Lesson 11: Rational Functions

Student Outcomes

- Students reduce rational expressions to lowest terms.
- Students determine the domain of rational functions.

Lesson Notes

In Algebra II, students reduced rational expressions to lowest terms and performed arithmetic operations with them, in preparation for solving rational equations. In the previous lesson, students verified that rational expressions are closed under addition, subtraction, multiplication, and division. In this lesson, students will first review the concept of equivalent rational expressions from Algebra II, Module 1, Lesson 22. They attend to precision (MP.6) in keeping track of the values of the variable that must be excluded from the domain to avoid division by zero. This lesson then introduces rational functions as functions that can be written as quotients of two polynomial functions. Then, students determine whether functions are rational and identify their domain (range will be addressed later when they graph rational functions). Reviewing the process of reducing rational expressions to lowest terms prepares students for later lessons in which they graph and compose rational functions.

Classwork

Opening Exercise (4 minutes)

The Opening Exercise gets students thinking about factoring polynomial expressions, which is a skill they need to complete their work with rational functions in this and subsequent lessons. The students should complete this exercise independently. After a few minutes, select students to share their solutions. The factored expressions could also be written on individual whiteboards for quick checks.

Scaffolding:
- Cue students to look for patterns that can help them factor the expressions, for example, difference of squares, common factors, or binomial theorem.
- Ask students to consider simpler examples, such as $x^2 - 9 = (x-3)(x+3)$; $y^2 + 2y - 15 = (y-3)(y+5)$; $a^2 + 5a + 4 = (a+4)(a+1)$.

Factor each expression completely:

a. $9x^4 - 16x^2$

$x^2(3x+4)(3x-4)$

b. $2x^3 + 5x^2 - 8x - 20$

$(x^2-4)(2x+5) = (x+2)(x-2)(2x+5)$

c. $x^3 + 3x^2 + 3x + 1$

$(x+1)^3$

d. $8x^3 - 1$

$(2x-1)(4x^2+2x+1)$

Discussion: Equivalent Rational Expressions (5 minutes)

In Algebra II, Module 1, Lesson 22, students practiced reducing rational expressions to lowest terms, taking care to note values of the variable that must be excluded to avoid division by zero. In this lesson, we extend this idea to finding the domain of a rational function. Use this discussion to reactivate students' knowledge of rational expressions and reducing a rational expression to lowest terms.

- Recall that in Algebra II and the previous lesson we described rational expressions as expressions that can be put into the form $\frac{P}{Q}$ where P and Q are polynomial expressions, and Q is not the zero polynomial. For example, $\frac{x}{x^2-3x+2}$, x^2+1, $\frac{x^3-1}{x^2+2}$, 0, and $1+\frac{3}{x}$ are all rational expressions.
- What does it mean for two rational expressions to be equivalent?
  - *That although the expressions may be in different forms, each expression takes on the same value for any value of the variables. That is, if we substitute a value such as 3 for x into each expression, the values of the expressions are the same.*
- Notice that $\frac{x}{x^3+x} = \frac{x}{x(x^2+1)}$. Are $\frac{x}{x^3+x}$ and $\frac{1}{x^2+1}$ equivalent rational expressions?
  - *No. The first expression is undefined for $x = 0$, but the second is defined for all values of x. Thus, they are not equivalent expressions.*
- What should we do to make these equivalent expressions?
  - *Excluding the value of 0 from the set of possible values of x makes both expressions equivalent because $\frac{x}{x^3+x} = \frac{1}{x^2+1}$ only for $x \neq 0$.*
- What does it mean to reduce a rational expression to lowest terms?
  - *We divide any common factors from the numerator and denominator, leaving polynomials of the lowest possible degree.*

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- What do we need to pay attention to in order to ensure that as we reduce a rational expression to lowest terms we ensure that the resulting rational expressions are equivalent to the original one?
  - *We need to exclude any value of the variable that caused division by zero in the original expression.*

Example 1 (5 minutes)

This example provides a review of reducing rational expressions to lowest terms from Algebra II, Module 1. It is important to emphasize excluding the value $x = 3$ from the possible values for x as we reduce the expression.

Example 1

Reduce the expression $\frac{x^2-5x+6}{x-3}$ to lowest terms and identify the value(s) of x that must be excluded to avoid division by zero.

Give students time to think about how to approach this task before leading them through a solution.

Since $x^2 - 5x + 6 = (x-2)(x-3)$, our original expression can be written as $\frac{x^2-5x+6}{x-3} = \frac{(x-2)(x-3)}{x-3}$. To reduce this expression to lowest terms, we need to divide the numerator and denominator by any common factors. The only common factor in this example is $x - 3$. However, we can only divide by $x - 3$ if $x - 3 \neq 0$, which means that we have to exclude 3 as a possible value of x.

Thus, if $x \neq 3$ we have

$$\begin{aligned}\frac{x^2-5x+6}{x-3} &= \frac{(x-2)(x-3)}{x-3} = \\ &= \frac{(x-2)(x-3)}{x-3} \cdot \frac{\frac{1}{x-3}}{\frac{1}{x-3}} \\ &= x-2\end{aligned}$$

So as long as $x \neq 3$, the expressions $\frac{x^2-5x+6}{x-3}$ and $x - 2$ are equivalent.

Exercise 1: Reducing Rational Expressions to Lowest Terms (6 minutes)

Exercise 1

1. Reduce each rational expression to lowest terms, specifying the values of x that must be excluded to avoid division by zero.

a. $\frac{x^2-6x+5}{x^2-3x-10}$

The denominator factors into $x^2 - 3x - 10 = (x-5)(x+2)$, so to avoid division by zero we must have $x \neq 5$ and $x \neq -2$. Thus, $\frac{x^2-6x+5}{x^2-3x-10} = \frac{(x-5)(x-1)}{(x-5)(x+2)} = \frac{x-1}{x+2}$, where $x \neq -2$ and $x \neq 5$.

b. $\frac{x^3+3x^2+3x+1}{x^3+2x^2+x}$

The denominator factors into $x^3 + 2x^2 + x = x(x+1)^2$, so to avoid division by zero we must have $x \neq 0$ and $x \neq -1$. Thus, $\frac{x^3+3x^2+3x+1}{x^3+2x^2+x} = \frac{(x+1)^3}{x(x+1)^2} = \frac{x+1}{x}$, where $x \neq 0$ and $x \neq 1$.

c. $\frac{x^2-16}{x^2+2x-8}$

The denominator factors into $x^2 + 2x - 8 = (x-2)(x+4)$, so to avoid division by zero we must have $x \neq 2$ and $x \neq -4$. Thus, $\frac{x^2-16}{x^2+2x-8} = \frac{(x-4)(x+4)}{(x-2)(x+4)} = \frac{x-4}{x-2}$, where $x \neq 2$ and $x \neq -4$.

d. $\frac{x^2-3x-10}{x^3+6x^2+12x+8}$

The denominator factors into $x^3 + 6x^2 + 12x + 8 = (x+2)^3$ so to avoid division by zero we must have $x \neq -2$. Thus, $\frac{x^2-3x-10}{x^3+6x^2+12x+8} = \frac{(x-5)(x+2)}{(x+2)^3} = \frac{x-5}{(x+2)^2}$, where $x \neq -2$.

e. $\frac{x^3+1}{x^2+1}$

While $x^3+1=(x+1)(x^2-x+1)$, the polynomial expression in the denominator does not factor. Thus, this expression is already reduced to lowest terms. Since $x^2+1\geq 1$ for all values of x, the denominator is never zero. Thus, there are no values of x that need to be excluded.

Discussion: Identifying Rational Functions (5 minutes)

This discussion describes rational functions as those that can be written as the quotient of two polynomial functions. Continue to emphasize the domain of a rational function through this discussion and throughout the lesson.

- We are now interested in using rational expressions to define functions.
- Remember that a function $f: X \to Y$ is a correspondence between two sets X and Y. To specify a function, we need to know its domain and the rule used to match elements of X to elements of Y. We now want to define functions whose rule of assignment can be described using rational expressions.
- A rational function is a function whose rule of assignment can be written in the form $f(x)=\frac{P(x)}{Q(x)}$ where P and Q are polynomial functions, and Q is not the zero polynomial. What can you recall about the structure of a polynomial function?
  - *It can be written in the form $p(x)=a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+\cdots a_1x+a_0$, where $a_n, a_{n-1}, \ldots, a_0$ are real numbers and n is a whole number.*
- Is the function $f(x)=\frac{x^2+5x+4}{x^2-16}$ a rational function? Explain.
  - *Yes: The numerator and denominator of f are both polynomial functions.*
- Let's see if we can use our definition to classify some more complicated functions. Which of the functions shown here are rational functions? Explain how you know.

$$f(x)=\frac{5x^3-6x+2}{\pi x^2}$$
$$g(x)=\frac{x^{200}}{x^{200}-1}$$
$$h(x)=\sqrt{2x+1}-2$$
$$j(x)=17$$
$$k(x)=\frac{\cos(x)}{x^2+1}$$

  - *Both f and g are rational functions because both the numerator and denominator of each function are polynomials, e.g., the terms have real-numbered coefficients and powers of x that are integers. The function h is not rational because it cannot be written as a quotient of polynomial functions. Function j is a rational function with numerator $P(x)=17$ and denominator $Q(x)=1$, and k is not a rational function because $P(x)=\cos(x)$ cannot be written as a polynomial function.*

Example 2 (5 minutes)

This example demonstrates how the rule of a rational function can be expressed in an equivalent form by dividing the numerator and denominator by common factors and explicitly stating a restricted domain. The exercise should be completed in pairs and, after a few minutes, the responses should be reviewed as part of a teacher-led discussion. Alternatively, the example could be completed as part of a teacher-led discussion.

- How can we simplify the expression of the function $f(x) = \frac{x^2+5x+4}{x^2-16}$?
  - *Factor the numerator and denominator, and look for common factors that can be divided out.*
- What characteristics of the denominator could help us to factor it?
  - *Answers may vary but should address the presence a difference of squares in the denominator.*
- Why can't we rewrite the equation for f as $f(x) = \frac{x+1}{x-4}$?
  - *Without indicating the restricted values on the domain of the function, there is no way to tell from the reduced expression that the function is undefined at $x = 4$. Thus, if we don't explicitly identify the additional restriction $x \neq 4$ on the domain, we don't have the same function.*
- And how do we know that 4 and -4 are restricted values not in the domain of f?
  - *The denominator of the function is 0 for each of these values of x, which results in the function being undefined.*
- Remember that a function is a rule and a domain, so if we change the domain we have substantially changed the function. When we reduce the expression that defines a rational function, how can we make sure that we do not change the domain?
  - *Answers may vary but should address identifying the restricted values from the factored form of the original function before it is reduced to lowest terms. The restricted values represent those numbers that, when substituted into the function, produce a fraction with a denominator equal to 0.*
- And how can we write a rational function so that its expression has been reduced to lowest terms and the restricted domain values are indicated?
  - *Answers will vary. Lead students toward the response that they can write the reduced expression of the function together with an explicit statement of the excluded values of the variable.*

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Example 2

Let $f(x) = \frac{2x^4+6x^3+6x^2+2x}{3x^2+3x}$. Reduce the rational expression $\frac{2x^4+6x^3+6x^2+2x}{3x^2+3x}$ to lowest terms and use the reduced form to express the rule of f. Be sure to indicate any restrictions on the domain.

$$\frac{2x^4+6x^3+6x^2+2x}{3x^2+3x} = \frac{2x(x^3+3x^2+3x+1)}{3x(x+1)} = \frac{2x(x+1)^3}{3x(x+1)} = \frac{2(x+1)^2}{3} \text{ if}$$

$x \neq -1$ and $x \neq 0$. Then $f(x) = \frac{2(x+1)^2}{3}$ for $x \neq 0$ and $x \neq -1$.

Scaffolding:

Cue students to look for a binomial pattern to help them factor the numerator of f.

Exercise 2 (7 minutes)

Have students complete this exercise in pairs. After a few minutes, select students to share their responses. If individual whiteboards are available, students could write their answers on the boards for quick checks.

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Exercise 2

2. Determine the domain of each rational function and express the rule for each function in an equivalent form in lowest terms.

a. $f(x) = \frac{(x+2)^2(x-3)(x+1)}{(x+2)(x+1)}$

The domain is all real numbers x so that $x \neq -1$ and $x \neq -2$.

$f(x) = \frac{(x+2)^2(x-3)(x+1)}{(x+2)(x+1)} = (x+2)(x-3)$ *for* $x \neq -1$ *and* $x \neq -2$

b. $f(x) = \frac{x^2-6x+9}{x-3}$

The domain is all real numbers x so that $x \neq 3$.

$f(x) = \frac{(x-3)^2}{x-3} = x-3$ for $x \neq 3$

c. $f(x) = \frac{3x^3-75x}{x^3+15x^2+75x+125}$

The domain is all real numbers x so that $x \neq -5$.

$f(x) = \frac{3x(x^2-25)}{(x+5)^3} = \frac{3x(x+5)(x-5)}{(x+5)^3} = \frac{3x(x-5)}{(x+5)^2}$ for $x \neq -5$

Scaffolding:

Have advanced students form conjectures about the range of each function.

Closing (3 minutes)

Have students reflect on the questions below. After a minute, ask them to share their thoughts with a partner.

- How do we identify the domain of a rational function?
  - *A rational function has the domain of all real numbers except for any value of x that causes division by zero.*
- Explain why the functions $f(x) = \frac{x}{x-3}$ and $g(x) = \frac{x(x-1)}{(x-1)(x-3)}$ are not the same function.
  - *The first function $f(x) = \frac{x}{x-3}$ is defined for all $x \neq 3$, but the second function $g(x) = \frac{x(x-1)}{(x-1)(x-3)}$ is defined for $x \neq 3$ and $x \neq 1$. Since the two functions do not agree for every value of x, they are not the same function.*
  - *The two functions have different domains so they are not the same function.*

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 11: Rational Functions

Exit Ticket

1. Identify whether the functions shown are rational:

   a. $f(x) = \frac{x}{x^2+1}$

   b. $f(x) = \frac{\sqrt{x}}{x^2+1}$

   c. $f(x) = \frac{x}{x^{2/3}+1}$

   d. $f(x) = \left(\frac{x}{x^2+1}\right)^2$

   e. $f(x) = \frac{\sqrt{2}x}{ex^2+\sqrt{\pi}}$

2. Anmol says $f(x) = \frac{x+1}{x^2-1}$ and $g(x) = \frac{1}{x-1}$ represent the same function. Is she correct? Justify your answer.

Exit Ticket Sample Solutions

1. Identify whether the functions shown are rational:

 a. $f(x) = \frac{x}{x^2+1}$

 Yes. Both $P(x) = x$ and $Q(x) = x^2 + 1$ are polynomial functions.

 b. $f(x) = \frac{\sqrt{x}}{x^2+1}$

 No. The function $P(x) = \sqrt{x}$ is not a polynomial function.

 c. $f(x) = \frac{x}{x^{0.4}+1}$

 No. The function $Q(x) = x^{0.4} + 1$ is not a polynomial function.

 d. $f(x) = \left(\frac{x}{x^2+1}\right)^2$

 Yes. When multiplied out, $f(x) = \frac{x^2}{x^4+2x^2+1}$, so f is the quotient of two polynomial functions.

 e. $f(x) = \frac{\sqrt{2}x}{ex^2+\sqrt{\pi}}$

 Yes. While the coefficients are not integers, $P(x) = \sqrt{2}x$ and $Q(x) = e\,x^2 + \sqrt{\pi}$ are both polynomial functions since all the powers of x are whole numbers.

2. Anmol says $f(x) = \frac{x+1}{x^2-1}$ and $g(x) = \frac{1}{x-1}$ represent the same function. Is she correct? Justify your answer.

 She is not correct.

 The function $f(x) = \frac{x+1}{x^2-1}$ is not defined for $x = 1$ and $x = -1$. However, the function $g(x) = \frac{1}{x-1}$ is not defined for $x = 1$. These two functions do not have the same domain, so they are not the same function.

Problem Set Sample Solutions

1. For each pair of functions f and g, find the domain of f and the domain of g. Indicate whether f and g are the same function.

 a. $f(x) = \frac{x^2}{x}$, $g(x) = x$

 The domain of f is all real numbers x with $x \neq 0$. The domain of g is all real numbers x.

 No, functions f and g are not the same function.

b. $f(x) = \frac{x}{x}$, $g(x) = 1$

The domain of f is all real numbers x with $x \neq 0$. The domain of g is all real numbers x.

No, functions f and g are not the same function.

c. $f(x) = \frac{2x^2+6x+8}{2}$, $g(x) = x^2 + 6x + 8$

The domain of f is all real numbers x. The domain of g is all real numbers x.

Yes, functions f and g are the same function.

d. $f(x) = \frac{x^2+3x+2}{x+2}$, $g(x) = x + 1$

The domain of f is all real numbers x with $x \neq -2$. The domain of g is all real numbers x.

No, functions f and g are not the same function.

e. $f(x) = \frac{x+2}{x^2+3x+2}$, $g(x) = \frac{1}{x+1}$

The domain of f is all real numbers x with $x \neq -2$ and $x \neq -1$. The domain of g is all real numbers x with $x \neq -1$.

No, functions f and g are not the same function.

f. $f(x) = \frac{x^4-1}{x^2-1}$, $g(x) = x^2 + 1$

The domain of f is all real numbers x with $x \neq 1$ and $x \neq -1$. The domain of g is all real numbers x.

No, functions f and g are not the same function.

g. $f(x) = \frac{x^4-1}{x^2+1}$, $g(x) = x^2 - 1$

Because $x^2 + 1$ is never zero, the domain of f is all real numbers x. The domain of g is all real numbers x.

Yes, functions f and g are the same function.

h. $f(x) = \frac{x^4-x}{x^2+x}$, $g(x) = \frac{x^3-1}{x+1}$

The domain of f is all real numbers x with $x \neq 0$ and $x \neq -1$. The domain of g is all real numbers x with $x \neq -1$.

No, functions f and g are not the same function.

i. $f(x) = \frac{x^4+x^3+x^2}{x^2+x+1}$, $g(x) = x^2$

Because $x^2 + x + 1$ doesn't factor, the denominator of f is never zero and the domain of f is all real numbers x. The domain of g is also all real numbers x.

Yes, functions f and g are the same function.

2. **Determine the domain of each rational function and express the rule for each function in an equivalent form in lowest terms.**

a. $f(x) = \frac{x^4}{x^2}$

The domain of f is all real numbers x with $x \neq 0$.

$f(x) = \frac{x^4}{x^2} = x^2$, *where* $x \neq 0$ *and* $x \neq -1$

b. $f(x) = \frac{3x+3}{15x-6}$

Because $15x - 6 = 3(5x - 2)$, *the domain of f is all real numbers x with* $x \neq \frac{2}{5}$.

$$f(x) = \frac{3(x+1)}{3(5x-2)} = \frac{x+1}{5x-2}, \text{ where } x \neq \frac{2}{5}$$

c. $f(x) = \frac{x^2-x-2}{x^2+x}$

Because $x^2 + x = x(x+1)$, *the domain of f is all real numbers x with* $x \neq 0$ *and* $x \neq -1$.

$$f(x) = \frac{x^2-x-2}{x^2+x} = \frac{(x-2)(x+1)}{x(x+1)} = \frac{x-2}{x}, \text{ where } x \neq 0 \text{ and } x \neq -1$$

d. $f(x) = \frac{8x^2+2x-15}{4x^2-4x-15}$

Because $4x^2 - 4x - 15 = (2x+3)(2x-5)$, *the domain of f is all real numbers x with* $x \neq -\frac{3}{2}$ *and* $x \neq \frac{5}{2}$.

$$f(x) = \frac{8x^2+2x-15}{4x^2-4x-15} = \frac{(2x+3)(4x-5)}{(2x+3)(2x-5)} = \frac{(4x-5)}{(2x-5)}, \text{ where } x \neq \frac{5}{2} \text{ and } x \neq -\frac{3}{2}$$

e. $f(x) = \frac{2x^3-3x^2-2x+3}{x^3-x}$

Because $\mathrm{x}^3 - \mathrm{x} = \mathrm{x}(\mathrm{x}-1)(\mathrm{x}+1)$, *the domain of f is all real numbers x with* $\mathrm{x} \neq 0$, $\mathrm{x} \neq 1$ *and* $\mathrm{x} \neq -1$.

$$f(x) = \frac{2x^3-3x^2-2x+3}{x^3-x} = \frac{(2x-3)(x^2-1)}{x(x^2-1)} = \frac{2x-3}{x}, \text{ where } x \neq 0 \text{ and } x \neq \pm 1$$

f. $f(x) = \frac{3x^3+x^2+3x+1}{x^3+x}$

Because $x^3 + x = x(x^2+1)$, *the domain of f is all real numbers x with* $x \neq 0$.

$$f(x) = \frac{3x^3+x^2+3x+1}{x^3+x} = \frac{(3x+1)(x^2+1)}{x(x^2+1)} = \frac{3x+1}{x}, \text{ where } x \neq 0$$

3. **For each pair of functions below, calculate $f(x)+g(x), f(x)-g(x), f(x)\cdot g(x)$ and $\frac{f(x)}{g(x)}$. Indicate restrictions on the domain of the resulting functions.**

a. $f(x)=\frac{2}{x}, g(x)=\frac{x}{x+2}$

$f(x)+g(x)=\frac{2}{x}+\frac{x}{x+2}=\frac{2(x+2)}{x(x+2)}+\frac{x(x)}{(x+2)(x)}=\frac{x^2+2x+4}{x(x+2)}$, *where* $x\neq 0,-2$

$f(x)-g(x)=\frac{2}{x}-\frac{x}{x+2}=\frac{2(x+2)}{x(x+2)}-\frac{x(x)}{(x+2)(x)}=\frac{-x^2+2x+4}{x(x+2)}$, *where* $x\neq 0,-2$

$f(x)\cdot g(x)=\frac{2}{x}\cdot\frac{x}{x+2}=\frac{2x}{x(x+2)}=\frac{2}{(x+2)}$, *where* $x\neq 0,-2$

$\frac{f(x)}{g(x)}=\frac{2}{x}\div\frac{x}{x+2}=\frac{2}{x}\cdot\frac{x+2}{x}=\frac{2x+4}{x^2}$, *where* $x\neq 0,-2$

b. $f(x)=\frac{3}{x+1}, g(x)=\frac{x}{x^3+1}$

$f(x)+g(x)=\frac{3}{x+1}+\frac{x}{x^3+1}=\frac{3(x^2-x+1)}{(x+1)(x^2-x+1)}+\frac{x}{x^3+1}=\frac{3x^2-2x+3}{x^3+1}$, *where* $x\neq -1$

$f(x)-g(x)=\frac{3}{x+1}-\frac{x}{x^3+1}=\frac{3(x^2-x+1)}{(x+1)(x^2-x+1)}-\frac{x}{x^3+1}=\frac{3x^2-4x+3}{x^3+1}$, *where* $x\neq -1$

$f(x)\cdot g(x)=\frac{3}{x+1}\cdot\frac{x}{x^3+1}=\frac{3x}{(x+1)(x^3+1)}$, *where* $x\neq -1$

$\frac{f(x)}{g(x)}=\frac{3}{x+1}\div\frac{x}{x^3+1}=\frac{3}{(x+1)}\cdot\frac{x^3+1}{x}=\frac{3(x+1)(x^2-x+1)}{x(x+1)}=\frac{3(x^2-x+1)}{x}$, *where* $x\neq -1, 0$