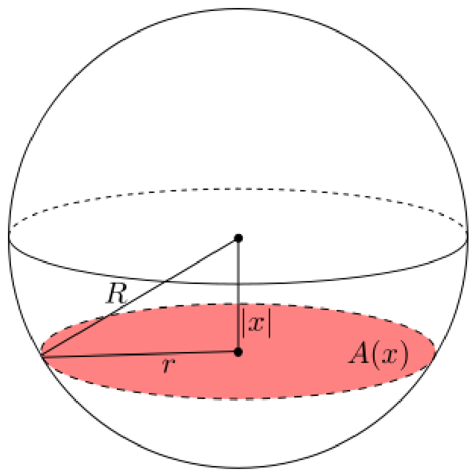
Lesson 9: Volume and Cavalieri's Principle

Classwork

Exercises 1–3

1. Let $R = 5$, and let $A(x)$ represent the area of a cross section for a circle at a distance x from the center of the sphere.



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a. Find $A(0)$. What is special about this particular cross section?

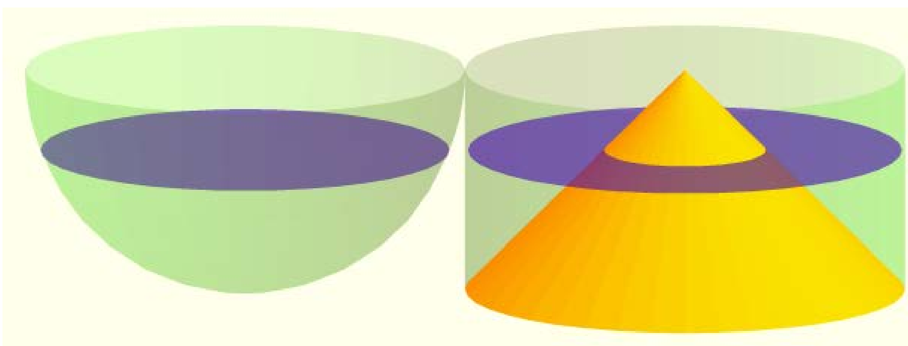
b. Find $A(1)$.

c. Find $A(3)$.

d. Find $A(4)$.

e. Find $A(5)$. What is special about this particular cross section?

2. Let the radius of the cylinder be $R = 5$, and let $B(x)$ represent the area of the blue ring when the slicing plane is at a distance x from the top of the cylinder.



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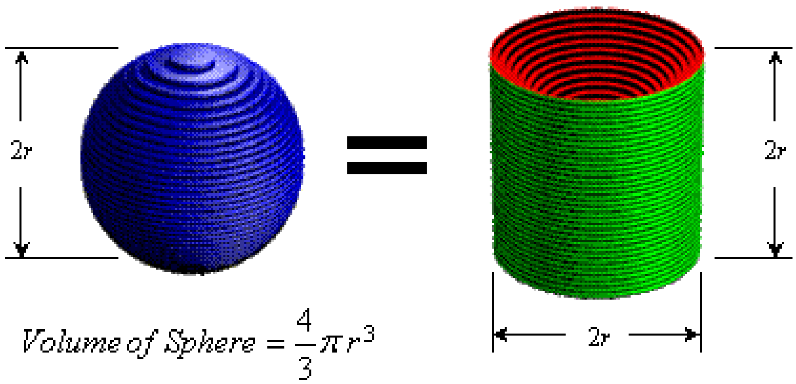
a. Find $B(1)$. Compare this area with $A(1)$, the area of the corresponding slice of the sphere.

b. Find $B(2)$. Compare this area with $A(2)$, the area of the corresponding slice of the sphere.

c. Find $B(3)$. Compare this area with $A(3)$, the area of the corresponding slice of the sphere.

3. Explain how to derive the formula for the volume of a sphere with radius r.



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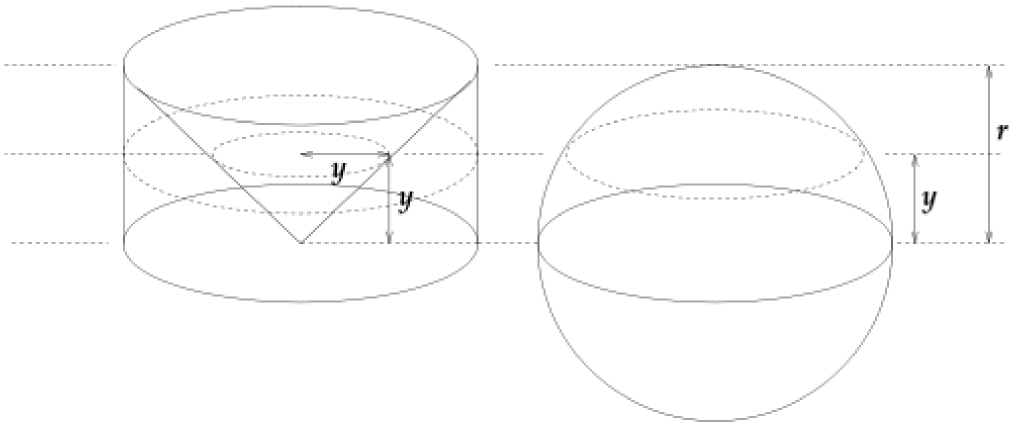
Problem Set

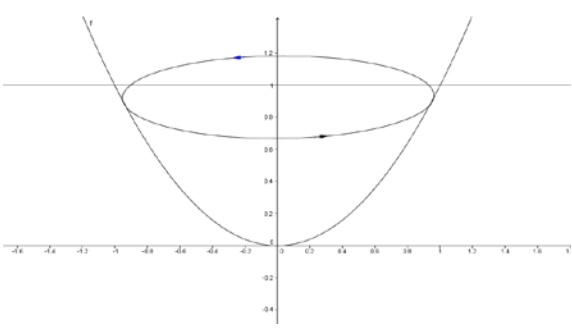
1. Consider the sphere with radius $r = 4$. Suppose that a plane passes through the sphere at a height $y = 2$ units above the center of the sphere, as shown in the figure below.

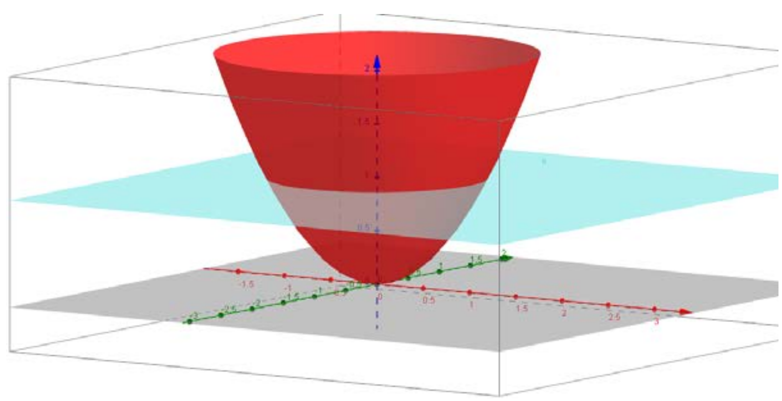


   a. Find the area of the cross section of the sphere.
   b. Find the area of the cross section of the cylinder that lies outside of the cone.
   c. Find the volume of the cylinder, the cone, and the hemisphere shown in the figure.
   d. Find the volume of the sphere shown in the figure.
   e. Explain using Cavalieri's principle the formula for the volume of any single solid.

2. Give an argument for why the volume of a right prism is the same as an oblique prism with the same height.

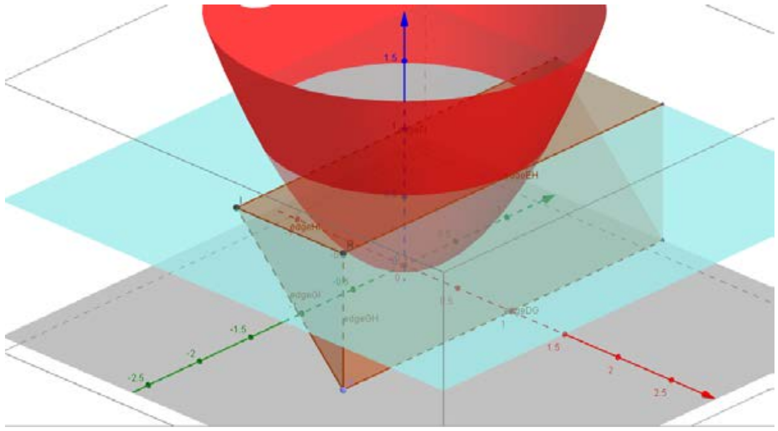
3. A *paraboloid of revolution* is a three-dimensional shape obtained by rotating a parabola around its axis. Consider the solid between a paraboloid described by the equation $y = x^2$ and the line $y = 1$.





   a. Cross sections perpendicular to the y-axis of this paraboloid are what shape?
   b. Find the area of the largest cross section of this solid, when $y = 1$.
   c. Find the area of the smallest cross section of this solid, when $y = 0$.

d. Consider a right triangle prism with legs of length 1, hypotenuse of length $\sqrt{2}$, and depth π as pictured below. What shape are the cross sections of the prism perpendicular to the y-axis?



e. Find the areas of the cross sections of the prism at $y = 1$ and $y = 0$.

f. Verify that at $y = y_0$, the areas of the cross sections of the paraboloid and the prism are equal.

g. Find the volume of the paraboloid between $y = 0$ and $y = 1$.

h. Compare the volume of the paraboloid to the volume of the smallest cylinder containing it. What do you notice?

i. Let V_{cyl} be the volume of a cylinder, V_{par} be the volume of the inscribed paraboloid, and V_{cone} be the volume of the inscribed cone. Arrange the three volumes in order from smallest to largest.

4. Consider the graph of f described by the equation $f(x) = \frac{1}{2}x^2$ for $0 \le x \le 10$.

a. Find the area of the 10 rectangles with height $f(i)$ and width 1, for $i = 1,2,3, \dots ,10$.

b. What is the total area for $0 \le x \le 10$? That is, evaluate: $\sum_{i=1}^{10} f(i) \cdot \Delta x$ for $\Delta x = 1$.

c. Draw a picture of the function and rectangles for $i = 1,2,3$.

d. Is your approximation an overestimate or an underestimate?

e. How could you get a better approximation of the area under the curve?

5. Consider the three-dimensional solid that has square cross sections and whose height y at position x is given by the equation $y = 2\sqrt{x}$ for $0 \le x \le 4$.

a. Approximate the shape with four rectangular prisms of equal width. What is the height and volume of each rectangular prism? What is the total volume?

b. Approximate the shape with eight rectangular prisms of equal width. What is the height and volume of each rectangular prism? What is the total volume?

c. How much did your approximation improve? The volume of the shape is 32 cubic units. How close is your approximation from part (b)?

d. How many rectangular prisms would you need to be able to approximate the volume accurately?