2x^3} = \frac{1}{2x}$ is an end-behavior model for $f(x)$.
- What conclusions can you draw about the end behavior of f based on your analysis?
 - We know that as $x \rightarrow \infty$, $\frac{1}{x} \rightarrow 0$, so we conclude that $f(x) \rightarrow 0$ as well.
 - Similarly, we know that as $x \rightarrow -\infty$, $\frac{1}{x} \rightarrow 0$, so we conclude that $f(x) \rightarrow 0$ as well.
- Use your calculator to confirm your results numerically.

x	-200	-100	100	200
$y = f(x)$	-0.0025	-0.0049	0.0050	0.0025

- The numerical data confirms our views about the end behavior of $f(x)$.

Exercises (5 minutes)

Give students time to work on the following exercises. Encourage students to compare their answers with a partner. Select three students to present their work to the class and to explain their thinking.

Determine the end behavior of each rational function below.

1. $f(x) = \frac{7x^5 - 3x + 1}{4x^3 + 2}$

This function has the same end behavior as

$$\frac{7x^5}{4x^3} = \frac{7}{4}x^2.$$

Using this model as a guide, we conclude that $f(x) \rightarrow \infty$ as $x \rightarrow \infty$, and that $f(x) \rightarrow \infty$ as $x \rightarrow -\infty$.

2. $f(x) = \frac{7x^3 - 3x + 1}{4x^3 + 2}$

This function has the same end behavior as

$$\frac{7x^3}{4x^3} = \frac{7}{4}.$$

Using this model as a guide, we conclude that $f(x) \rightarrow \frac{7}{4}$ as $x \rightarrow \infty$, and that $f(x) \rightarrow \frac{7}{4}$ as $x \rightarrow -\infty$.

3. $f(x) = \frac{7x^3 + 2}{4x^5 - 3x + 1}$

This function has the same end behavior as

$$\frac{7x^3}{4x^5} = \frac{7}{4x^2}.$$

Using this model as a guide, we conclude that $f(x) \rightarrow 0$ as $x \rightarrow \infty$, and that $f(x) \rightarrow 0$ as $x \rightarrow -\infty$.

Closing (1 minutes)

- Take a minute to write a summary in your notebook of what you learned today.
 - *We can use what we know about the end behavior of polynomials to analyze the end behavior of rational functions. The key point to understand is that the term with the highest exponent determines the end behavior of a polynomial.*
 - *We also explored how to describe the behavior of rational functions as they approach restricted input values.*

Exit Ticket (5 minutes)

Name _____

Date _____

Lesson 12: End Behavior of Rational Functions

Exit Ticket

Given $f(x) = \frac{x+2}{x^2-1}$, find the following, and justify your findings.

- The end-behavior model for the numerator.
- The end-behavior model for the denominator.
- The end-behavior model for $f(x)$.
- What is the value of $f(x)$ as $x \rightarrow \infty$?
- What is the value of $f(x)$ as $x \rightarrow -\infty$?
- What is the value of $f(x)$ as $x \rightarrow 1$ from the positive and negative sides?
- What is the value of $f(x)$ as $x \rightarrow -1$ from the positive and negative sides?

Exit Ticket Sample Solutions

Given $f(x) = \frac{x+2}{x^2-1}$ find the following and justify your findings.

- a. The end-behavior model for the numerator.

$$x$$

The end-behavior model is $g(x) = x^n$ where n is the greatest power of the expression. Here, the numerator is $x + 2$, so $n = 1$, and the model is $g(x) = x$.

- b. The end-behavior model for the denominator.

$$x^2$$

The end-behavior model is $g(x) = x^n$ where n is the greatest power of the expression. Here, the denominator is $x^2 - 1$, so $n = 2$, and the model is $g(x) = x^2$.

- c. The end-behavior model for $f(x)$.

$$\frac{x}{x^2} = \frac{1}{x}$$

I replaced the numerator and denominator with the end-behavior models of each and simplified.

- d. What is the value of $f(x)$ as $x \rightarrow \infty$?

$$f(x) \rightarrow 0$$

As

$$x \rightarrow \infty, f(x) \rightarrow \frac{1}{\infty} = 0$$

- e. What is the value of $f(x)$ as $x \rightarrow -\infty$?

$$f(x) \rightarrow 0$$

As

$$x \rightarrow -\infty, f(x) \rightarrow \frac{1}{-\infty} = 0$$

- f. What is the value of $f(x)$ as $x \rightarrow 1$ from the positive and negative sides?

$f(x) \rightarrow \infty$ as $x \rightarrow 1$ from the positive side. As x nears 1 from the positive side, the numerator of the function approaches 3, and the denominator becomes a tiny positive number. The ratio of these is a very large positive number that exceeds any bounds.

$f(x) \rightarrow -\infty$ as $x \rightarrow 1$ from the negative side. As x nears 1 from the negative side, the numerator of the function approaches 3, and the denominator becomes a tiny negative number. The ratio of these is a very large negative number that exceeds any bounds.

- g. What is the value of $f(x)$ as $x \rightarrow -1$ from the positive and negative sides?

$f(x) \rightarrow -\infty$ as $x \rightarrow -1$ from the positive side. As x nears -1 from the positive side, the numerator of the function approaches 1, and the denominator becomes a tiny negative number. The ratio of these is a very large negative number that exceeds any bounds.

$f(x) \rightarrow \infty$ as $x \rightarrow -1$ from the negative side. As x nears -1 from the negative side, the numerator of the function approaches 1, and the denominator becomes a tiny positive number. The ratio of these is a very large positive number that exceeds any bounds.

Problem Set Sample Solutions

1. Analyze the end behavior of both functions.

a. $f(x) = x$, $g(x) = \frac{1}{x}$.

For $f(x)$: $f(x) \rightarrow \infty$ as $x \rightarrow \infty$, and $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$.

For $g(x)$: $g(x) \rightarrow \infty$ as $x \rightarrow 0$, and $g(x) \rightarrow -\infty$ as $x \rightarrow 0$.

b. $f(x) = x^3$, $g(x) = \frac{1}{x^3}$.

For $f(x)$: $f(x) \rightarrow \infty$ as $x \rightarrow \infty$, and $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$.

For $g(x)$: $g(x) \rightarrow \infty$ as $x \rightarrow 0$, and $g(x) \rightarrow -\infty$ as $x \rightarrow 0$.

c. $f(x) = x^2$, $g(x) = \frac{1}{x^2}$.

For $f(x)$: $f(x) \rightarrow \infty$ as $x \rightarrow \infty$, and $f(x) \rightarrow \infty$ as $x \rightarrow -\infty$.

For $g(x)$: $g(x) \rightarrow \infty$ as $x \rightarrow 0$, and $g(x) \rightarrow \infty$ as $x \rightarrow 0$.

d. $f(x) = x^4$, $g(x) = \frac{1}{x^4}$.

For $f(x)$: $f(x) \rightarrow \infty$ as $x \rightarrow \infty$, and $f(x) \rightarrow \infty$ as $x \rightarrow -\infty$.

For $g(x)$: $g(x) \rightarrow \infty$ as $x \rightarrow 0$, and $g(x) \rightarrow \infty$ as $x \rightarrow 0$.

e. $f(x) = x - 1$, $g(x) = \frac{1}{x-1}$.

For $f(x)$: $f(x) \rightarrow \infty$ as $x \rightarrow \infty$, and $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$.

For $g(x)$: $g(x) \rightarrow \infty$ as $x \rightarrow 1$, and $g(x) \rightarrow -\infty$ as $x \rightarrow 1$.

f. $f(x) = x + 2$, $g(x) = \frac{1}{x+2}$.

For $f(x)$: $f(x) \rightarrow \infty$ as $x \rightarrow \infty$, and $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$.

For $g(x)$: $g(x) \rightarrow 0$ as $x \rightarrow \infty$, and $g(x) \rightarrow 0$ as $x \rightarrow -\infty$.

g. $f(x) = x^2 - 4$, $g(x) = \frac{1}{x^2 - 4}$.

For $f(x)$: $f(x) \rightarrow \infty$ as $x \rightarrow \infty$, and $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$.

For $g(x)$: $g(x) \rightarrow \infty$ as $x \rightarrow 0$, and $g(x) \rightarrow -\infty$ as $x \rightarrow 0$.

2. For the following functions, determine the end behavior. Confirm your answer with a table of values.

a. $f(x) = \frac{3x-6}{x+2}$.

$f(x)$ has the same end behavior as $\frac{3x}{x} = 3$. Using this model as a guide, we conclude that

$f(x) \rightarrow 3$ as $x \rightarrow \infty$, and $f(x) \rightarrow 3$ as $x \rightarrow -\infty$.

x	100	500	1000	5000
$y = f(x)$	2.882	2.976	2.988	2.998

x	-100	-500	-1000	-5000
$y = f(x)$	-3.122	3.024	3.012	3.002

b. $f(x) = \frac{5x+1}{x^2-x-6}$.

$f(x)$ has the same end behavior as $\frac{5x}{x^2} = \frac{5}{x}$. Using this model as a guide, we conclude that

$f(x) \rightarrow 0$ as $x \rightarrow \infty$, and $f(x) \rightarrow 0$ as $x \rightarrow -\infty$.

x	100	500	1000	5000
$y = f(x)$	0.051	0.010	0.005	0.001

x	-100	-500	-1000	-5000
$y = f(x)$	-0.049	-0.010	-0.005	-0.001

c. $f(x) = \frac{x^3-8}{x^2-4}$.

$f(x)$ has the same end behavior as $\frac{x^3}{x^2} = x$. Using this model as a guide, we conclude that

$f(x) \rightarrow \infty$ as $x \rightarrow \infty$, and $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$.

x	100	500	1000	5000
$y = f(x)$	100	500	1000	5000

x	-100	-500	-1000	-5000
$y = f(x)$	-100	-500	-1000	-5000

d. $f(x) = \frac{x^3-1}{x^4-1}$.

$f(x)$ has the same end behavior as $\frac{x^3}{x^4} = \frac{1}{x}$. Using this model as a guide, we conclude that

$f(x) \rightarrow 0$ as $x \rightarrow \infty$, and $f(x) \rightarrow 0$ as $x \rightarrow -\infty$.

x	100	500	1000	5000
$y = f(x)$	0.010	0.002	0.001	0.0002

x	-100	-500	-1000	-5000
$y = f(x)$	-0.010	-0.002	-0.001	-0.0002

e. $f(x) = \frac{(2x+1)^3}{(x^2-x)^2}$.

$f(x)$ has the same end behavior as $\frac{8x^3}{x^4} = \frac{8}{x}$. Using this model as a guide, we conclude that

$f(x) \rightarrow 0$ as $x \rightarrow \infty$, and $f(x) \rightarrow 0$ as $x \rightarrow -\infty$.

x	100	500	1000	5000
$y = f(x)$	0.083	0.016	0.008	0.0016

x	-100	-500	-1000	-5000
$y = f(x)$	-0.077	-0.016	-0.008	-0.0016

3. For the following functions, determine the end behavior.

a. $f(x) = \frac{5x^6-3x^3+x-2}{5x^4-3x^3+x-2}$.

$f(x)$ has the same end behavior as $\frac{5x^6}{5x^4} = x^2$. Using this model as a guide, we conclude that

$f(x) \rightarrow \infty$ as $x \rightarrow \infty$, and $f(x) \rightarrow \infty$ as $x \rightarrow -\infty$.

b. $f(x) = \frac{5x^4-3x^3+x-2}{5x^6-3x^3+x-2}$.

$f(x)$ has the same end behavior as $\frac{5x^4}{5x^6} = \frac{1}{x^2}$. Using this model as a guide, we conclude that

$f(x) \rightarrow 0$ as $x \rightarrow \infty$, and $f(x) \rightarrow 0$ as $x \rightarrow -\infty$.

c. $f(x) = \frac{5x^4-3x^3+x-2}{5x^4-3x^3+x-2}$.

$f(x)$ has the same end behavior as $\frac{5x^4}{5x^4} = 1$. Using this model as a guide, we conclude that

$f(x) \rightarrow 1$ as $x \rightarrow \infty$, and $f(x) \rightarrow 1$ as $x \rightarrow -\infty$.

d. $f(x) = \frac{\sqrt{2}x^2 + x + 1}{3x + 1}$.

$f(x)$ has the same end behavior as $\frac{\sqrt{2}x^2}{3x} = \frac{\sqrt{2}}{3}x$. Using this model as a guide, we conclude that

$f(x) \rightarrow \infty$ as $x \rightarrow \infty$, and $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$.

e. $f(x) = \frac{4x^2 - 3x - 7}{2x^3 + x - 2}$.

$f(x)$ has the same end behavior as $\frac{4x^2}{2x^3} = \frac{2}{x}$.

$f(x) \rightarrow 0$ as $x \rightarrow \infty$, and $f(x) \rightarrow 0$ as $x \rightarrow -\infty$.

4. Determine the end behavior of each function.

a. $f(x) = \frac{\sin x}{x}$.

$-1 \leq \sin x \leq 1$,

$f(x) \rightarrow 0$ as $x \rightarrow \infty$, and $f(x) \rightarrow 0$ as $x \rightarrow -\infty$.

b. $f(x) = \frac{\cos x}{x}$.

$-1 \leq \cos x \leq 1$,

$f(x) \rightarrow 0$ as $x \rightarrow \infty$, and $f(x) \rightarrow 0$ as $x \rightarrow -\infty$.

c. $f(x) = \frac{2^x}{x}$.

$f(x) \rightarrow \infty$ as $x \rightarrow \infty$, and $f(x) \rightarrow 0$ as $x \rightarrow -\infty$.

d. $f(x) = \frac{x}{2^x}$.

$f(x) \rightarrow 0$ as $x \rightarrow \infty$, and $f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$.

e. $f(x) = \frac{4}{1 + e^{-x}}$.

$f(x) \rightarrow 4$ as $x \rightarrow \infty$, and $f(x) \rightarrow 0$ as $x \rightarrow -\infty$.

f. $f(x) = \frac{10}{1 + e^{-x}}$.

$f(x) \rightarrow 10$ as $x \rightarrow \infty$, and $f(x) \rightarrow 0$ as $x \rightarrow -\infty$.

5. Consider the functions $f(x) = x!$ and $g(x) = x^5$ for natural numbers x .

- a. What are the values of $f(x)$ and $g(x)$ for $x = 5, 10, 15, 20, 25$?

$$f(x) = \{120, 3628800, 1307674368000, 2.4 \times 10^{18}, 1.6 \times 10^{25}\}$$

$$g(x) = \{3125, 100000, 759375, 3200000, 9765625\}$$

- b. What is the end behavior of $f(x)$ as $x \rightarrow \infty$.

$$f(x) \rightarrow \infty$$

- c. What is the end behavior of $g(x)$ as $x \rightarrow \infty$.

$$g(x) \rightarrow \infty$$

- d. Make an argument for the end behavior of $\frac{f(x)}{g(x)}$ as $x \rightarrow \infty$.

Since $y = f(x)$ increases so much faster than $y = g(x)$, $f(x)$ will overpower the division by $g(x)$. By $x = 25$, $\frac{f(x)}{g(x)} \approx 1.6 \times 10^{18}$. Thus, $\frac{f(x)}{g(x)} \rightarrow \infty$.

- e. Make an argument for the end behavior of $\frac{g(x)}{f(x)}$ as $x \rightarrow \infty$.

For the same reasons as in part (d), division by $f(x)$ will overpower $g(x)$ in the numerator. By $x = 25$, $\frac{g(x)}{f(x)} \approx 0.0000000000000000$. Thus, $\frac{g(x)}{f(x)} \rightarrow 0$.

6. Determine the end behavior of the functions.

- a. $f(x) = \frac{x}{x^2}$, $g(x) = \frac{1}{x}$.

$$f(x) \rightarrow 0 \text{ as } x \rightarrow \infty, \text{ and } f(x) \rightarrow 0 \text{ as } x \rightarrow -\infty.$$

$$g(x) \rightarrow 0 \text{ as } x \rightarrow \infty, \text{ and } g(x) \rightarrow 0 \text{ as } x \rightarrow -\infty.$$

- b. $f(x) = \frac{x+1}{x^2-1}$, $g(x) = \frac{1}{x-1}$.

$$f(x) \rightarrow 0 \text{ as } x \rightarrow \infty, \text{ and } f(x) \rightarrow 0 \text{ as } x \rightarrow -\infty.$$

$$g(x) \rightarrow 0 \text{ as } x \rightarrow \infty, \text{ and } g(x) \rightarrow 0 \text{ as } x \rightarrow -\infty.$$

- c. $f(x) = \frac{x-2}{x^2-x-2}$, $g(x) = \frac{1}{x+1}$.

$$f(x) \rightarrow 0 \text{ as } x \rightarrow \infty, \text{ and } f(x) \rightarrow 0 \text{ as } x \rightarrow -\infty.$$

$$g(x) \rightarrow 0 \text{ as } x \rightarrow \infty, \text{ and } g(x) \rightarrow 0 \text{ as } x \rightarrow -\infty.$$

d. $f(x) = \frac{x^3-1}{x-1}$, $g(x) = x^2 + x + 1$.

$f(x) \rightarrow \infty$ as $x \rightarrow \infty$, and $f(x) \rightarrow \infty$ as $x \rightarrow -\infty$.

$g(x) \rightarrow \infty$ as $x \rightarrow \infty$, and $g(x) \rightarrow \infty$ as $x \rightarrow -\infty$.

7. Use a graphing utility to graph the following functions $f(x)$ and $g(x)$. Explain why they have the same graphs. Determine the end behavior of the functions and whether the graphs have any horizontal asymptotes.

a. $f(x) = \frac{x+1}{x-1}$, $g(x) = 1 + \frac{2}{x-1}$.

The graphs are the same because the expressions are equivalent: $1 + \frac{2}{x-1} = \frac{x-1}{x-1} + \frac{2}{x-1} = \frac{x+1}{x-1}$ for all $x \neq 1$.

$f(x) \rightarrow 1$ as $x \rightarrow \infty$, and $f(x) \rightarrow 1$ as $x \rightarrow -\infty$.

$g(x) \rightarrow 1$ as $x \rightarrow \infty$, and $g(x) \rightarrow 1$ as $x \rightarrow -\infty$.

b. $f(x) = \frac{-2x+1}{x+1}$, $g(x) = \frac{3}{x+1} - 2$.

The graphs are the same because the expressions are equivalent: $\frac{3}{x+1} - 2 = \frac{3}{x+1} - \frac{2(x+1)}{x+1} = \frac{3-2x-2}{x+1} = \frac{-2x+1}{x+1}$ for all $x \neq -1$.

$f(x) \rightarrow -2$ as $x \rightarrow \infty$, and $f(x) \rightarrow -2$ as $x \rightarrow -\infty$.

$g(x) \rightarrow -2$ as $x \rightarrow \infty$, and $g(x) \rightarrow -2$ as $x \rightarrow -\infty$.